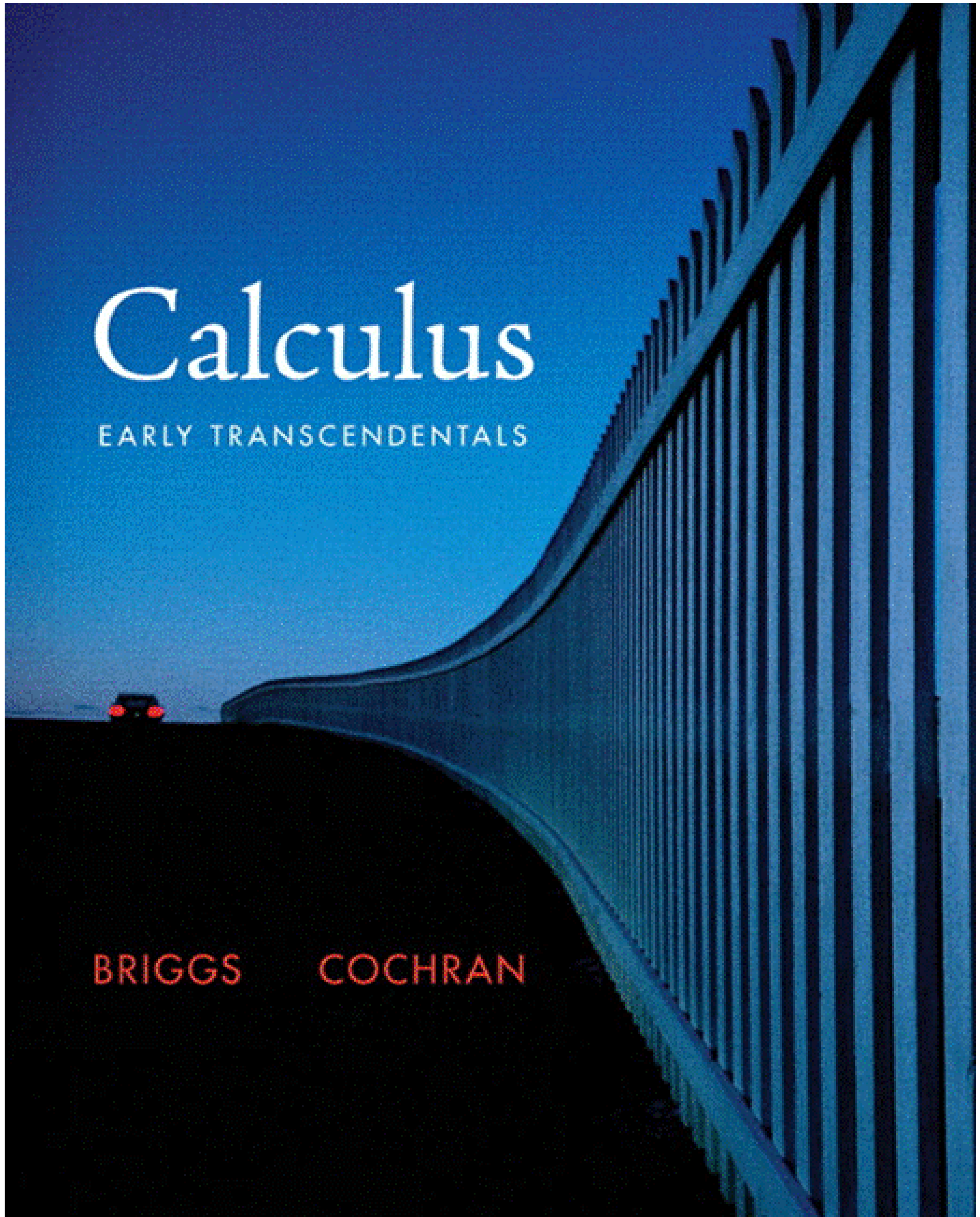
Calculus
EARLY TRANSCENDENTALS
BRIGGS
COCHRAN

Chapter 13

Multiple Integration

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13.1

Double Integrals over Rectangular Regions

PEARSON

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Table 13.1

	Derivatives	Integrals
Single variable: $f(x)$	$f'(x)$	$\int_a^b f(x)\,dx$
Several variables: $f(x, y)$ and $f(x, y, z)$	$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$	$\iint_R f(x, y)\,dA$, $\iiint_D f(x, y, z)\,dV$

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A three-dimensional solid bounded by $z = f(x, y)$ and a region R in the xy-plane is approximated by a collection of boxes.

