The Birth of Period Three

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1. The logistic map The logistic map is one of the most far-reaching examples in all of mathematics [1–8]. It is given by the difference equation

$$x_{n+1} = rx_n(1 - x_n), \tag{1}$$

where $0 \le x_n \le 1$ and $0 \le r \le 4$. In other words, given some starting number $0 \le x_1 \le 1$, we generate a new number x_2 by the rule $x_2 = rx_1(1 - x_1)$, and then repeat the process to generate x_3 from x_2, and so on.

This equation has many virtues:

1) It is *accessible*. High school students can explore its patterns, as long as they have access to a hand calculator or a small computer.

2) It is *exemplary*. This single example illustrates many of the fundamental notions of nonlinear dynamics, such as equilibrium, stability, periodicity, chaos, bifurcations, and fractals. May [6] was the first to stress the pedagogical value of (1).

3) It is *living mathematics*. Most of the important discoveries about the logistic map are less than 20 years old. Certain aspects of (1) are still not understood rigorously, and are being pursued by a few of the finest living mathematicians.

4) It is *relevant to science*. Predictions derived from the logistic map have been verified in experiments on weakly turbulent fluids, oscillating chemical reactions, nonlinear electronic circuits, and a variety of other systems [8].

2. Period-3 cycle and tangent bifurcation This paper is concerned with one aspect of the logistic map, namely the value of r at which a period-3 cycle is created in a tangent bifurcation. To explain what this mouthful means (and why anyone might care!), we begin with an example. If we set $r = 3.835$ and then generate the sequence $\{x_n\}$, we find that x_n eventually repeats every three iterations. This is shown graphically in Figure 1. For a typical choice of x_1, the sequence bobbles around for a few iterations and finally approaches a *period*-3 *cycle*, for which $x_{n+3} = x_n$.

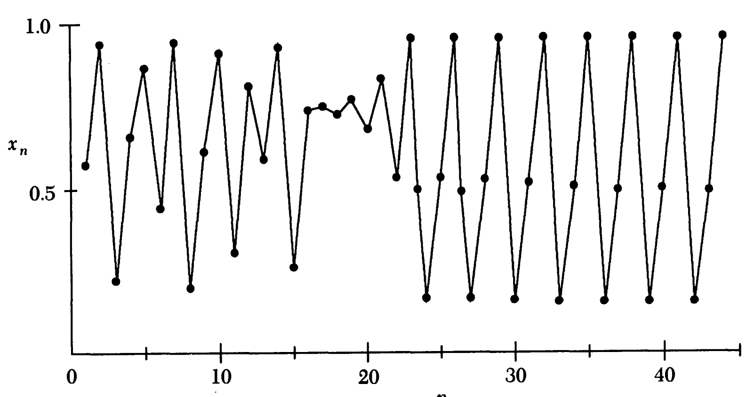

FIGURE 1
Time series of x_n, for $r = 3.835$.

Of course, the parameter value $r = 3.835$ was cunningly chosen; for different choices of r, one can see completely different long-term behavior of x_n. To see the behavior for all values of r at the same time, we plot the well-known "orbit diagram" [2, 3] for the system (FIGURE 2). This picture should be regarded as a stack of vertical lines, one above each r. For a given r, we start at some random x_1, and then iterate for 300 cycles or so, to allow the system to settle down to its eventual behavior. Now that the transients have presumably decayed, we plot many points, say $x_{301}, \ldots, x_{600}$ above that r. Then we move on to the next r and repeat, eventually sweeping across the whole picture.

FIGURE 2 shows the most interesting part of the diagram, in the region $3.4 \leq r \leq 4$. At $r = 3.4$, the system exhibits a period-2 cycle, as indicated by the two branches. As r increases, these branches split, yielding a period-4 cycle. This splitting is called a "period-doubling bifurcation". A cascade of further period-doublings occurs as r increases, yielding period-8, period-16,..., until at $r \approx 3.57$ the map becomes "chaotic". The orbit diagram seems to have degenerated into a featureless mass of dots. Yet order sometimes re-emerges from chaos as we move to still larger r. This is seen most dramatically in the period-3 window marked on FIGURE 2. This region includes the value $r = 3.835$ used earlier in FIGURE 1.

