The Nonlinear Pendulum

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1 The Simple Plane Pendulum

A *simple plane pendulum* consists, ideally, of a point mass connected by a light rod of length L to a frictionless pivot. The mass is displaced from its natural vertical position and released, after which it swings back and forth. There are two major questions we would like to answer:

1. What is the angle θ of the pendulum from the vertical at any time t?
2. What is the period of the motion?

For such a simple system, the simple plane pendulum has a surprisingly complicated solution. We'll first derive the differential equation of motion to be solved, then find both the approximate and exact solutions.

2 Differential Equation of Motion

To derive the differential equation of motion for the pendulum, we begin with Newton's second law in rotational form:

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2}, \tag{1}$$

where τ is the torque, I is the moment of inertia, α is the angular acceleration, and θ is the angle from the vertical. In the case of the pendulum, the torque is given by

$$\tau = -mgL\sin\theta, \tag{2}$$

and the moment of inertia is

$$I = mL^2. \tag{3}$$

Substituting these expressions for τ and I into Eq. (1), we get the second-order differential equation

$$-mgL\sin\theta = mL^2\frac{d^2\theta}{dt^2}, \tag{4}$$

which simplifies to give the differential equation of motion,

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta} \tag{5}$$

3 Approximate Solution

3.1 Equation of Motion

The easy way to solve Eq. (5) is to restrict the solution to cases where the angle θ is small. In that case, we can make the *linear* approximation

$$\sin\theta \approx \theta, \tag{6}$$

where θ is measured in *radians*. In this case, Eq. (5) becomes the differential equation for a simple harmonic oscillator,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta. \tag{7}$$

The solution to this differential equation is

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}}\, t + \delta\right), \tag{8}$$

as may be verified by direct substitution. Here θ_0 and δ are arbitrary constants that depend on the initial conditions. The angle θ_0 is called the *amplitude* of the motion, and is the maximum displacement of the pendulum from the vertical. The constant δ is called the *phase constant*, and represents where in its motion the pendulum is at time $t = 0$.

3.2 Period

Eq. (8) implies that the angular frequency of the motion is $\omega = \sqrt{g/L}$; since the period $T = 2\pi/\omega$, we find the period for small amplitudes to be

$$T_0 = 2\pi\sqrt{\frac{L}{g}}. \tag{9}$$

4 Exact Solution

While the small-angle approximate solution to Eq. (5) is fairly straightforward, finding an exact solution for angles that are *not* necessarily small is considerably more difficult. We won't go through the derivations here—we'll just look at the results. Here we'll assume the amplitude of the motion $\theta_0 < \pi$, so that the pendulum does *not* spin in complete circles around the pivot, but simply oscillates back and forth.

4.1 Equation of Motion

When the amplitude θ_0 is not necessarily small, the angle θ from the vertical at any time t is found to be

$$\theta(t) = 2\sin^{-1}\left\{k \,\mathrm{sn}\left[\sqrt{\frac{g}{L}}\,(t - t_0); k\right]\right\}. \tag{10}$$

where $\mathrm{sn}(x; k)$ is a *Jacobian elliptic function* with modulus $k = \sin(\theta_0/2)$. The time t_0 is a time at which the pendulum is vertical ($\theta = 0$).

The Jacobian elliptic function is one of a number of so-called "special functions" that often appear in mathematical physics. In this case, the function $\mathrm{sn}(x;k)$ is defined as a kind of inverse of an integral. Given the function

$$u(y;k) = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{11}$$

the Jacobian elliptic function is defined as the inverse of u:

$$y = \mathrm{sn}(u;k). \tag{12}$$

Values of $\mathrm{sn}(x;k)$ may be found in tables of functions or computed by specialized mathematical software libraries.

4.2 Period

Eq. (9) is really only an approximate expression for the period of a simple plane pendulum; the smaller the amplitude of the motion, the better the approximation. An *exact* expression for the period is given by

$$T = 4\sqrt{\frac{L}{g}} \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{13}$$

which is a type of integral known as a *complete elliptic integral of the first kind*.

The integral in Eq. (13) cannot be evaluated in closed form, but it *can* be expanded into an infinite series. The result is

$$\begin{aligned} T &= 2\pi\sqrt{\frac{L}{g}}\left\{1 + \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 \sin^{2n}\left(\frac{\theta_0}{2}\right)\right\} && (14) \\ &= 2\pi\sqrt{\frac{L}{g}}\left\{1 + \sum_{n=1}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \sin^{2n}\left(\frac{\theta_0}{2}\right)\right\} && (15) \end{aligned}$$