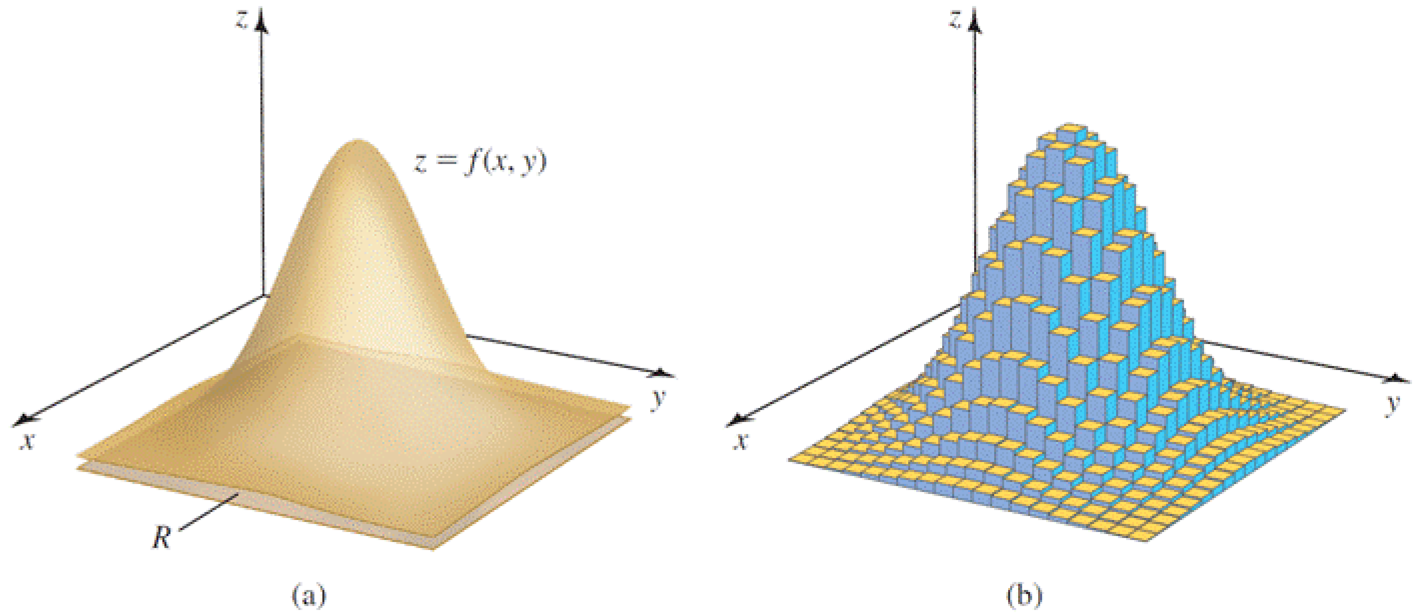

FIGURE 13.1

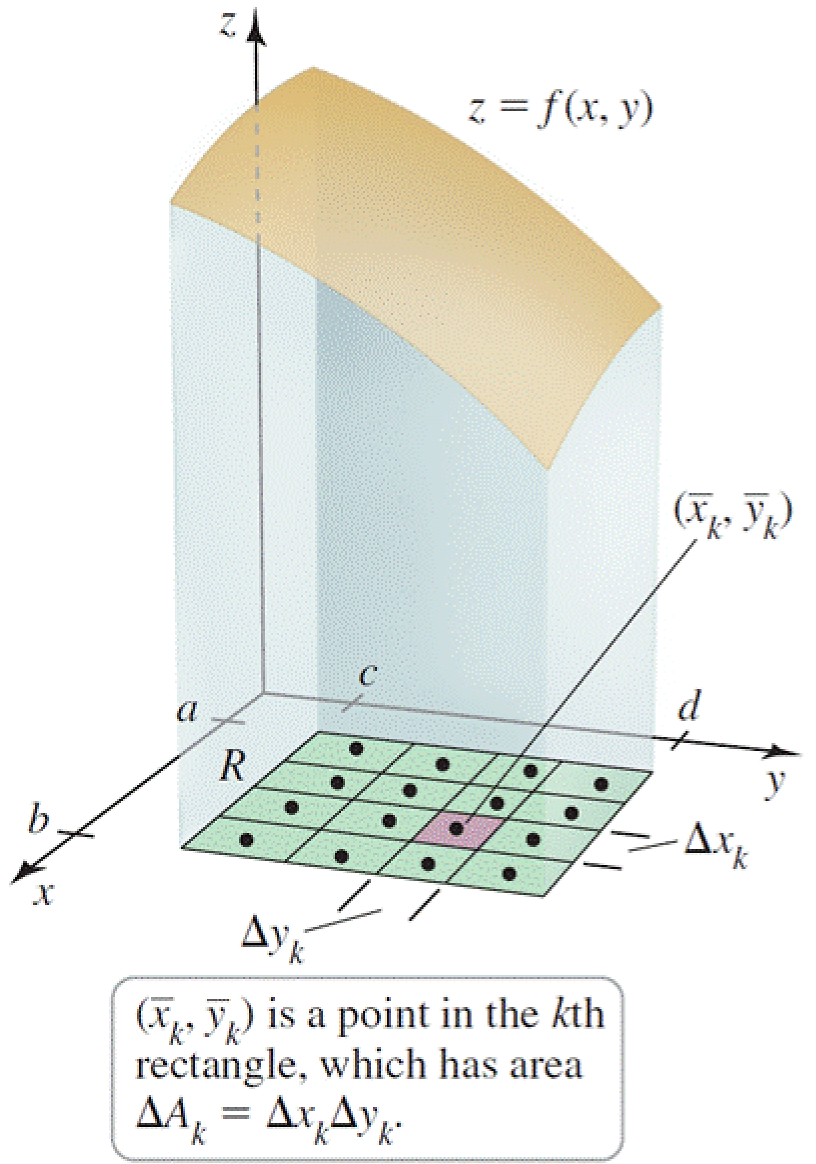

FIGURE 13.2