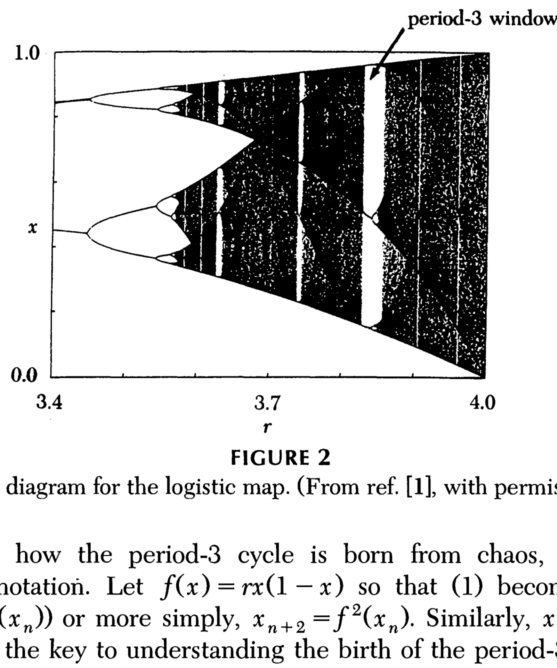

FIGURE 2

Orbit diagram for the logistic map. (From ref. [1], with permission.)

To understand how the period-3 cycle is born from chaos, we first need to introduce some notation. Let $f(x) = rx(1-x)$ so that (1) becomes $x_{n+1} = f(x_n)$. Then $x_{n+2} = f(f(x_n))$ or more simply, $x_{n+2} = f^2(x_n)$. Similarly, $x_{n+3} = f^3(x_n)$. The function $f^3(x)$ is the key to understanding the birth of the period-3 cycle. Any point p in a period-3 cycle repeats every three iterates by definition, so such points satisfy $p = f^3(p)$. Since $f^3(x)$ is an eighth-degree polynomial, this equation is not explicitly solvable. But a graph provides sufficient insight. FIGURE 3 plots $f^3(x)$ for $r = 3.835$. Intersections between the graph and the diagonal line correspond to solutions of $f^3(x) = x$. There are eight solutions, six of interest to us and marked with dots, and two imposters that are not genuine period-3; they are actually fixed points, i.e. period-1 points for which $f(x^*) = x^*$. The black dots in FIGURE 3 correspond to the stable period-3 cycle seen in FIGURE 1, whereas the open dots correspond to an unstable cycle that is not observed numerically.

Now suppose we decrease r into the chaotic regime—how does the graph change? FIGURE 4 shows that when $r = 3.8$, the six marked intersections have vanished.

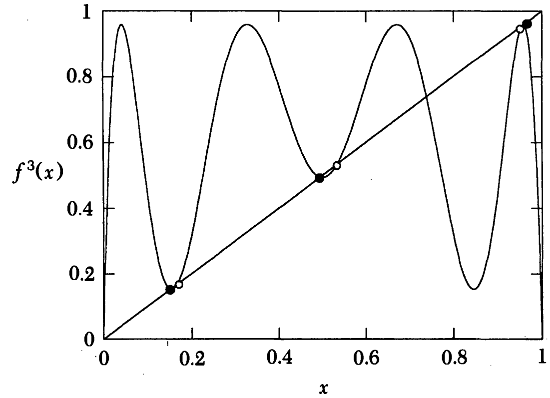

FIGURE 3

Graph of $f^3(x)$, for $r = 3.835$. Black dots, stable period-3 points; open dots, unstable period-3 points.

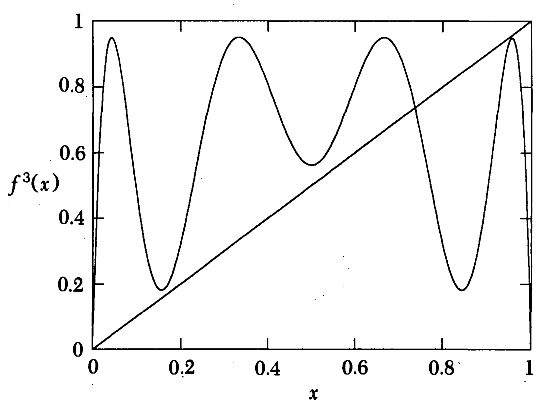

FIGURE 4

Graph of $f^3(x)$ for $r = 3.8$. The period-3 cycle has disappeared.

