MTH 452/552 – Homework 3

Do either 1 or 2.

1. (40 points) [Use of `ode113` and `ode45`]

   This problem can be solved by a modifying the m-files odesample.m and odesampletest.m available from the author's webpage.

   Consider the third order initial value problem

   $$\begin{aligned} &v'''(t) + v''(t) + 4v'(t) + 4v(t) = 4t^2 + 8t - 10, \\ &v(0) = -3, \quad v'(0) = -2, \quad v''(0) = 2. \end{aligned}$$

   (a) Verify that the function

   $$v(t) = -\sin(2t) + t^2 - 3$$

   is a solution to this problem. How do you know it is the unique solution?

   (b) Rewrite this problem as a first order system of the form $u'(t) = f(u(t), t)$ where $u(t) \in \mathbb{R}^3$. Make sure you also specify the initial condition $u(0) = \eta$ as a 3-vector.

   (c) Use the MATLAB function `ode113` to solve this problem over the time interval $0 \le t \le 2$. Plot the true and computed solutions to make sure you've done this correctly.

   (d) Test the MATLAB solver by specifying different tolerances spanning several orders of magnitude. Create a table showing the maximum error in the computed solution for each tolerance and the number of function evaluations required to achieve this accuracy.

   (e) Repeat part (d) using the MATLAB function `ode45`, which uses an embedded pair of Runge-Kutta methods instead of Adams-Bashforth-Moulton methods.

   (f) Make as many useful observations on the results in your tables as possible. Attempt to provide explanations. At least discuss which method performs better and why.

2. (40 points) [Use of explicit RK1, RK2 and RK4]

   (a) (**452:**) Write a code in the form of a `.m` function file for each of the following methods:

   I Forward Euler

   II Explicit Midpoint

   III Classical RK4 (5.33 in the text)

Each code should accept as input: the function defining the right-hand side of an ode (may be a vector valued function of a vector) $\vec{f}(t, \vec{u})$, an interval of time on which to find a numerical solution, the initial condition $\vec{u}_0$, and the time-step k. Each code should output the solution and the times on which the solutions were approximated, i.e., $\{t_n\}_{n=0}^N$ and $\{\vec{U}_n\}_{n=0}^N$. If we define `f=@(t,u) t^2*sin(u)` (see Example 5.10 in the text), an example of a calling sequence might be `[t,u]=rk4(f,[0 1], 2, .001)`.

(b) (**552:**) Write *one* code in the form of a `.m` function file which accepts each of the inputs in (a) plus a matrix A, and vectors $\vec{b}$ and $\vec{c}$ where $A, \vec{b}$ and $\vec{c}$ define the RK method to be used to compute the numerical solution. An example calling sequence might be `[t,u]=rk(f,[0 1], 2, .001, A, b, c)`.

(c) Use your code(s) from either (a) or (b) to answer the following questions. Consider the initial value problem

$$\begin{aligned} u'(t) &= -5tu^2 + \frac{5}{t} - \frac{1}{t^2}, \quad 1 \le t \le 25 \\ u(1) &= 1. \end{aligned}$$

Note that the exact solution is $1/t$.

i. Solve the IVP over the time interval $1 \le t \le 25$ with Forward Euler using time-steps ranging from 2^{-2} to 2^{-9} halving the time-step for each subsequent case (i.e., `j=[-2:-1:-9]` and `k=2^j`).

ii. Repeat (i) for Explicit Midpoint method.

iii. Repeat (i) for classical RK4.

iv. Make a table of the maximum absolute value of the error for each method above using each time-step. Also include the ratio of errors (see HW1 for an example of making a table in MATLAB). The header of your table may look like

k	RK1 error	RK1 ratio	RK2 error	RK2 ratio	RK4 error	RK4 ratio

You may instead simply make one table for each method.

v. Make as many useful observations on the results in your table(s) as possible. Attempt to provide explanations. At least discuss any rates of convergence that do not agree with theoretical expectations. Plotting solutions may help (e.g., Forward Euler using $k = .25$ and $k = .125$ in the same figure).

vi. (**552:**) Comparing errors for methods with different numbers of stages is not fair. For each case above, compute the number of function evaluations required to find the solution (alternatively, you may use `tic` and `toc` to find the actual runtime). The ratio of error to cost gives a measure of the *efficiency* of a method for a given time-step k. Plot this efficiency ratio as a function of k for each method in the same figure. Discuss which method is most efficient (for this IVP).