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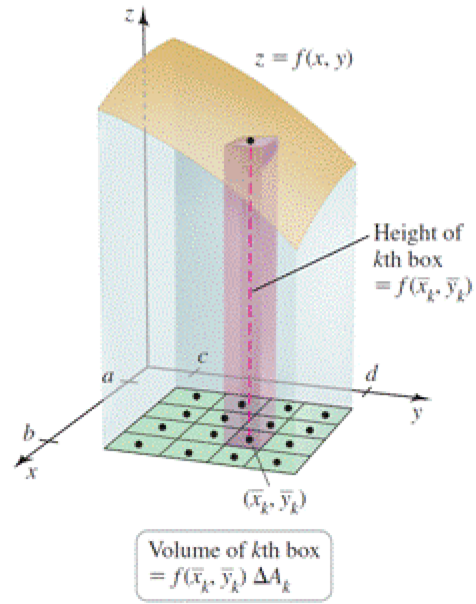
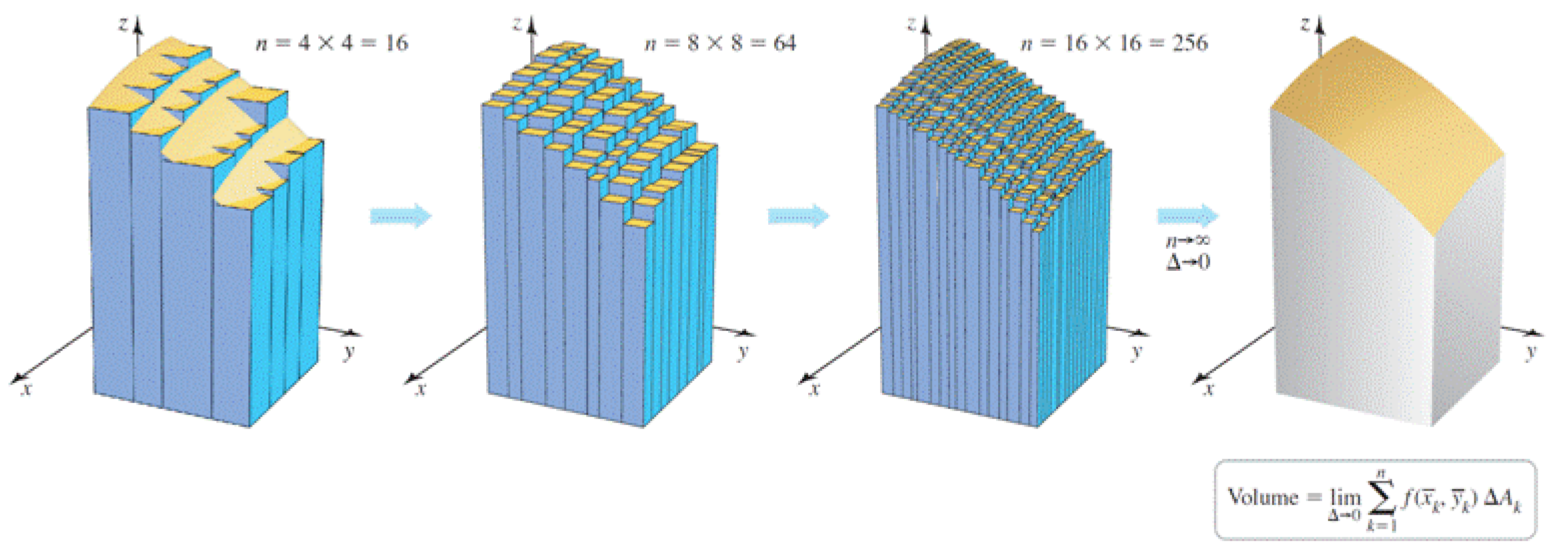


