MTH 453/553 – Homework 1

1. (20 points) [Solution to advection system]

   Consider the wave equation

   $$u_{tt} = c^2 u_{xx}$$

   written as a first order system of two equations in the form

   $$\mathbf{y}_t + A\mathbf{y}_x = 0,$$

   where

   $$A = \begin{bmatrix} 0 & -c \\ -c & 0 \end{bmatrix}.$$

   Diagonalize A and decouple the system. Write this decoupled system in terms of the variable $\mathbf{w}$. Determine the characteristics and, hence, $\mathbf{w}$. Knowing $\mathbf{w}$, determine $u(x,t)$ such that it satisfies the initial data

   $$u(x,0) = \eta(x), \qquad u_t(0,x) = \mu(x).$$

2. (20 points) [Well-posedness of advection-diffusion]

   Consider the *advection-diffusion* equation

   $$u_t + au_x = \nu u_{xx}$$

   with constant coefficients $\nu > 0$ and a. Show that the Cauchy problem is well-posed. What happens as $\nu \to 0$ (in particular, what happens to the bound on the norm of the solution)?