MTH 323: Mathematical Modeling

Lab 1

The objective of this project is to help students familiarize themselves with basic concepts of harmonic oscillators. In addition, students will practice using MATLAB to perform simulations of second order, constant coefficient, forced and unforced ODEs.

Consider the following mathematical model for a spring-mass-dashpot system (using Hooke's law and viscous air damping):

$$m\frac{d^2x}{dt^2}(t) + c\frac{dx}{dt}(t) + kx(t) = f(t)$$

$$x(0) = x_0 \ , \ \dot{x}(0) = v_0$$

$$(1)$$

where m, c and k respectively denote the mass, damping and stiffness coefficients and $x(t)$ is the vertical displacement of the mass about the equilibrium position. We know that the analytic solution to (1) with $f(t) = 0$ is

$$x(t) = e^{-ct/2m}[\alpha\cos(\omega t) + \beta\sin(\omega t)]$$

where

$$\omega = \frac{\sqrt{4km - c^2}}{2m} \ , \ \alpha = x_0 \ , \ \beta = \left(v_0 + \frac{c}{2m}x_0\right)/\omega.$$

To numerically approximate the solution to (1), re-write as a first order linear system

$$\frac{d\vec{y}}{dt}(t) = A\vec{y} + \vec{F}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -K & -C \end{bmatrix} \ , \ \vec{F}(t) = \begin{bmatrix} 0 \\ F(t) \end{bmatrix}$$

and $\vec{y} = [x \, , \, \dot{x}]^T$, with the substitutions $C = c/m$, $K = k/m$, and $F = f/m$.

1. Unforced harmonic oscillator

    (a) Simulate the system corresponding to $m = 4$, $c = 1$, and $k = 6$ with $x_0 = 10$ and $v_0 = 25$ on the interval $[0, 20]$ without forcing ($f(t) = 0$).

    (b) Increase c from the above value just until x does not cross zero. The system is now overdamped. How does the value of c you just found compare with the analytical definition of critical damping?

    (c) Change the initial velocity to be -25 for this overdamped case. Comment on any zero-crossings.

    (d) Increase c in the previous scenario to avoid zero-crossings. What value is necessary? Compare these three plots of overdamped systems to those in Figure 13-1.

    (e) Now change x_0 to -10 in your most recent model. Is this qualitatively different from the three plots in Figure 13-1 or essentially the same as one of them (if so, which one)?

2. Resonance

    (a) Simulate the undamped, forced system corresponding to $m = 1$, $c = 0$, and $k = 1$ with $x_0 = 0$ and $v_0 =$ on the interval $[0, 40]$ with forcing function $f(t) = \cos(t/2)$. What is the period of the solution?

    (b) Repeat with $f(t) = \cos(t)$. What is the period of the solution?

    (c) Repeat with $f(t) = \cos(2t)$. What is the period of the solution?

    (d) Which one of the above exhibits resonance? What is the harmonic frequency of the system?

3. Beats

    (a) Simulate the undamped, forced system corresponding to $m = 1$, $c = 0$, and $k = 1$ with $x_0 = 0$ and $v_0 =$ on the interval $[0, 125]$ with forcing function $f(t) = \cos(.85t)$. Discuss what you observe.

    (b) Repeat with $f(t) = \cos(.9t)$.

    (c) Repeat with $f(t) = \cos(.95t)$.

    (d) Repeat with $f(t) = \cos(.975t)$.

    (e) Explain what happens in the limit as the forcing frequency approaches the harmonic frequency.

    (f) Add a small amount of damping, say $c = 1$ (what do we mean by small?) to one of the above. Is resonance possible in the presence of damping?

4. Continuous approximation to a square wave

   (a) The usefulness of numerical methods is in depicting solutions to systems which are not easily solved analytically, e.g., corresponding to $f(t) = \exp(-\log(\sin(t+1)^2)^n)$ for some value of n. To see what this function is try the following:

   ```
   ezplot('exp(-log(sin(t)^2)^2)')
   ezplot('exp(-log(sin(t)^2)^10)')
   ezplot('exp(-log(sin(t)^2)^100)')
   ```

   (b) Simulate the undamped, forced system corresponding to $m = 1$, $c = 0$, and $k = 1$ with $x_0 = 0$ and $v_0 =$ on the interval $[0, 40]$ with forcing function $f(t) = \exp(-\log(\sin(t+1)^2)^{100})$. Qualitatively describe the solution.

   (c) Repeat with $c = 2$, $c = 3$ and $c = 4$. Discuss what you observe. How is this problem related to previous problems? Is resonance possible with discontinuous forcing?