FIGURE 13.3



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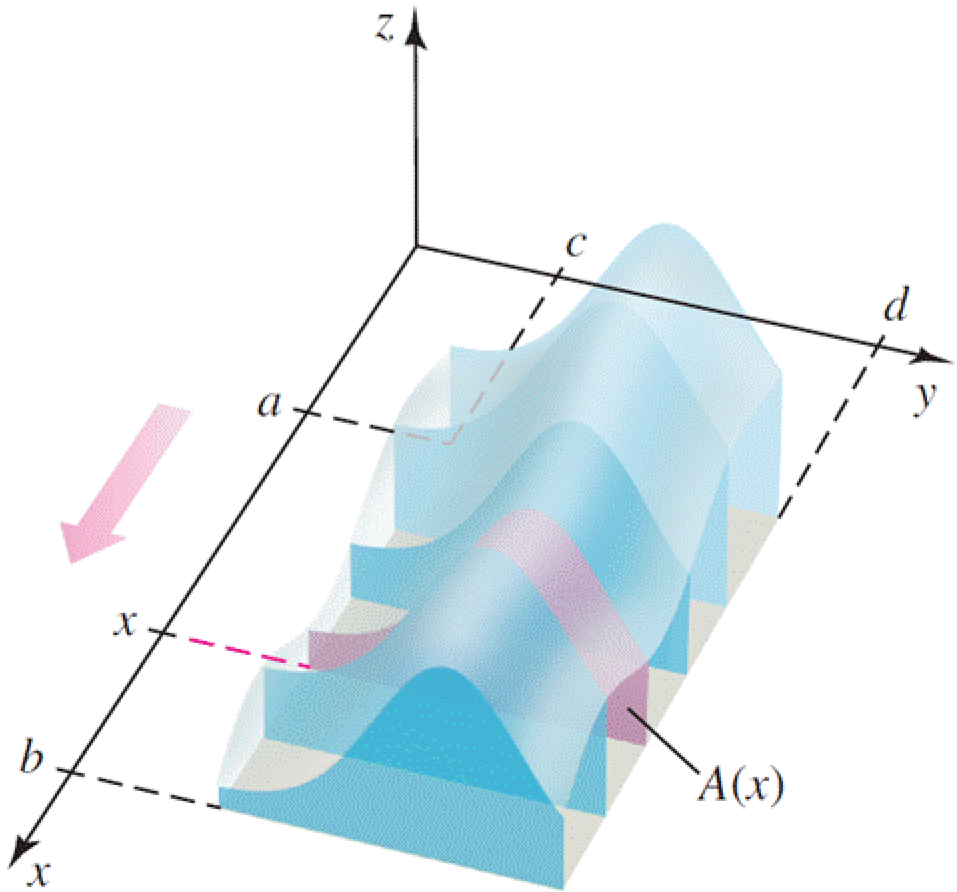