Somewhere between $r = 3.8$ and $r = 3.835$, the graph of $f^3(x)$ must have become tangent to the diagonal. At this critical value of r, the period-3 cycles are created in a *tangent bifurcation*.

Finally we come to the point of this paper. In several texts on chaos, it is mentioned that the value of r at the tangent bifurcation is given exactly by $1 + 2\sqrt{2} \approx 3.8284\ldots$ (e.g. [5, p. 169], [7, p. 289], [8, p. 83]). Given the beautiful simplicity of this result, one of us (Strogatz) assumed it should be easy to derive, and assigned it as a homework problem in a class on nonlinear dynamics. Of course, there were grounds for suspicion: The result is always stated without proof in the references we have seen. A few students in the class managed to derive the result with the help of Maple, MACSYMA or Mathematica—but these solutions were unsatisfying. One student (Saha) found an elementary solution that we present here. The solution exploits the symmetries of the period-3 cycle, and illustrates the importance of finding the right change of variables.

3. The period-3 conditions The period-3 conditions can be expressed in terms of the three points x, y, z in the cycle:

$$y = rx(1-x) = f(x), \tag{2}$$

$$z = ry(1-y) = f(y) = f^2(x), \tag{3}$$

$$x = rz(1-z) = f(z) = f^2(y) = f^3(x). \tag{4}$$

We are also given that the onset of period-three is heralded by a tangent bifurcation. Hence f^3 has slope 1 at each intersection with the diagonal. At x, this yields

$$\begin{aligned}\frac{d(f^3(x))}{dx} &= \frac{d(f^3(x))}{d(f^2(x))} \cdot \frac{d(f^2(x))}{d(f(x))} \cdot \frac{d(f(x))}{dx} \\ &= \frac{d(f(z))}{dz} \cdot \frac{d(f(y))}{dy} \cdot \frac{d(f(x))}{dx} \\ &= r^3(1-2z)(1-2y)(1-2x) \\ &= 1.\end{aligned} \tag{5}$$

Equations (2)–(5) are four equations in four unknowns: x, y, z, r. Can we solve for r analytically? The answer is yes, though straightforward attempts (like collapsing the four equations into two in x and r) quickly get out of hand. The system suggests changes of variables that considerably simplify the process. We show how.

4. Two smaller problems Two subsequent changes of variables break the problem into two easily manageable ones. We first notice that the right-hand sides of (2)–(4) suggest a certain symmetry of the three variables x, y, and z about the value $1/2$. Accordingly, we define the variables $p = x - \frac{1}{2}$, $q = y - \frac{1}{2}$, and $t = z - \frac{1}{2}$. Another change of variables, $A = rp$, $B = rq$, $C = rt$, renders (2)–(4) very simple:

$$\frac{r^2}{4} - \frac{r}{2} = A^2 + B = B^2 + C = C^2 + A. \tag{6}$$

Equation (5) is even simpler:

$$8ABC = -1. \tag{7}$$

We thus have two smaller problems to solve. If we let R denote the common value of the terms in (6), we get

$$R = A^2 + B = B^2 + C = C^2 + A, \tag{8}$$

and a quadratic equation in r,

$$\frac{r^2}{4} - \frac{r}{2} = R. \tag{9}$$

Our strategy is to solve (7) and (8) for R, and then invoke (9) to obtain r.

