MTH 452/552 – Homework 5

1. (15 points) [Absolute stability region for LMM]

   Determine the characteristic polynomials $\rho(\zeta)$ and $\sigma(\zeta)$ for any **three** of the following linear multistep methods. Find and plot the region of absolute stability of these methods. You may wish to use `plotS.m` or `plotBL.m` from the author's website.

   (a) The 2-step Adams-Bashforth method

   $$U_{n+2} = U_{n+1} + \frac{k}{2}\left(-f(U_n) + 3f(U_{n+1})\right)$$

   (b) The 2-step Adams-Moulton method

   $$U_{n+2} = U_{n+1} + \frac{k}{12}\left(-f(U_n) + 8f(U_{n+1}) + 5f(U_{n+2})\right)$$

   (c) The 2-step Nyström method (explicit midpoint)

   $$U_{n+2} = U_n + 2kf(U_{n+1})$$

   (d) The 2-step Milne-Simpson method (implicit Nyström)

   $$U_{n+2} = U_n + \frac{k}{3}\left(f(U_n) + 4f(U_{n+1}) + f(U_{n+2})\right)$$

   (e) The 2-step Backward Differentiation Formula method (BDF)

   $$U_{n+2} = \frac{4}{3}U_{n+1} - \frac{1}{3}U_n + \frac{2k}{3}f(U_{n+2})$$

2. (10 points) [θ-method]

   For a given ODE $u'(t) = f(u)$, consider the θ-*method*

   $$U_{n+1} = U_n + k\left(\theta f(U_{n+1}) + (1-\theta)f(U_n)\right)$$

   for some value of θ, $0 \le \theta \le 1$.

   (a) Sketch (or plot) the region of absolute stability for some θ in

      i. $(0, \frac{1}{2})$

      ii. $(\frac{1}{2}, 1)$

   (b) Explain if and when one of the above values of θ would be preferred over the typical choices of 0, $\frac{1}{2}$, and 1 (consider cost and accuracy in your answer).

(c) Describe the changes in the qualitative behavior of the regions as θ varies from 0 to 1, especially the transition that occurs as θ crosses $\frac{1}{2}$.

(d) For which values of θ is the method A-stable?

Do one of the following two.

3. (10 points) [$R(z)$ for Runge-Kutta methods]

Any r-stage Runge-Kutta method applied to $u' = \lambda u$ will give an expression of the form

$$U^{n+1} = R(z)U^n$$

where $z = \lambda k$.

Since $u(t_{n+1}) = e^z u(t_n)$ for this problem, we expect that a pth order accurate method will give a function $R(z)$ satisfying

$$R(z) = e^z + \mathcal{O}(z^{p+1}) \quad \text{as } z \to 0. \tag{1}$$

One can determine the value of p in (1) by expanding e^z in a Taylor series about $z = 0$, writing the $\mathcal{O}(z^{p+1})$ term as

$$Cz^{p+1} + \mathcal{O}(z^{p+2}),$$

multiplying through by the denominator of $R(z)$, and then collecting terms.

(a) Determine $R(z)$ and p for the classical RK4 (5.33).

(b) Determine $R(z)$ and p for the TR-BDF2 method (8.6).

For each of the above, find and plot the stability region (you may wish to use `plotS.m` from the author's website).

(Note: all fourth order, explicit 4-stage RK methods have the same $R(z)$, in fact, all s order, explicit s-stage RK methods with $s \leq 4$ have $R(z)$ in a similar, and obvious, form. Further, these agree with the s order Taylor series methods. Therefore you can use the $R(z)$ for the Taylor series methods in `plotS.m`*. What significant property occurs for $s \geq 5$? What would the stability region look like for $s = 6$ if $R(z)$ followed the same pattern?)*

4. (10 points) [fixed point iteration of implicit methods]

Consider a predictor-corrector method (see Section 5.9.4) consisting of forward Euler as the predictor and backward Euler as the corrector, and suppose we make N correction iterations, i.e., we set

$$\hat{U}_0 = U_n + kf(U_n)$$
for $j = 0,\ 1,\ \ldots,\ N-1$

$$\hat{U}_{j+1} = U_n + kf(\hat{U}_j)$$
end
$$U_{n+1} = \hat{U}_N.$$

Note that this can be interpreted as a *fixed point iteration* for solving the nonlinear equation

$$U^{n+1} = U_n + kf(U^{n+1})$$

of the backward Euler method. Since the backward Euler method is implicit and has a stability region that includes the entire left half plane, as shown in Figure 7.1(b), one might hope that this predictor-corrector method also has a large stability region.

(a) Find the polynomial $R_N(z)$ such that

$$U_{n+1} = R_N(z)U_n$$

for arbitrary N.

(b) Plot the stability region S_N of this method for $N = 2,\ 5,\ 10,\ 20$ (perhaps using `plotS.m` from the author's website) and comment on any change in the size of the stability region.

(c) Note that the fixed point iteration above can only be expected to converge for the test problem if $|k\lambda| < 1$. (Why?) Based on this result, and considering the shape of the stability region of Backward Euler, what do you expect the stability region S_N of part (4b) to converge to as $N \to \infty$?