

# Math 452/552: Numerical solution of ordinary differential equations: Midterm Review

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## Part 1: Finite Difference Approximations to Derivatives

### § 1.1: Truncation Error (*Order of Accuracy*)

Definitions:

- Finite difference approximations to derivatives
  - various basic examples: forward, backward, centered
- Taylor series with remainder term
- Order of accuracy

Important Skills:

- Determine order of accuracy of an approximation
  - via Taylor expansions
  - via ratios of errors
- Higher orders of accuracy
  - via Taylor expansions of more function evaluations (undetermined coefficients)
  - via higher order quadrature rules
  - via higher order polynomial interpolation

### § 1.2,1.5: Undetermined Coefficients

Definitions:

- Vandermonde system

Important Skills:

- Perform Undetermined Coefficients method

## § 1.3,1.4: Higher Order Derivatives

Definitions:

- one-sided approximations
- centered approximations

Important Skills:

- Construct higher order derivative approximations
  - via polynomial interpolation (e.g., linear)
  - via composition of first order approximations
  - via Undetermined Coefficients

## Part 2: Background on ODE Theory

### IVP vs BVP

Definitions:

- Direction (Slope) Field,
- Initial Value Problem (IVP)
- Classification: ODE vs. PDE, order, linear vs. nonlinear, autonomous
- (Linearized) Pendulum model

Important Skills:

- Convert n-th order ODE to a system

## § 5.2: Existence, Uniqueness, Well-posedness

Definitions:

- Integrating factor
- Lipschitz continuous/constant
- Continuous dependence on initial data
- Well-posedness of an IVP

Theorems:

- Existence and Uniqueness of Solutions
  - first order linear

- $f$  and  $\partial f/\partial u$  continuous
- Lipschitz continuous on unbounded  $D$
- Lipschitz continuous on bounded  $D$
- Systems of ODEs

Important Skills:

- Determine the existence and uniqueness of solutions to differential equations, including interval of definition (e.g.,  $t^*$ ).
- Find Lipschitz constant for a function on  $D$

## [AP] Chapter 2: Stability of Solutions (*Stability Region*)

Definitions:

- stable, asymptotically stable and unstable solutions
- stability region (e.g.,  $S = \{\lambda \in \mathbb{C} | \Re(\lambda) \leq 0\}$ )
- diagonalizable systems
- spectral absciss

Theorems:

- Stability of solutions to linear, constant coefficient ODEs

Important Skills:

- use Taylor expansion of the matrix exponential  $e^{At}$
- diagonalize simple matrices

## Part 3: Numerical Methods for ODEs

### § 5.3-5.6: Taylor Series Methods

Definitions:

- Basic methods: Forward/Backward Euler, leapfrog, trapezoidal, etc.
- explicit/implicit
- local truncation error, one-step error
- order of accuracy, consistency

Important Skills:

- find the LTE for a method
- apply one step of a method

## § 5.7: Multistage Methods

Definitions:

- Basic methods: FE/BE, explicit midpoint, explicit trapezoidal, classical RK4, etc.
- row-sum condition, order conditions (upto third order)
- embedded methods

Theorems:

- maximum order of an  $r$ -stage explicit RK method

Important Skills:

- apply one step of a method
- explain what each stage of a RK method is doing
- convert between definition and Butcher array
- determine order of accuracy
- derive a method of desired accuracy
- why choose explicit vs. implicit?

## § 5.8-5.9: Multistep Methods

Definitions:

- Basic methods: leapfrog, trapezoidal, AB2, Simpson, etc.
- Families: Adams, Adams-Bashforth, Adams-Moulton, etc.
- characteristic polynomial

Important Skills:

- apply one step of a method
- convert between definition and characteristic polynomial
- determine LTE and order of accuracy
- derive a method of desired accuracy
- why choose explicit vs. implicit?
- why choose multistage vs. multistep?
- understand interpolation interpretation
- startup

## § 5.9.4: Predictor-Corrector Methods

Definitions:

- Adams-Bashforth-Moulton
- PE(CE) $^{\mu}$

Theorems:

- order of PE(CE) $^{\mu}$  with a P of order  $p^*$

Important Skills:

- apply one step of a method
- recognize similarity to RK methods
- application to adaptive time-stepping