Introduction to Mathematical Modeling

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MTH 323: MATHEMATICAL MODELING

Outline

What is a Mathematical Model?

1. Models are abstractions of reality!
2. Models are a representation of a particular thing, idea, or condition.
3. Mathematical Models are simplified representations of some real-world entity or process
   - can be in equations or computer code
   - are intended to mimic essential features while leaving out inessentials
4. Mathematical models are characterized by assumptions about:
   - Variables (the things which change)
   - Parameters (the things which do not change)
   - Functional forms (the relationship between the two)

The Modeling Process...

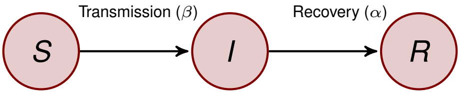
...is a series of steps taken to convert an idea first into a *conceptual* model and then into a *quantitative* model.

1. A *conceptual* model represents our ideas about how the system works. It can be expressed visually in a model diagram, for example, involving boxes (state variables) and arrows (material flows or causal effects).
2. Equations are developed to form a *quantitative model*, for example, consisting of dynamic (i.e., varying with time) equations for each state variable involving the rates of each process.
3. The equations can then be studied mathematically or translated into computer code to obtain *numerical solutions*, for example, state variable trajectories.

A Conceptual Model Diagram

Spread of an infectious disease.

The SIR Epidemic Model



State Variables

- Susceptibles (S): Individuals susceptible to the disease
- Infectious (I): Infected Individuals able to transmit the parasite to others
- Recovered (R): Individuals that have recovered, are immune or have died from the disease and do not contribute to the transmission of the disease

Parameters: α, β

Functional Forms?

Why do Mathematical Modeling?

- Scientific Understanding
  - A model embodies a *hypothesis* about the study system, and lets you compare that hypothesis with data.
  - A model can still be useful when it *fails to fit the data*, because that says that some of your ideas about the study system are wrong.
  - Mathematical models and computer simulations are useful experimental tools for building and testing theories, assesing quantitative conjectures, answering specific questions, determining *sensitivities to changes* in parameter values and estimating key parameters from data.

Why do Mathematical Modeling? (cont)

- Clarification
  - The model formulation process clarifies assumptions, variables, and parameters
  - The process of formulating a model is extremely helpful for organizing one's thinking, bringing hidden assumptions to light and identifying data needs.... do you really have all the necessary pieces?!
- Using our Scientific Understanding to Manage the World
  - Forecasting disease outbreaks or weather
  - Designing man-made systems, for example, bioengineering, nanodevices, electronics
  - Managing existing systems such as agriculture, health care, reservoirs
  - Regulating policies like medical treatments, pollution, natural resources

Why do Mathematical Modeling? (cont)

- Simulated Experimentation
  Realistic experimenting may be impossible
  - Experiments with infectious disease spread in human populations are often impossible, unethical or expensive.
  - We cannnot manage endangered species by trial and error.
  - We dare not set dosage for clinical trials of new drugs on humans or set safe limits for exposure to toxic substances without proper knowledge of the consequences.
- The curse of dimensionality
  - Sometimes a purely experimental approach is not feasible because the data requirements for estimating a model grow rapidly in the number of variables.
  - Modeling using computer programs is cheap.

Types of Mathematical Models

- Qualitative vs. Quantitative Models
  - Qualitative models lead to a detailed, numerical predicition about responses, whereas qualitative models lead to general descriptions about the responses.
- Static vs. Dynamic Models
  - Static models are at an equilibrium or steady state, as opposed to dynamic models which change with respect to time.
- Continuous vs. Discrete Models
  - Differential vs. difference equations

Types of Mathematical Models (cont)

- Individual vs. Structured Models
  - Structured models based on age, size, stage, etc.
- Mechanistic vs. Statistical Models
  - Statistical or empirical models are usually regression based. They provide a quantitative summary of the observed relationships among a set of measured variables.
  - A mechanistic or scientific model begins with a description of how nature might work, and proceeds from this description to a set of predictions relating the independent and dependent variables.
- Deterministic vs. Stochastic models
  - Deterministic models have no components that are inherently uncertain, i.e., no parameters in the model are characterized by probability distributions, as opposed to stochastic models.
  - For fixed starting values, a deterministic model will always produce the same result. A stochastic model will produce many different results depending on the actual values that the random variables take in each realization.

Conclusions

- Modeling clarifies what the underlying assumptions are.
- Model analysis and/or simulation predictions can suggest crucial data that should be gathered.
- Model analysis and/or simulation can suggest control strategies that could be implemented.
- Models are abstractions of reality; modeling is essentially a tradeoff between: Generality, Realism and Precision.
- The usefulness of any particular model depends on the modeler's goals. To describe general principles, it is usually necessary to sacrifice realism and precision. To describe a particular situation precisely, it is usually necessary to sacrifice generality.
- Modeling is not perfect and usually is a simplification of reality. Remember a model is only as good as its assumptions are.

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