MTH 452/552 – Homework 4

1. (15 points) [Characteristic polynomials for LMM]

   For the following three methods determine the first characteristic polynomial $\rho(\zeta)$ and find if the method is zero-stable, and if so, strongly or weakly stable.

   (a) The 2-step Nyström method (explicit midpoint)

   $$U_{n+2} = U_n + 2kf(U_{n+1})$$

   (b) The 2-step Backward Differentiation Formula method (BDF)

   $$3U_{n+2} = 4U_{n+1} - U_n + 2kf(U_{n+2})$$

   (c) The 3-step Backward Differentiation Formula method (BDF)

   $$11U_{n+3} = 18U_{n+2} - 9U_{n+1} + 2U_n + 6kf(U_{n+3})$$

2. (15 points) [Deriving a 3-step method]

   Use the method of Undetermined Coefficients to derive the scheme with the highest possible order of accuracy given the following form

   $$\sum_{j=0}^{3} \alpha_j U_{n+j} = k\beta_2 f(U_{n+2})$$

   (note that this is explicit whereas BDF methods are implicit).

   Determine the order of accuracy of your method and whether it is convergent.

Do one of the following three.

3. (10 points) [Lipschitz constant for a one-step method]

   For the two-stage explicit RK method (6.17) show that the Lipschitz constant is $\tilde{L} = L + \frac{1}{2}kL^2$ where L is the Lipschitz constant for the right-hand side of $u' = f$.

4. (10 points) [Convergence of backward Euler method)]

   Suppose the function $f(u)$ is Lipschitz continuous over some domain $|u - \eta| \leq a$ with Lipschitz constant L. Let $g(u) = u - kf(u)$ and let $\Phi(v) = g^{-1}(v)$, the inverse function.

   Show that for $k < 1/L$, the function $\Phi(v)$ is Lipschitz continuous over some domain $|v - f(\eta)| \leq b$ and determine a Lipschitz constant.

Hint: Suppose $v = u - kf(u)$ and $v^* = u^* - kf(u^*)$ and obtain an upper bound on $|u - u^*|$ in terms of $|v - v^*|$.

Note: The backward Euler method (5.21) takes the form

$$U^{n+1} = \Phi(U^n)$$

and so this shows that the implicit backward Euler method is convergent.

5. (10 points) [Forward Euler with variable time-step)]

   Show that the Forward Euler with variable time-step k_n is zero-stable.

   Hint: You will need to show that

$$(1 + k_{j+1}L)\dots(1 + k_n)L \le \mathrm{e}^{L(t_n - t_j)}, \quad 0 \le j \le n.$$

   and

$$k_j \mathrm{e}^{L(t_n - t_j)} \le \int_{t_{j-1}}^{t_j} \mathrm{e}^{L(t_n - t)} dt.$$