We can explicitly write out the first few terms of this series; the result is

$$\begin{aligned} T = 2\pi\sqrt{\frac{L}{g}}\Bigg[& 1 + \frac{1}{4}\sin^2\left(\frac{\theta_0}{2}\right) + \frac{9}{64}\sin^4\left(\frac{\theta_0}{2}\right) + \frac{25}{256}\sin^6\left(\frac{\theta_0}{2}\right) \\ & + \frac{1225}{16384}\sin^8\left(\frac{\theta_0}{2}\right) + \frac{3969}{65536}\sin^{10}\left(\frac{\theta_0}{2}\right) + \frac{53361}{1048576}\sin^{12}\left(\frac{\theta_0}{2}\right) + \frac{184041}{4194304}\sin^{14}\left(\frac{\theta_0}{2}\right) \\ & + \frac{41409225}{1073741824}\sin^{16}\left(\frac{\theta_0}{2}\right) + \frac{147744025}{4294967296}\sin^{18}\left(\frac{\theta_0}{2}\right) + \frac{2133423721}{68719476736}\sin^{20}\left(\frac{\theta_0}{2}\right) + \cdots\Bigg]. \end{aligned} \tag{16}$$

If we wish, we can write out a series expansion for the period in another form—one which does not involve the sine function, but only involves powers of the amplitude θ_0. To do this, we expand $\sin(\theta_0/2)$ into a Taylor series:

$$\begin{aligned} \sin\frac{\theta_0}{2} &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\theta_0^{2n-1}}{2^{2n-1}(2n-1)!} && (17) \\ &= \frac{\theta_0}{2} - \frac{\theta_0^3}{48} + \frac{\theta_0^5}{3840} - \frac{\theta_0^7}{645120} + \frac{\theta_0^9}{185794560} - \frac{\theta_0^{11}}{81749606400} + \cdots && (18) \end{aligned}$$

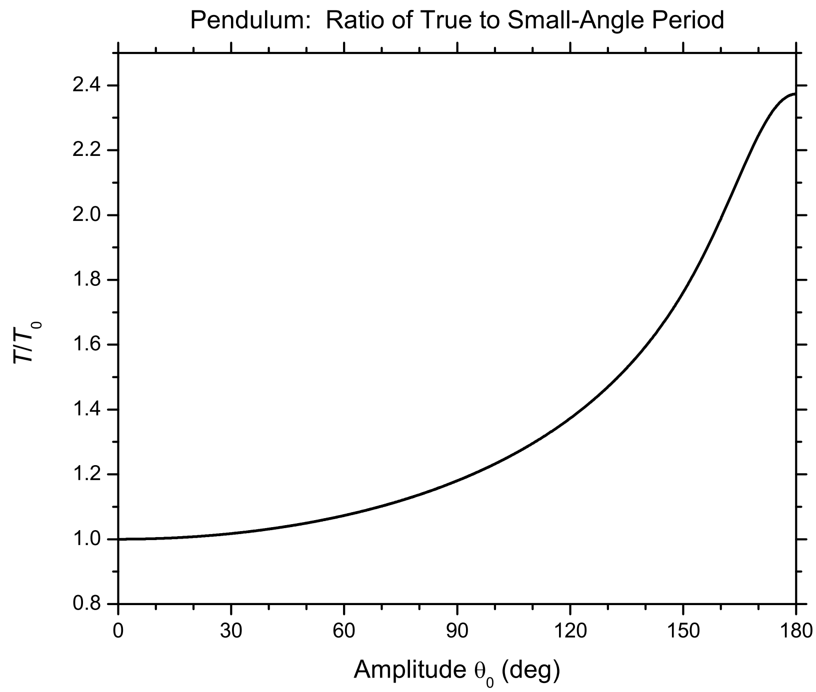

Figure 1: Ratio of a pendulum's true period T to its small-angle period $T_0 = \sqrt{L/g}$, as a function of amplitude θ_0. For small amplitudes, this ratio is near 1; for larger amplitudes, the true period is longer than predicted by the small-angle approximation.

Now substitute this series into the series of Eq. (14) and collect terms. The result is

$$
\begin{aligned}
T = 2\pi\sqrt{\frac{L}{g}}\Bigg(1 &+ \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \frac{173}{737280}\theta_0^6 + \frac{22931}{1321205760}\theta_0^8 + \frac{1319183}{951268147200}\theta_0^{10} \\
&+ \frac{233526463}{2009078326886400}\theta_0^{12} + \frac{2673857519}{265928913086054400}\theta_0^{14} \\
&+ \frac{39959591850371}{44931349155019751424000}\theta_0^{16} + \frac{8797116290975003}{10999194273148835148595 2000}\theta_0^{18} \\
&+ \frac{4872532317019728133}{668751011807449177034588160000}\theta_0^{20} + \cdots\Bigg).
\end{aligned}
\tag{19}
$$

5 Plot of Period vs. Amplitude

Shown in Fig. 1 is a plot of the ratio of the pendulum's true period T to its small-angle period T_0 ($T/(2\pi\sqrt{L/g})$) vs. amplitude θ_0 for values of the amplitude between 0 and 180°, using Eq. (15). As you can see, the ratio is 1 for small amplitudes (as expected), and increasingly deviates from 1 for large amplitudes. The true period will always be longer than the small-angle period T_0.

6 References

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