The Birth of Period 3, Revisited

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Introduction Over the last twenty years, the logistic map,

$$x_{n+1} = rx_n(1 - x_n), \tag{1}$$

has served as an exemplar of nonlinear dynamics [**1**, **2**]. As May stressed some 20 years ago [**3**], the patterns formed by iterates of the logistic map are simple to compute but illustrate the complexities possible in nonlinear dynamics. The bifurcation of the logistic map, which summarizes the long-time dynamics as a function of the control parameter r in Equation 1, is one of the most commonly reproduced images of dynamical systems. (See FIGURE 1.) Also, using this map, Feigenbaum derived his famous renormalization-group theory of scaling exponents. These results were soon shown to apply to real experimental systems, such as fluid undergoing thermal convection [**2**]. Given both the pedagogical value and the scientific importance of the logistic map, exact analytic results concerning its solutions have been collected with great care. One such result is that a period-3 orbit (or 3-cycle), the most prominent of the "periodic windows" in FIGURE 1, is born via a tangent bifurcation at the control-parameter value $r = 1 + \sqrt{8}$.

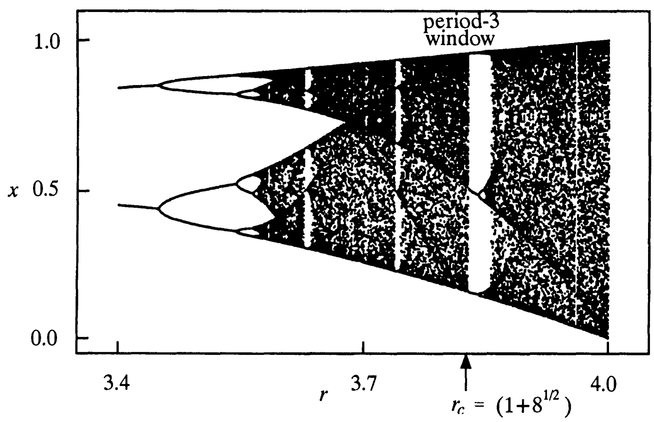

FIGURE 1
Bifurcation diagram of the logistic map. The 3-cycle is indicated on the figure.

A proof of this statement was recently given in these pages by Saha and Strogatz [**1**]. Their method, although straightforward, involves fairly complex algebra and a number of intermediate steps, which led the latter author to wonder whether a simpler proof might not exist. My purpose here is to give just such a proof. An independent (and different!) proof is given in the next article. More precisely, I shall show that given that a tangent bifurcation creates a period-3 cycle somewhere, that 3-cycle must be created at $r = 1 + \sqrt{8}$.

The Proof We first introduce some notation. Let $f(x) = rx(1-x)$ so that Equation 1 becomes $x_{n+1} = f(x_n)$. The second iterate of f, $f(f(x))$, will be denoted as $f^2(x)$, and similarly for higher iterates. Thus, any point of a period-3 orbit p will satisfy $p = f^3(p)$. In FIGURE 2, we plot $f^3(x)$ for a value of r just larger than the onset value for the 3-cycle, r_c. Note that the function $y = f^3(x)$ crosses the diagonal line ($y = x$) eight times. Three crossings, denoted by filled circles, correspond to the stable 3-cycle seen in the long-time dynamics. The nearby set of three crossings denoted by hollow circles correspond to an unstable 3-cycle. If we lower the control parameter r to $r_c = 1 + \sqrt{8}$, the stable and unstable 3-cycles will "collide" and have identical values of x. Finally, the two squares denote values of x that are fixed points of the direct map $f(x)$ (and hence, of course, fixed points of $f^3(x)$).

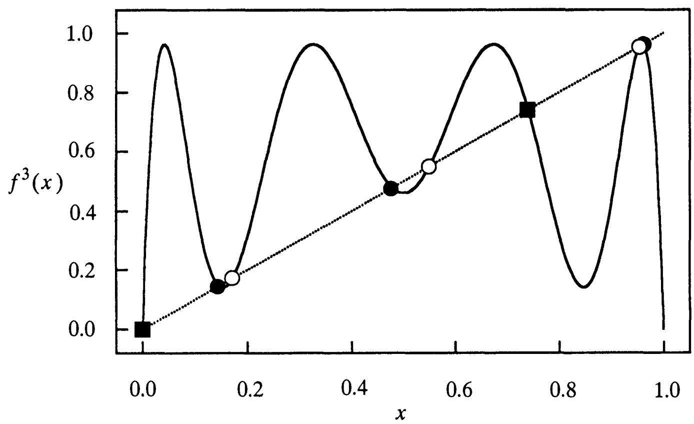

FIGURE 2

Graph of $f^3(x)$ for $r = 3.85$. Filled circles, stable 3-cycles; hollow circles, unstable 3-cycle; filled squares, fixed points of f.

