NUMERICAL ANALYSIS: This refers to the analysis of mathematical problems by numerical means, especially mathematical problems arising from models based on calculus.

Effective numerical analysis requires several things:

- An understanding of the computational tool being used, be it a calculator or a computer.

- An understanding of the problem to be solved.

- Construction of an algorithm which will solve the given mathematical problem to a given desired accuracy and within the limits of the resources (time, memory, etc) that are available.

This is a complex undertaking. Numerous people make this their life's work, usually working on only a limited variety of mathematical problems.

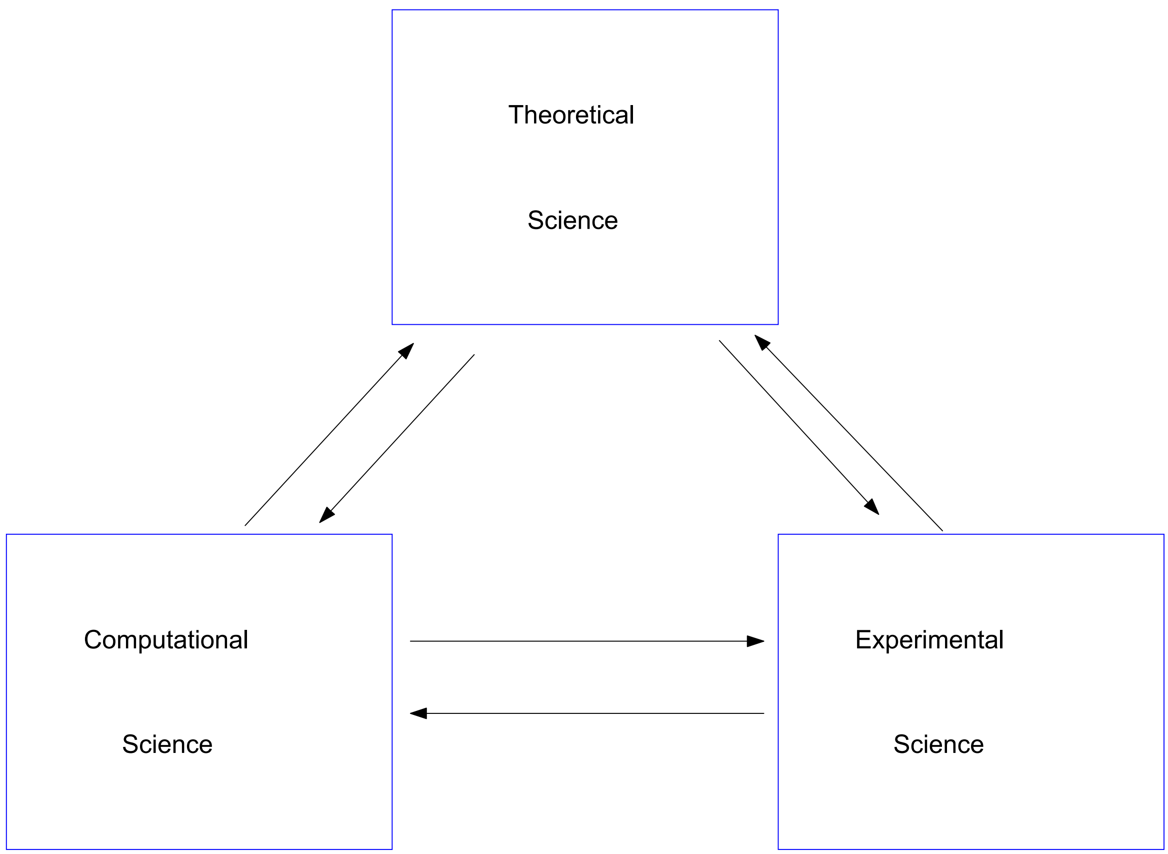
Within this course, we attempt to show the spirit of the subject. Most of our time will be taken up with looking at algorithms for solving basic problems such as *rootfinding* and *numerical integration*; but we will also look at the structure of computers and the implications of using them in numerical calculations.

We begin by looking at the relationship of numerical analysis to the larger world of science and engineering.

SCIENCE

Traditionally, engineering and science had a two-sided approach to understanding a subject: the *theoretical* and the *experimental*. More recently, a third approach has become equally important: the *computational*.

Traditionally we would build an understanding by building theoretical *mathematical models*, and we would solve these for special cases. For example, we would study the flow of an incompressible irrotational fluid past a sphere, obtaining some idea of the nature of fluid flow. But more practical situations could seldom be handled by direct means, because the needed equations were too difficult to solve. Thus we also used the experimental approach to obtain better information about the flow of practical fluids. The theory would suggest ideas to be tried in the laboratory, and the experiemental results would often suggest directions for a further development of theory.


