MTH 452/552 – Homework 2

1. (10 points) (**452:**) [Uniqueness for an ODE]

   Prove that the ODE
   $$u'(t) = \frac{1}{t^2 + u(t)^2}, \quad \text{for } t \geq 1$$
   has a unique solution for all time from any initial value $u(1) = \eta$.

2. (10 points) [Well-posedness for an ODE]

   Discuss the well-posedness of the IVP
   $$\begin{aligned} u'(t) &= u(t)^{\frac{1}{3}} \\ u(0) &= \eta. \end{aligned}$$

3. [Lipschitz constant for an ODE]

   Let $f(u) = \log(u)$.

   (a) (10 points) Determine the best possible Lipschitz constant for this function over $2 \leq u < \infty$.

   (b) (5 points) Is $f(u)$ Lipschitz continuous over $0 < u < \infty$? Explain.

   (c) (10 points) Consider the initial value problem
   $$\begin{aligned} u'(t) &= \log(u(t)), \\ u(0) &= 2. \end{aligned}$$
   Explain why we know that this problem has a unique solution for all $t \geq 0$ based on the existence and uniqueness theory described in Section 5.2.1. (Hint: Argue that f is Lipschitz continuous in a domain that the solution never leaves, though the domain is not symmetric about $\eta = 2$ as assumed in the theorem quoted in the book.)

4. (20 points) [Lipschitz constant for a system of ODEs]

   Consider the system of ODEs
   $$\begin{aligned} u_1' &= 3u_1 + 4u_2, \\ u_2' &= 5u_1 - 6u_2. \end{aligned}$$
   Determine the Lipschitz constant for this system in the max-norm $\|\cdot\|_\infty$ and the 1-norm $\|\cdot\|_1$. (See Appendix A.3.)

5. (10 points) (**552:**) [Duhamel's principle]

   Check that the solution $u(t)$ given by

$$u(t) = \mathrm{e}^{A\cdot(t-t_0)}\eta + \int_{t_0}^{t} \mathrm{e}^{A\cdot(t-\tau)} g(\tau)d\tau$$

   satisfies the constant coefficient system

$$u'(t) = Au(t) + g(t)$$

   with initial condition $u(0) = \eta$. Hint: To differentiate the matrix exponential you can differentiate the Taylor series expansion of e^{At} (D.31) term by term. Note: the use of $\cdot$ in $A \cdot (t-\tau)$ above is meant to distinguish multiplication from the possibility of the matrix A being time-varying and evaluated at time $t-\tau$, i.e., $A(t-\tau)$.

6. (**452:**) [Problem stability] For each of the following matrices A, determine whether the system $u'(t) = Au(t)$ is *unstable*, *stable*, or *asymptotically stable*.

   (a) (5 points) $\begin{bmatrix} -1 & 10 \\ 0 & -2 \end{bmatrix}$

   (b) (5 points) $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

   (c) (5 points) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

7. (**552:**) [Problem stability] Consider the following variable coefficient matrix

$$\begin{bmatrix} -\frac{1}{4} + \frac{3}{4}\cos 2t & -\frac{3}{4}\sin 2t \\ -\frac{3}{4}\sin 2t & -\frac{1}{4} - \frac{3}{4}\cos 2t \end{bmatrix}$$

   (a) (10 points) Determine the eigenvalues of the matrix. (Hint: you may want to use $X = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$.)

   (b) (5 points) Determine whether the system $u'(t) = A(t)u(t)$ is *unstable*, *stable*, or *asymptotically stable*.