

# MTH 453/553 – Homework 3

1. Consider the following initial boundary value problem for the heat equation

$$\begin{aligned}u_t &= u_{xx}, \text{ for } x \in (0, 1), t > 0 \\u(0, t) &= u(1, t) = 0 \\u(x, 0) &= f(x),\end{aligned}$$

with the function  $f$  defined as

$$f = \begin{cases} 2x & x \leq 1/2, \\ 2(1-x) & x \geq 1/2. \end{cases}$$

A formal solution of this problem can be derived using Fourier's method and is given as

$$u(x, t) = \frac{8}{\pi^2} \sum_{p=1}^{\infty} \left( \frac{\sin(p\pi/2)}{p^2} \right) e^{-(p\pi)^2 t} \sin(p\pi x).$$

- (a) Plot the Fourier series solution as a function of  $x$  for  $t = 0.1$ , by truncating the series after 200 terms.
- (b) Implement the explicit method (forward Euler in MOL), the implicit method (backward Euler in MOL) and the Crank Nicolson method (trapezoidal in MOL) in MATLAB. (Download the code `ImplicitHE.m` from my webpage. This implements the implicit scheme. Modify this to implement the explicit and Crank-Nicolson methods.)
- (c) Using  $h = 0.02$  and  $k = 0.000201$ , so that  $r = k/h^2 = 0.5025$ , show that the numerical solution to the explicit method exhibits instability, whereas the numerical solutions produced by the implicit and Crank Nicolson methods are well behaved. Plot the three numerical solutions and the exact solution on the same graph.
- (d) Choose  $h = k = 0.02$  so that  $r = 50$  in the implicit and Crank Nicolson schemes. What can you say about these numerical solutions? Plot the solutions and the exact solution on the same graph.
2. Consider the following nonlinear heat equation

$$\begin{aligned}u_t &= (\alpha(u)u_x)_x, \text{ for } x \in (0, 1), t > 0 \\u(0, t) &= u(1, t) = 0 \\u(x, 0) &= f(x),\end{aligned}$$

where  $\alpha(u)$  is a given strictly positive and smooth function.

- (a) Let  $v = \alpha(u)u_x$  and justify the following approximations, in particular, what are the errors?

$$u_t(x, t) \approx \frac{u(x, t+k) - u(x, t)}{k}$$

$$v_x(x, t) \approx \frac{v(x+h/2, t) - v(x-h/2, t)}{h}$$

- (b) **(553:)** Show the following approximation. What is the order of the error?

$$v(x+h/2, t) \approx \frac{1}{2} [\alpha(u(x+h, t)) + \alpha(u(x, t))] \frac{u(x+k, t) - u(x, t)}{h}$$

- (c) Use these approximations to derive the scheme

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{\alpha_{j+1/2}^n (U_{j+1}^n - U_j^n) - \alpha_{j-1/2}^n (U_j^n - U_{j-1}^n)}{h^2},$$

where  $\alpha_{j+1/2}^n = (\alpha(U_{j+1}^n) + \alpha(U_j^n))/2$ .

3. Consider the nonlinear heat equation

$$u_t = (uu_x)_x, \text{ for } x \in (0, 1), 0 < t \leq 1$$

$$u(0, t) = t, \quad u(1, t) = 1 + t$$

$$u(x, 0) = x$$

- (a) Verify that the exact solution to this problem is  $u(x, t) = x + t$ .
- (b) **(Bonus):** Show, by induction, that the explicit scheme in Problem 2, part c, gives the exact solution at each grid point, i.e., show that  $U_j^n = x_j + t_n$  for any grid sizes.
- (c) Compute the numerical solution at  $t = 1$  using this explicit scheme. Try the following grid parameters:

- i.  $m = 4$  and  $k = 1/40$
- ii.  $m = 30$  and  $k = 1/3000$

(Note:  $h = 1/(m + 1)$ .) Discuss your observations in light of the result in part b. (Download the code `NonlinearHE.m` from my webpage. This implements the nonlinear heat equation for the example problem that we saw in class. Modify this code for the present problem.)

- (d) From the numerical results obtained in part c, it is clear that some kind of stability condition is needed. Try to come up with a stability condition for this problem. Run some numerical experiments with mesh parameters satisfying this condition. Are the numerical solutions well-behaved if the conditions on the mesh parameters are satisfied?