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DEFINITION Volumes and Double Integrals

Let f be defined on a rectangular region R in the xy-plane. If the limit

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k)\, \Delta A_k$$

exists for all partitions of R and for all choices of $(\bar{x}_k, \bar{y}_k)$ within those partitions, it is called the **double integral of f over R**, denoted $\iint_R f(x, y)\, dA$, and f is said to be **integrable** on R. If f is nonnegative over R, then the double integral equals the **volume** of the solid bounded by $z = f(x, y)$ and the xy-plane over R.

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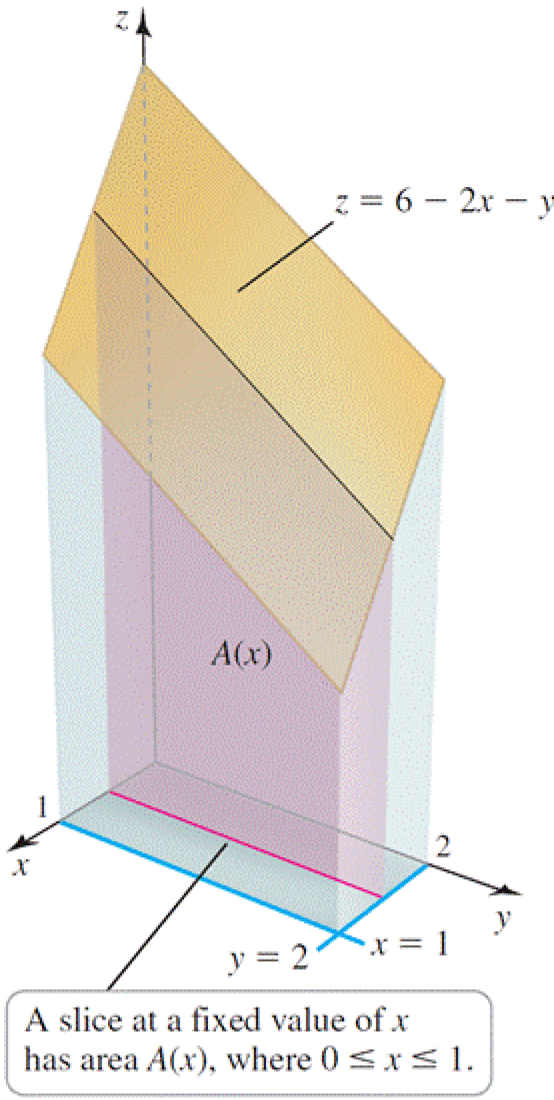

FIGURE 13.5

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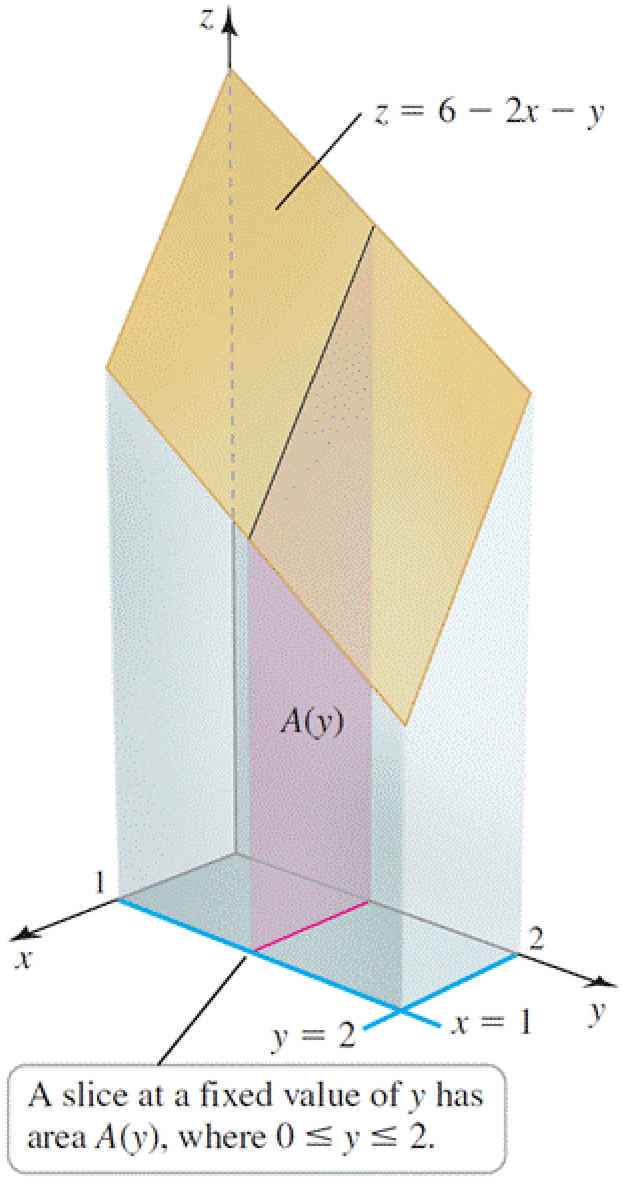

