MTH 453/553 – Homework 1 Solutions

1. (20 points) [Solution to advection system]

   Consider the wave equation
$$u_{tt} = c^2 u_{xx}$$
   written as a first order system of two equations in the form
$$\mathbf{y}_t + A\mathbf{y}_x = 0, \tag{1}$$
   where
$$A = \begin{bmatrix} 0 & -c \\ -c & 0 \end{bmatrix}.$$
   Diagonalize A and decouple the system. Write this decoupled system in terms of the variable $\mathbf{w}$. Determine the characteristics and, hence, $\mathbf{w}$. Knowing $\mathbf{w}$, determine $u(x,t)$ such that it satisfies the initial data
$$u(x,0) = \eta(x), \qquad u_t(x,0) = \mu(x).$$

   Answer: Let
$$\mathbf{y}(x,t) = \begin{bmatrix} y_1(x,t) \\ y_2(x,t) \end{bmatrix} = \begin{bmatrix} u_t(x,t) \\ cu_x(x,t) \end{bmatrix} \tag{2}$$
   Using this definition of $\mathbf{y}$, the system of two first order equations written in vector form is given as above in (1).

   The matrix A can be diagonalized as follows:
$$A = P\Lambda P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$
   Let $\mathbf{w} = P^{-1}\mathbf{y} \implies \mathbf{y} = P\mathbf{w}$. Multiplying (1) by P^{-1} we get the decoupled system
$$P^{-1}\mathbf{y}_t + P^{-1}A\mathbf{y}_x = 0 \implies \mathbf{w}_t + \Lambda\mathbf{w}_x = 0$$
   This can be written in scalar form as
$$\begin{aligned} (w_1)_t + c(w_1)_x &= 0 \\ (w_2)_t - c(w_2)_x &= 0 \end{aligned}$$
   The solution to this system is
$$\begin{aligned} w_1(x,t) &= w_1^0(x-ct) \\ w_2(x,t) &= w_2^0(x+ct) \end{aligned}$$
   where
$$\mathbf{w}(x,0) = \begin{bmatrix} w_1^0(x) \\ w_2^0(x) \end{bmatrix}$$

Since $\mathbf{w} = P^{-1}\mathbf{y}$, this implies that

$$\mathbf{w}(x,0) = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \mathbf{y}(x,0) = \begin{bmatrix} \frac{1}{2}\left(y_1^0(x) - y_2^0(x)\right) \\ \frac{1}{2}\left(y_1^0(x) + y_2^0(x)\right) \end{bmatrix}$$

Hence,

$$\mathbf{y}(x,t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{w}(x,t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{2}\left(y_1^0(x-ct) - y_2^0(x-ct)\right) \\ \dfrac{1}{2}\left(y_1^0(x+ct) + y_2^0(x+ct)\right) \end{bmatrix}$$

So we have

$$\mathbf{y}(x,t) = \frac{1}{2} \begin{bmatrix} \left(y_1^0(x-ct) - y_2^0(x-ct)\right) - \left(y_1^0(x+ct) + y_2^0(x+ct)\right) \\ \left(y_1^0(x-ct) - y_2^0(x-ct)\right) + \left(y_1^0(x+ct) + y_2^0(x+ct)\right) \end{bmatrix}$$

In terms of initial data for the function $u(x,t)$, note that

$$y_1^0(x) = y_1(x,0) = u_t(x,0) = \mu(x)$$

and

$$y_2^0(x) = y_2(x,0) = cu_x(x,0) = c\eta'(x).$$

2. Consider the *advection-diffusion* equation

$$u_t + au_x = \nu u_{xx}$$

with constant coefficients $\nu > 0$ and a. Show that the Caucy problem is well-posed. What happens as $\nu \to 0$?

Answer: Fourier transform the advection-diffusion equation to get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(u_t + au_x - \nu u_{xx}\right) \mathrm{e}^{-i\xi x} dx = 0$$

This implies that

$$\hat{u}_t(\xi,t) + a(i\xi)\hat{u}(\xi,t) - \nu(i\xi)^2\hat{u}(\xi,t) = 0$$

which can be written as

$$\hat{u}_t(\xi,t) = -\left(ai\xi + \nu\xi^2\right)\hat{u}(\xi,t). \tag{3}$$

This is a time dependent ODE for the evolution of $\hat{u}(\xi,t)$ in time. The ODEs for different values of ξ are decoupled from one another. We have to solve an infinite number of ODEs, one for each value of ξ, but they are decoupled scalar equations rather than a coupled system. To solve (3) we need initial data $\hat{u}(\xi,0)$ at $t=0$ for every value of ξ. Fourier trasform the initial data $u(x,0) = \eta(x)$ to get

$$\hat{u}(\xi,0) = \hat{\eta}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-i\xi x}\eta(x) dx.$$

Thus, we now need to solve the initial value problem

$$\hat{u}_t(\xi,t) = -\left(ai\xi + \nu\xi^2\right)\hat{u}(\xi,t) \tag{4}$$
$$\hat{u}(\xi,0) = \hat{\eta}(\xi). \tag{5}$$

The solution to (4)-(5) is

$$\hat{u}(\xi,t) = \mathrm{e}^{-(ai\xi+\nu\xi^2)t}\hat{\eta}(\xi).$$

Using Parseval's identity we have

$$||u(t)||^2 = ||\hat{u}(t)||^2 = \int_{-\infty}^{\infty} |\hat{u}(\xi,t)|^2 d\xi.$$

Thus,

$$\begin{aligned} ||u(t)||^2 &= \int_{-\infty}^{\infty} |\mathrm{e}^{-ai\xi t}|^2 |\mathrm{e}^{-\nu\xi^2 t}|^2 |\hat{\eta}(\xi)|^2 d\xi \\ &\leq \left(\sup_{-\infty<\xi<\infty} |\mathrm{e}^{-\nu\xi^2 t}|^2\right) \int_{-\infty}^{\infty} |\hat{\eta}(\xi)|^2 d\xi \\ &\leq \int_{-\infty}^{\infty} |\hat{\eta}(\xi)|^2 d\xi \\ &= \int_{-\infty}^{\infty} |u(x,0)|^2 dx = ||u(\cdot,0)||^2. \end{aligned}$$

Thus, the PDE is well posed (with $K=1$ and $\alpha=0$).

When $\nu \to 0$, we obtain the advection equation. In this case we have

$$||u(\cdot,t)||^2 = ||u(\cdot,0)||^2.$$