Math 452/552: Numerical solution of ordinary differential equations: Review

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Part 1: Finite Difference Approximations to Derivatives

§ 1.1: Truncation Error (*Order of Accuracy*)

Definitions:

- Finite difference approximations to derivatives
  - various basic examples: forward, backward, centered
- Taylor series with remainder term
- Order of accuracy

Important Skills:

- Determine order of accuracy of an approximation
  - via Taylor expansions
  - via ratios of errors
- Higher orders of accuracy
  - via Taylor expansions of more function evaluations (undetermined coefficients)
  - via higher order quadrature rules
  - via higher order polynomial interpolation

§ 1.2,1.5: Undetermined Coefficients

Definitions:

- Vandermonde system

Important Skills:

- Perform Undetermined Coefficients method

§ 1.3,1.4: Higher Order Derivatives

Definitions:

- one-sided approximations
- centered approximations

Important Skills:

- Construct higher order derivative approximations
  - via polynomial interpolation (e.g., linear)
  - via composition of first order approximations
  - via Undetermined Coefficients

Part 2: Background on ODE Theory

IVP vs BVP

Definitions:

- Direction (Slope) Field,
- Initial Value Problem (IVP)
- Classification: ODE vs. PDE, order, linear vs. nonlinear, autonomous
- (Linearized) Pendulum model

Important Skills:

- Convert n-th order ODE to a system

§ 5.2: Existence, Uniqueness, Well-posedness

Definitions:

- Integrating factor
- Lipschitz continuous/constant
- Continuous dependence on initial data
- Well-posedness of an IVP

Theorems:

- Existence and Uniqueness of Solutions
  - first order linear

- f and $\partial f/\partial u$ continuous
- Lipschitz continuous on unbounded D
- Lipschitz continuous on bounded D
- Systems of ODEs

Important Skills:

- Determine the existence and uniqueness of solutions to differential equations, including interval of definition (e.g., t^*).
- Find Lipschitz constant for a function on D

[AP] Chapter 2: Stability of Solutions (***Stability Region***)

Definitions:

- stable, asymptotically stable and unstable solutions
- stability region (e.g., $S = \{\lambda \in \mathbb{C} | \Re(\lambda) \leq 0\}$)
- diagonalizable systems
- spectral absciss

Theorems:

- Stability of solutions to linear, constant coefficient ODEs

Important Skills:

- use Taylor expansion of the matrix exponential e^{At}
- diagonalize simple matrices

Part 3: Numerical Methods for ODEs

§ 5.3-5.6: Taylor Series Methods

Definitions:

- Basic methods: Forward/Backward Euler, leapfrog, trapezoidal, etc.
- explicit/implicit
- local truncation error, one-step error
- order of accuracy, consistency

Important Skills:

- find the LTE for a method
- apply one step of a method

§ 5.7: Multistage Methods

Definitions:

- Basic methods: FE/BE, explicit midpoint, explicit trapezoidal, classical RK4, etc.
- row-sum condition, order conditions (upto third order)
- embedded methods

Theorems:

- maximum order of an r-stage explicit RK method

Important Skills:

- apply one step of a method
- explain what each stage of a RK method is doing
- convert between definition and Butcher array
- determine order of accuracy
- derive a method of desired accuracy
- why choose explicit vs. implicit?

§ 5.8-5.9: Multistep Methods

Definitions:

- Basic methods: leapfrog, trapezoidal, AB2, Simpson, etc.
- Families: Adams, Adams-Bashforth, Adams-Moulton, Nystrom, Milne-Simpson, BDF, etc.
- characteristic polynomials

Important Skills:

- apply one step of a method
- convert between definition and characteristic polynomials
- determine LTE and order of accuracy
- derive a method of desired accuracy using undetermined coefficients
- why choose explicit vs. implicit?
- why choose multistage vs. multistep?
- understand interpolation interpretation
- startup

§ **5.9.4: Predictor-Corrector Methods**

Definitions:

- Adams-Bashforth-Moulton
- PE(CE)$^\mu$

Theorems:

- order of PE(CE)$^\mu$ with a P of order p^*

Important Skills:

- apply one step of a method
- recognize similarity to RK methods
- recognize difference from embedded RK methods
- application to adaptive time-stepping

Part 4: Analysis of Methods

§ **5.7 & 5.9.1: Consistency**

Definitions:

- consistent method
- one-step error

Theorems:

- order conditions for RK methods
- order constants for LMM

Important Skills:

- relation to characteristic polynomials of LMM

§ **6.1-6.3: Convergence**

Definitions:

- consistent starting values for LMM
- convergence
- test problem

Theorems:

- convergence of Euler on test problem
- convergence of Euler on Lipschitz problem
- convergence of explicit one-step method
- 6.3 (LMM): consistent + zero-stable = convergent

§ 6.4: Zero-stability

Definitions:

- zero-stable
- strongly/weakly zero-stable
- characteristic polynomial
- root condition
- principal/parasitic roots

Theorems:

- root condition implies zero-stable

Important Skills:

- find solutions to linear difference equation
- determine zero-stability of a LMM
- understand rationale for choice of α_j's in Adams methods

Chapter 7: Absolute Stability (*Absolute Stability Region*)

Definitions:

- absolute stability requirement
- absolute stability region
- stability polynomial for LMM
- amplification function for one-step methods
- BDF methods

Theorems:

- stability region of LMM is where stability polynomial satisfies root condition

Important Skills:

- determine stability regions
- understand boundary locus method for LMM
- find $R(z)$ for one-step methods
- determine largest possible k ensuring stability
- understand consequences of absolute stability region not agreeing with problem stability
- recognize methods appropriate for various scenarios
- understand cost of improving stability

Part 5: Chapter 8: Stiff ODEs

Definitions:

- transients
- stiffness
- stiffness ratio
- $A-$stable
- $A(\alpha)-$stable
- $L-$stable
- BDF methods

Important Skills:

- recognize stiffness in a direction field plot
- recognize stiffness in a linear system
- explain rationale for choice of β_j's in BDF methods
- understand necessity for $A(\alpha)-$stability instead of A-stability
- understand problem with explicit methods (and Adams-Moulton methods)