MTH 351 Fall 2006 – Lab 5

Due: Before class November 29

Suppose it were known that a certain function $y(x)$ satisfied the following three properties:

$$\text{(i.)}\ y''(x) = -\sin(x) \qquad \text{(ii.)}\ y(0) = 0 \qquad \text{(iii.)}\ y(\pi) = \pi$$

The solution to this *two-point boundary value problem* (*cf.* Sec 8.8) can be found numerically by means of a *finite-difference approximation* for the derivative and the solution to a *linear system of equations*.

1. Let $x_j = hj$, for $j = 0, \ldots, n$ where $h = \pi/n$, be a discretization of the domain $[0, \pi]$. Then let $y_j = y(x_j)$ be the solution of the boundary value problem (BVP) evaluated at the discretization nodes $\{x_j\}$. Show that applying the *second order centered difference formula* (5.91) for approximating the second derivative in (i.), evaluated at x_j, results in
$$\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = -\sin(x_j). \tag{1}$$

2. Show that equation (1) evaluated for $j = 1, \ldots, n-1$, combined with $y_0 = 0$ and $y_n = \pi$, gives the following matrix equation for $y = \{y_j\}$:
$$Ay = b \tag{2}$$
where
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ & & \ddots & & & \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ h^2\sin(x_1) \\ h^2\sin(x_2) \\ \vdots \\ h^2\sin(x_{n-1}) \\ \pi \end{bmatrix}$$
It is sufficient to show that it is true for the case where $n = 4$. (See (8.134) for help.)

3. Use Gaussian elimination to solve the system in (2) with $n = 4$. Either write your own code to define A and b or copy/paste the following:

```
n=4;
h=pi/n;
xi=[0:h:pi]';
b=h^2*sin(xi);
b(1)=0; b(n+1)=pi;
e = ones(n+1,1);
A = 2*diag(e,0)-diag(e(1:n),-1)-diag(e(1:n),1);
A(1,1)=1; A(1,2)=0;
A(n+1,n+1)=1; A(n+1,n)=0;
```

You may use either the `GEpivot` code from the publisher's website or write your own. Compare your solution graphically to the exact solution using something similar to the following:

```
plot(xi,yi,'--o')
hold on
ezplot('x+sin(x)',[0 pi])
```

4. Suppose you only need an estimate to the solution with a relative error of the residual less than $\delta = 10^{-3}$. Solve (2) for $n = 1000$ using Gaussian elimination with your code from 3 as well as with the following iterative methods:

   (a) Jacobi iteration (see `Jacobi`)

   (b) Gauss-Seidel iteration (see `GS`)

   and compare the runtimes (see `help tic` for help on computing runtimes). You do not need to display or plot your solutions. Edit the files *Jacobi.m* and *GS.m* by adding a ";" at the end of any line that contains `x =` or `x0 =` to avoid having each iteration print to the screen. Use `x0 = b` and `max_it = 1000`.

5. (**Bonus**) Solve (2) with $n = 1000$ using Crout factorization. Modify the definition of A above in order to use `tridiag` from the publisher's website (and described in the book). Compare the runtime for the case where the factorization needs to be computed to the case where the factorization is given (`iflag` $= 0$ and 1, respectively).

Turn in a hard copy of the Matlab output, including figures, along with your answers to the questions for each problem. Email a copy of the script which runs all the commands. If you prefer to create one file with commands and comments, then begin your comment lines with %.