3. (10 points) [Order conditions]

   Find a family of *explicit third order accurate three-stage methods* by filling in the Butcher array below

$$\begin{array}{c|ccc} 0 & ? & ? & ? \\ ? & ? & ? & ? \\ 2/3 & ? & ? & ? \\ \hline & 1/4 & ? & \alpha \end{array}$$

4. (10 points) [Order of accuracy of Runge-Kutta methods]

   Consider the Runge-Kutta methods defined by the tableaux below. In each case show that the method is third order accurate in two different ways: (optional) First by checking that the order conditions (5.35), (5.37), and (5.38) are satisfied, and (required) then by applying one step of the method to $u' = \lambda u$ and verifying that the Taylor series expansion of $e^{k\lambda}$ is recovered to the expected order.

   (a) Runge's 3rd order method:

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1 & 0 & 1 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 2/3 & 0 & 1/6 \end{array}$$

   (b) Heun's 3rd order method:

$$\begin{array}{c|ccc} 0 & & & \\ 1/3 & 1/3 & & \\ 2/3 & 0 & 2/3 & \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

5. (10 points) [Row-sum condition]

   The *row-sum condition* is actually not necessary for consistency. The one-stage Radau IA method defined as

$$\begin{array}{c|c} 0 & 1 \\ \hline & 1 \end{array}$$

   is first order accurate (the proof involves Taylor expansions in two dimensions).

   Sketch one step of the method using $k = 1$ applied the sample problem $u' = t + 1/u$, $u(0) = 1$. Include in your sketch the point where the direction field is sampled. Comment on whether this method seems like a good idea in general.

6. (15 points) [θ-method]

   For a given ODE $u'(t) = f(u)$, consider the θ-*method*

   $$U_{n+1} = U_n + k\left(\theta f(U_{n+1}) + (1-\theta) f(U_n)\right)$$

   for some value of θ, $0 \le \theta \le 1$.

   (a) What are the common names of the methods obtained by value
      i. $\theta = 0$
      ii. $\theta = 1$
      iii. $\theta = 1/2$

   (b) Write the θ-method as a general Runge-Kutta method using a Butcher array.

   (c) What is the order of the method (based on order conditions as given in the text)?

7. (5 points) [Use of implicit RK for linear systems]

   Write the explicit update step (in terms of the inverse of a matrix, do not invert the matrix), for the Backward Euler method applied to the linearized pendulum model (converted to a system of first order ODEs).

8. (5 points) [Use of implicit RK for non-linear equations]

   Write the explicit update step for the Backward Euler method applied to the non-dimensionalized logistic population model

   $$u'(t) = r(1-u)u$$

9. (10 points) [Derivation of Adams-Moulton]

   Determine the coefficients $\beta_0,\ \beta_1,\ \beta_2$ for the third order, 2-step Adams-Moulton method using the expression for the local truncation error (from Section 5.9.1)

   $$\tau(t_{n+r}) = \frac{1}{k}\left(\sum_{j=0}^{r} \alpha_j u(t_{n+j}) - k\sum_{j=0}^{r} \beta_j u'(t_{n+j})\right)$$

   and taking a Taylor expansion of each function around t_n.

10. (15 points) [Characteristic polynomials and LTE for LMM]

    Determine the characteristic polynomials $\rho(\zeta)$ and $\sigma(\zeta)$ for any **three** the following linear multistep methods. Also, verify that

    $$\sum_{j=0}^{r}\left(\frac{1}{q!} j^q \alpha_j - \frac{1}{(q-1)!} j^{q-1}\beta_j\right) = 0$$

    for $q = 1..p$, and is not zero for $q = p+1$, where p is the order of the method.

(a) The 2-step Adams-Bashforth method

$$U_{n+2} = U_{n+1} + \frac{k}{2}\left(-f(U_n) + 3f(U_{n+1})\right)$$

(b) The 2-step Adams-Moulton method

$$U_{n+2} = U_{n+1} + \frac{k}{12}\left(-f(U_n) + 8f(U_{n+1}) + 5f(U_{n+2})\right)$$

(c) The 2-step Nyström method (explicit midpoint)

$$U_{n+2} = U_n + 2kf(U_{n+1})$$

(d) The 2-step Milne-Simpson method (implicit Nyström)

$$U_{n+2} = U_n + \frac{k}{3}\left(f(U_n) + 4f(U_{n+1}) + f(U_{n+2})\right)$$

(e) The 2-step Backward Differentiation Formula method (BDF)

$$U_{n+2} = \frac{4}{3}U_{n+1} - \frac{1}{3}U_n + \frac{2k}{3}f(U_{n+2})$$

11. (10 points) [Predictor-corrector methods]

Show that the one-step Adams-Bashforth-Moulton method is actually an RK method (which one?). What is the order of this method? Is this consistent with expected order for a predictor-corrector method?