As Saha and Strogatz point out, the condition for a tangent bifurcation may be expressed by the tangency of $y = f^3(x)$ to the line $y = x$. The stability of an orbit is given by the derivative of the map evaluated at any one of the points in the orbit [**2**]. The cycle just goes unstable (via a tangent bifurcation) when that derivative is 1. At a tangent bifurcation, f^3 has slope 1 at each of the 3 points a, b, c of the 3-cycle at $r = r_c$. Thus,

$$\begin{aligned}\frac{d(f^3(x))}{dx} &= \frac{d(f^3(x))}{d(f^2(x))} \cdot \frac{d(f^2(x))}{d(f(x))} \cdot \frac{d(f(x))}{dx} \\ &= \left.\frac{d(f(x))}{dx}\right|_{x=c} \cdot \left.\frac{d(f(x))}{dx}\right|_{x=b} \cdot \left.\frac{d(f(x))}{dx}\right|_{x=a} \\ &= r^3(1-2a)(1-2b)(1-2c) \\ &= 1. \end{aligned} \tag{2}$$

Multiplying this condition out, we have

$$r^3[1 - 2(a+b+c) + 4(ab+bc+ac) - 8(abc)] = 1. \tag{3}$$

Thus, as noted by Saha and Strogatz, we need to find the combinations $\alpha \equiv a+b+c$, $\beta \equiv ab+bc+ac$, and $\gamma \equiv abc$, but *not* a, b, and c individually.

The other condition that r_c must satisfy is that the third iterate of $f(x)$ have a fixed point at $x=a$, b, and c. This motivates us to define an auxiliary function $g(x)=f^3(x)-x$, which has roots at the fixed points of $f(x)$. FIGURE 2 thus shows that for r just larger than r_c, the function $g(x)$ has eight real roots. That is in fact the maximum possible, since $g(x)$ is an eighth-order polynomial ($f^3(x)$ is the third iterate of a quadratic function).

Now the important step: at the value $r=r_c$, the values of x for the stable and unstable 3-cycle collide at a, b, and c. The function $g(x)$ thus has double roots at a, b, and c. This accounts for six of the eight roots of g. Recalling that the two additional fixed points of f are at $x=0$ and $x=1-1/r$, we know that $g(x)$ must be of the form

$$g(x) \propto x(x-1+1/r)[(x-a)(x-b)(x-c)]^2. \tag{4}$$

Multiplying this out, we find

$$\begin{aligned} g(x) \propto x^8 &- [2\alpha+1-1/r]x^7 + [2\beta+\alpha^2+2(1-1/r)\alpha]x^6 \\ &- [2\gamma-2\alpha\beta+2(1-1/r)\beta+(1-1/r)\alpha^2]x^5 + \dots \end{aligned} \tag{5}$$

Lower-order terms will not be needed. Notice that the combinations of a, b, and c generated by the expansion of Equation 4 involve only the α, β and γ required for our tangency condition (3).

On the other hand, we may compute $g(x)$ directly by iterating the map f. This gives

$$\begin{aligned} g(x) &= r^3x(1-x)[1-rx(1-x)][1-r^2x(1-x)(1-rx(1-x))]-x \\ &= -r^7[x^8-4x^7+(6+2/r)x^6-(4+6/r)x^5+\dots]. \end{aligned} \tag{6}$$

Matching coefficients by starting with x^8 and descending in powers, we easily find α, β, and γ:

$$2\alpha = 3+\frac{1}{r} \tag{7}$$

$$4\beta = \frac{3}{2}+\frac{5}{r}+\frac{3}{2r^2} \tag{8}$$

$$8\gamma = -\frac{1}{2}+\frac{7}{2r}+\frac{5}{2r^2}+\frac{5}{2r^3}. \tag{9}$$

Putting these back in Equation 3 yields

$$r^2-2r-7=0 \tag{10}$$

whose sole positive root is $r_c = 1+\sqrt{8}$, which is our result.

Conclusions Although our proof is simpler than the one given by Saha and Strogatz, it does not readily lend itself to generalization. The result takes advantage of the coincidence of the number of roots of $g(x)$ with the number of points on cycles of immediate interest. Were one to try to use the same method to find the birth of a 5-cycle, for example, one would be faced with a 32nd-order polynomial. But the stable and unstable 5-cycles plus 2 fixed points account for only 12 of those roots. Nonetheless, for the 3-cycle problem, with suitable hints to get students started, the method outlined here should simplify greatly what had been a difficult homework problem for a course in nonlinear dynamics.

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Period Three Trajectories of the Logistic Map

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A recent Note in this MAGAZINE [3] was concerned with locating the "tangent bifurcation" to the logistic map

$$f(x) = rx(1-x). \tag{1}$$

From graphical considerations, this problem amounts to finding the smallest value of $r \in (0,4)$ for which the map f has a non-trivial 3-periodic orbit. In [3] this value is shown to be $r = r_1$, where

$$r_1 = 1 + 2\sqrt{2} \approx 3.8284\,2712\,4746. \tag{2}$$

The purpose of this note is to show how the result (2) can be easily obtained by exploiting the fact that every 3-periodic sequence $\{x(n)\}$ can be written in the form

$$x(n) = \mu + \beta\omega^n + \bar{\beta}\bar{\omega}^n, \tag{3}$$

where μ and β are constants, ω is a complex cube root of unity, and the overbars indicate complex conjugation. We shall also give an upper bound for the r-values that support stable 3-periodic orbits, *viz.*, $r = r_2$, where

$$r_2 = 1 + \sqrt{\left[\frac{11}{3} + \left(\frac{1915}{54} + \frac{5\sqrt{201}}{2}\right)^{1/3} + \left(\frac{1915}{54} - \frac{5\sqrt{201}}{2}\right)^{1/3}\right]}$$
$$\approx 3.8414\,9900\,7543. \tag{4}$$

A different proof of (2) is given in this current issue of the MAGAZINE by Bechhoefer [1]. We also note that (3) can be viewed as a discrete Fourier transform representation of the 3-periodic orbit $x(n)$; and that discrete Fourier transform techniques have been used in the study of periodic orbits of the Hénon map [2].