MTH 452/552 – Homework 5

Do one of the following two.

1. (10 points) [second order methods]

   Implement the following methods for solving the damped linear pendulum system (written in matrix form) (7.11) of Example 7.11.

   (a) midpoint (leapfrog)

   (b) trapezoidal (AM1)

   (c) 2-step explicit Adams-Bashforth method (AB2)

   (d) 2-stage explicit RK method (RK2) (5.30 in the text)

   Test the midpoint, trapezoid, AB2, and RK2 methods (all of which are second order accurate) for each of the following cases (and perhaps others of your choice) and comment on the behavior of each method. Include a discussion of quality of solution versus cost of each method.

   i. $a = 100,\ b = 0$ (undamped)

   ii. $a = 100,\ b = 3$ (damped)

   iii. $a = 100,\ b = 10$ (more damped)

2. (10 points) [two-step methods]

   Implement the following two-step methods on the problem

   $$u' = \lambda\,(u - \sin(t)) + \cos(t), \quad u(0) = 1, \quad 0 \le t \le \pi$$

   (a) midpoint (leapfrog)

   (b) 2-step explicit Adams-Bashforth method (AB2)

   (c) 2-step Backward Differentiation Formula method (BDF)

   $$3U_{n+2} = 4U_{n+1} - U_n + 2kf_{n+2}$$

   (d) The explicit two-step method with the highest possible order of accuracy (Bonus: what is the order?)

   $$U_{n+2} = -4U_{n+1} + 5U_n + 4kf_{n+1} + 2kf_n$$

   For each of the following cases, graph the numerical solutions using $k = .01$ on the same plot as the exact solution $u(t) = \mathrm{e}^{\lambda t} + \sin(t)$.

i. $\lambda = 10$

ii. $\lambda = -10$

iii. $\lambda = -500$

Note: use the exact value $u(k)$ for the starting value U_1. Discuss the expected quality of the approximations, first without doing the calculations, then confirm with your numerical experiments. Comment on the quality versus order of accuracy of each method.

3. (15 points) [Absolute stability region for LMM]

Determine the characteristic polynomials $\rho(\zeta)$ and $\sigma(\zeta)$ for any **three** of the following linear multistep methods. Find and plot the region of absolute stability of these methods. You may wish to use `plotS.m` or `plotBL.m` from the author's website.

(a) The 2-step Adams-Bashforth method

$$U_{n+2} = U_{n+1} + \frac{k}{2}\left(-f(U_n) + 3f(U_{n+1})\right)$$

(b) The 2-step Adams-Moulton method

$$U_{n+2} = U_{n+1} + \frac{k}{12}\left(-f(U_n) + 8f(U_{n+1}) + 5f(U_{n+2})\right)$$

(c) The 2-step Nyström method (explicit midpoint)

$$U_{n+2} = U_n + 2kf(U_{n+1})$$

(d) The 2-step Milne-Simpson method (implicit Nyström)

$$U_{n+2} = U_n + \frac{k}{3}\left(f(U_n) + 4f(U_{n+1}) + f(U_{n+2})\right)$$

(e) The 2-step Backward Differentiation Formula method (BDF)

$$U_{n+2} = \frac{4}{3}U_{n+1} - \frac{1}{3}U_n + \frac{2k}{3}f(U_{n+2})$$

Do one of the following two.

4. (10 points) [θ-method]

For a given ODE $u'(t) = f(u)$, consider the *θ-method*

$$U_{n+1} = U_n + k\left(\theta f(U_{n+1}) + (1-\theta)f(U_n)\right)$$

for some value of θ, $0 \le \theta \le 1$.

(a) Sketch (or plot) the region of absolute stability for some θ in

i. $(0, \frac{1}{2})$

ii. $(\frac{1}{2}, 1)$

(b) Explain if and when one of the above values of θ would be preferred over the typical choices of 0, $\frac{1}{2}$, and 1 (consider cost and accuracy in your answer).

(c) Describe the changes in the qualitative behavior of the regions as θ varies from 0 to 1, especially the transition that occurs as θ crosses $\frac{1}{2}$.