If we now try to find the values for A, B, and C, we will run into an avoidable complication. We must first realize that period three, by definition, implies three distinct values of A, B, and C. Looking at (8), we notice that cyclic interchanges in A, B, and C (i.e. $A \to B \to C \to A$) leave it unchanged. Thus if we solved for the variables A, B, and C, we would be forced to find not only the different triplets of numbers allowed *but their cyclic reassignments as well*! This clutter of solutions can clearly be reduced by another change of variables. We realize that each triplet and all its cyclic variations satisfy a *single cubic equation* of the form, $(x-A)(x-B)(x-C) = 0$. The coefficients of the cubic equation are independent of the cyclic reassignments, and are thus the variables we should use. Expanding and collecting powers of x of this cubic, we see that the coefficients are:

$$a = A + B + C, \tag{10}$$

$$b = AB + BC + CA, \tag{11}$$

$$c = ABC. \tag{12}$$

Our ultimate task is to solve an equation in one of the new variables; for reasons that become clear later, we will choose a. Equality in any of the two variables among A, B, and C would imply equality in all three, as can be seen from (8). This would result in $8A^3 = -1$, which (since $a = 3A$ in this case) becomes $8a^3 + 27 = 0$ or

$$(2a+3)(4a^2 - 6a + 9) = 0. \tag{13}$$

What happens when $A = B = C$? We get the period-1 condition mentioned in Section 2. Thus, when we finally derive an equation for a, we expect it to contain these factors, corresponding to the period-1 "imposters" mentioned earlier. We will be interested in any *additional* factors that may be present, as they will be related to the genuine period-3 solutions.

5. Manipulations For much of this section, we will be manipulating algebraic expressions. There is really nothing conceptual involved—just lots of *careful* algebraic maneuvers. The goal is to derive an equation in one of the variables among a, b, and c.

We will need the following algebraic identities:

$$A^2 + B^2 + C^2 = a^2 - 2b, \tag{14}$$

$$A^3 + B^3 + C^3 = a^3 - 3ab + 3c, \tag{15}$$

$$(AB)^2 + (BC)^2 + (CA)^2 = b^2 - 2ca. \tag{16}$$

Adding the three expressions for R in (8) and using the identities (14)–(16), we obtain

$$R = \tfrac{1}{3}(a^2 + a - 2b). \tag{17}$$

Now multiplying the first expression for R in (8) by A, the second by B, and the third by C, adding and using (14)–(17), we get

$$2a^3 - 7ab - a^2 + 9c + 3b = 0. \tag{18}$$

Next we multiply the first expression for R by C, the second by A, and the third by B, add and compare the resulting equation with what we had after a similar process just before—this establishes

$$A^3 + B^3 + C^3 = A^2C + B^2A + C^2B = a^3 - 3ab + 3c. \tag{19}$$

To obtain the next equation, we combine two of the three expressions for R as

$$(A^2 + B)(B^2 + C) = R^2, \quad \text{etc.} \tag{20}$$

In a manner we have followed before, we add the above three equations, and then use (14)–(17) and (19) to obtain:

$$a^4 - 4a^3 + 14ab + a^2 + b^2 + 6ac - 4a^2b - 3b - 18c = 0. \tag{21}$$

6. Solving for r Now it's time to reap the fruits of our labors. Equation (7) gives us $c = -1/8$ which, when put in (18), lets us solve for b:

$$b = \frac{16a^3 - 8a^2 - 9}{56a - 24}. \tag{22}$$

Substituting the above b in (21), we get the long-awaited equation in a:

$$1536a^6 - 3072a^5 + 4608a^4 + 3456a^3 - 10368a^2 + 15552a - 5832 = 0. \tag{23}$$

The above factors into:

$$24(2a-1)(2a+3)(4a^2 - 6a + 9)^2 = 0. \tag{24}$$

The factors $2a+3$ and $4a^2 - 6a + 9$ were expected from (13). The only value of interest for a is then $1/2$ and thus, from (22), $b = -9/4$. Using (17), we finally obtain $R = 7/4$ and now going back to the quadratic equation in r (9), we see that the only nonnegative root is $r = 1 + 2\sqrt{2}$, as desired.

7. Generalizations After this paper was completed, we discovered that similar methods have been used by Hitzl and Zele [4] in their study of the Henon map, a two-dimensional generalization of the logistic map. They find conditions for the existence of periods 1–6. Their analysis could be used as a challenging follow-up to the simpler case considered here.

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