FIGURE 13.6

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THEOREM 13.1 (Fubini) Double Integrals on Rectangular Regions

Let f be continuous on the rectangular region $R = \{(x, y): a \leq x \leq b, c \leq y \leq d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

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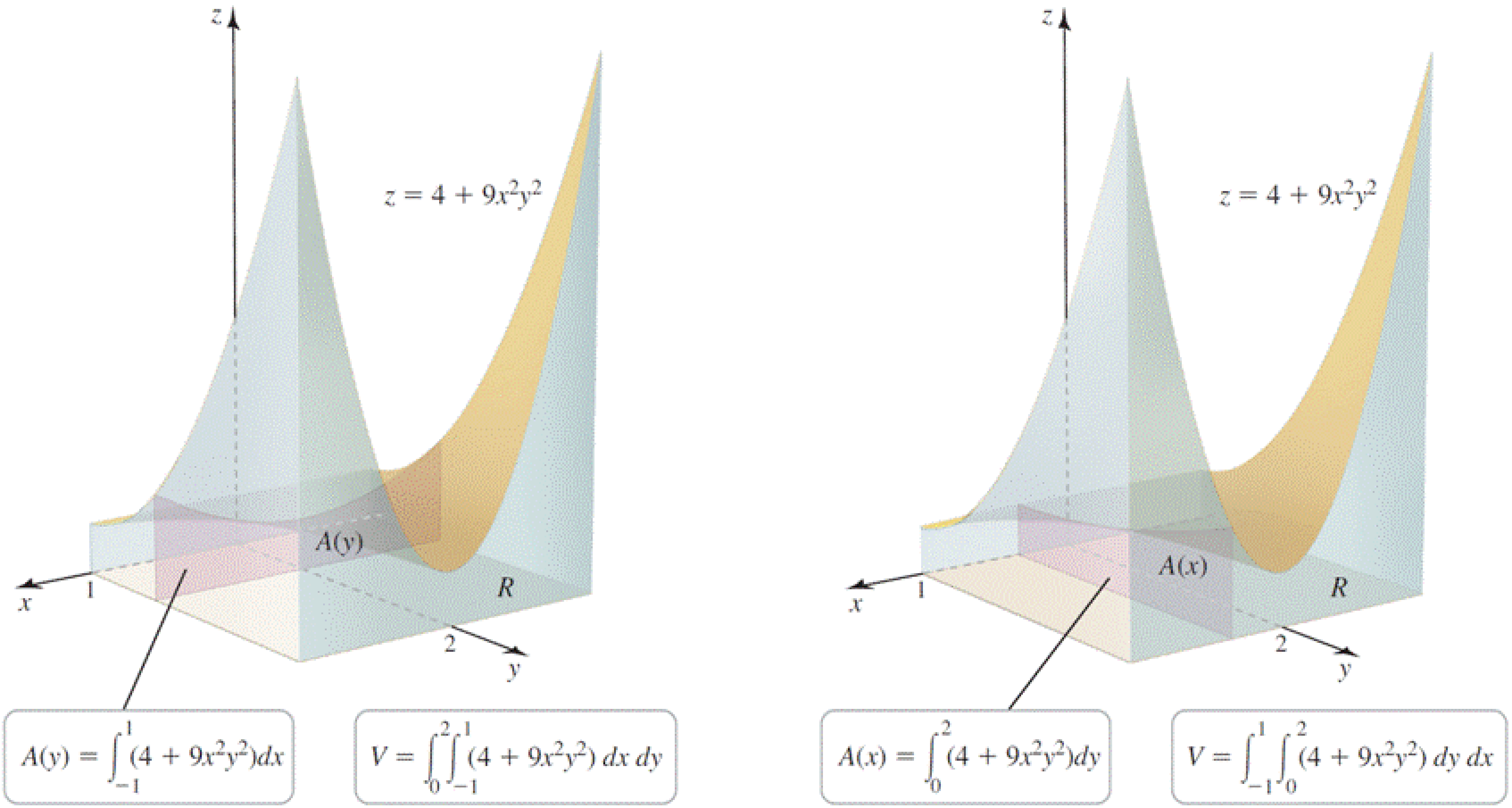