(d) For which values of θ is the method A-stable?

5. (10 points) [Convergence of midpoint method]

Consider the midpoint method $U^{n+1} = U^{n-1} + 2kf(U^n)$ applied to the test problem $u' = \lambda u$. The method is zero-stable and second order accurate, and hence convergent.

If $\lambda < 0$ then the true solution is exponentially decaying.On the other hand, for $\lambda < 0$ and $k > 0$ the point $z = k\lambda$ is never in the region of absolute stability of this method (see Example 7.7), and hence the numerical solution should be growing exponentially for any nonzero time step. (And yet it converges to a function that is exponentially decaying.)

Suppose we take $U^0 = \eta$, use Forward Euler to generate U^1, and then use the midpoint method for $n = 2,\ 3,\ \ldots$. Work out the exact solution U^n by solving the linear difference equation and explain how the apparent paradox described above is resolved.

Do one of the following two.

6. (10 points) [$R(z)$ for Runge-Kutta methods]

Any r-stage Runge-Kutta method applied to $u' = \lambda u$ will give an expression of the form

$$U^{n+1} = R(z)U^n$$

where $z = \lambda k$.

Since $u(t_{n+1}) = e^z u(t_n)$ for this problem, we expect that a pth order accurate method will give a function $R(z)$ satisfying

$$R(z) = e^z + \mathcal{O}(z^{p+1}) \quad \text{as } z \to 0. \tag{1}$$

One can determine the value of p in (1) by expanding e^z in a Taylor series about $z = 0$, writing the $\mathcal{O}(z^{p+1})$ term as

$$Cz^{p+1} + \mathcal{O}(z^{p+2}),$$

multiplying through by the denominator of $R(z)$, and then collecting terms.

(a) Determine $R(z)$ and p for the classical RK4 (5.33).

(b) Determine $R(z)$ and p for the TR-BDF2 method (8.6).

For each of the above, find and plot the stability region (you may wish to use `plotS.m` from the author's website).

(Note: all fourth order, explicit 4-stage RK methods have the same $R(z)$, in fact, all s order, explicit s-stage RK methods with $s \leq 4$ have $R(z)$ in a similar, and obvious, form. Further, these agree with the s order Taylor series methods. Therefore you can use the $R(z)$ for the Taylor series methods in `plotS.m`*. What significant property occurs for $s \geq 5$? What would the stability region look like for $s = 6$ if $R(z)$ followed the same pattern?)*

7. (10 points) [fixed point iteration of implicit methods]

Consider a predictor-corrector method (see Section 5.9.4) consisting of forward Euler as the predictor and backward Euler as the corrector, and suppose we make N correction iterations, i.e., we set

$$\begin{aligned}
&\hat{U}_0 = U_n + kf(U_n)\\
&\text{for } j = 0,\ 1,\ \ldots,\ N-1\\
&\qquad \hat{U}_{j+1} = U_n + kf(\hat{U}_j)\\
&\qquad \text{end}\\
&U_{n+1} = \hat{U}_N.
\end{aligned}$$

Note that this can be interpreted as a *fixed point iteration* for solving the nonlinear equation

$$U^{n+1} = U_n + kf(U^{n+1})$$

of the backward Euler method. Since the backward Euler method is implicit and has a stability region that includes the entire left half plane, as shown in Figure 7.1(b), one might hope that this predictor-corrector method also has a large stability region.

(a) Find the polynomial $R_N(z)$ such that

$$U_{n+1} = R_N(z)U_n$$

for arbitrary N.

(b) Plot the stability region S_N of this method for $N = 2,\ 5,\ 10,\ 20$ (perhaps using `plotS.m` from the author's website) and comment on any change in the size of the stability region.

(c) Note that the fixed point iteration above can only be expected to converge for the test problem if $|k\lambda| < 1$. (Why?) Based on this result, and considering the shape of the stability region of Backward Euler, what do you expect the stability region S_N of part (7b) to converge to as $N \to \infty$?