Theoretical
Science
Computational
Science
Experimental
Science

With the rapid advance in powerful computers, we now can augment the study of fluid flow by directly solving the theoretical models of fluid flow as applied to more practical situations; and this area is often referred to as "computational fluid dynamics". At the heart of computational science is numerical analysis; and to effectively carry out a computational science approach to studying a physical problem, we must understand the numerical analysis being used, especially if improvements are to be made to the computational techniques being used.

MATHEMATICAL MODELS

A *mathematical model* is a mathematical description of a physical situtation. By means of studying the model, we hope to understand more about the physical situation. Such a model might be very simple. For example,

$$A = 4\pi R_e^2, \qquad R_e \doteq 6,371\ km$$

is a formula for the surface area of the earth. How accurate is it? First, it assumes the earth is sphere, which is only an approximation. At the equator, the radius is approximately 6,378 km; and at the poles, the radius is approximately 6,357 km. Next, there is experimental error in determining the radius; and in addition, the earth is not perfectly smooth. Therefore, there are limits on the accuracy of this *model* for the surface area of the earth.

AN INFECTIOUS DISEASE MODEL

For rubella measles, we have the following model for the spread of the infection in a population (subject to certain assumptions).

$$\begin{aligned}\frac{ds}{dt} &= -a\,s\,i \\ \frac{di}{dt} &= a\,s\,i - b\,i \\ \frac{dr}{dt} &= b\,i\end{aligned}$$

In this, s, i, and r refer, respectively, to the proportions of a total population that are *susceptible*, *infectious*, and *removed* (from the susceptible and infectious pool of people). All variables are functions of time t. The constants can be taken as

$$a = \frac{6.8}{11}, \qquad b = \frac{1}{11}$$

The same model works for some other diseases (e.g. flu), with a suitable change of the constants a and b. Again, this is an approximation of reality (and a useful one).

But it has its limits. Solving a bad model will not give good results, no matter how accurately it is solved; and the person solving this model and using the results must know enough about the formation of the model to be able to correctly interpret the numerical results.

THE LOGISTIC EQUATION

This is the simplest model for population growth. Let $N(t)$ denote the number of individuals in a population (rabbits, people, bacteria, etc). Then we model its growth by

$$N'(t) = cN(t), \qquad t \geq 0, \qquad N(t_0) = N_0$$

The constant c is the *growth constant*, and it usually must be determined empirically. Over short periods of time, this is often an accurate model for population growth. For example, it accurately models the growth of US population over the period of 1790 to 1860, with $c = 0.2975$.

THE PREDATOR-PREY MODEL

Let $F(t)$ denote the number of foxes at time t; and let $R(t)$ denote the number of rabbits at time t. A simple model for these populations is called the *Lotka-Volterra predator-prey model*:

$$\begin{aligned} \frac{dR}{dt} &= a\,[1 - bF(t)]\,R(t) \\ \frac{dF}{dt} &= c\,[-1 + dR(t)]\,F(t) \end{aligned}$$

with a, b, c, d positive constants. If one looks carefully at this, then one can see how it is built from the logistic equation. In some cases, this is a very useful model and agrees with physical experiments. Of course, we can substitute other interpretations, replacing foxes and rabbits with other predator and prey. The model will fail, however, when there are other populations that affect the first two populations in a significant way.

NEWTON'S SECOND LAW

Newton's second law states that the force acting on an object is directly proportional to the product of its mass and acceleration,

$$F \propto ma$$

With a suitable choice of physical units, we usually write this in its scalar form as

$$F = ma$$

Newton's law of gravitation for a two-body situation, say the earth and an object moving about the earth is then

$$m\frac{d^2\mathbf{r}(t)}{dt^2} = -\frac{Gm\,m_e}{|\mathbf{r}(t)|^2} \cdot \frac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$$

with $\mathbf{r}(t)$ the vector from the center of the earth to the center of the object moving about the earth. The constant G is the gravitational constant, not dependent on the earth; and m and m_e are the masses, respectively of the object and the earth.

This is an accurate model for many purposes. But what are some physical situations under which it will fail?

When the object is very close to the surface of the earth and does not move far from one spot, we take $|\mathbf{r}(t)|$ to be the radius of the earth. We obtain the new model

$$m\frac{d^2\mathbf{r}(t)}{dt^2} = -mg\mathbf{k}$$

with $\mathbf{k}$ the unit vector directly upward from the earth's surface at the location of the object. The gravitational constant

$$g \doteq 9.8 \text{ meters/second}^2$$

Again this is a model; it is not physical reality.