FIGURE 13.7

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DEFINITION **Average Value of a Function over a Plane Region**

The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x, y)\, dA.$$

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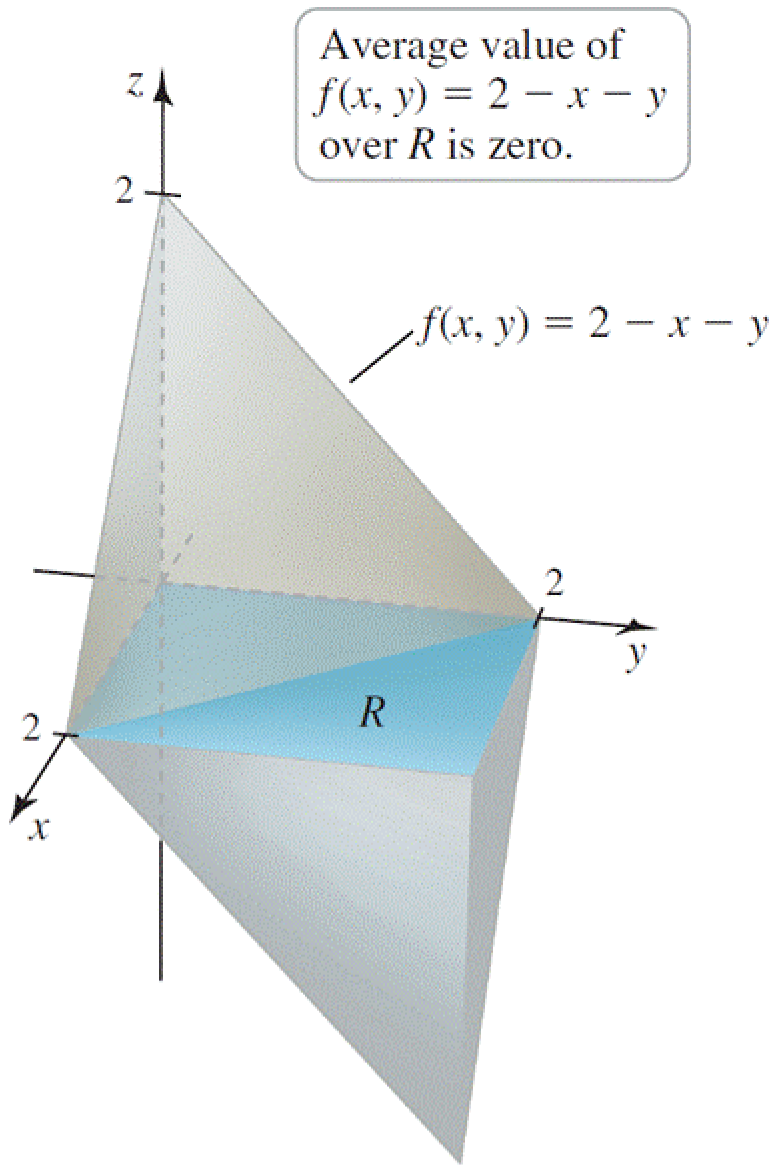

FIGURE 13.8

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13.2

Double Integrals over General Regions

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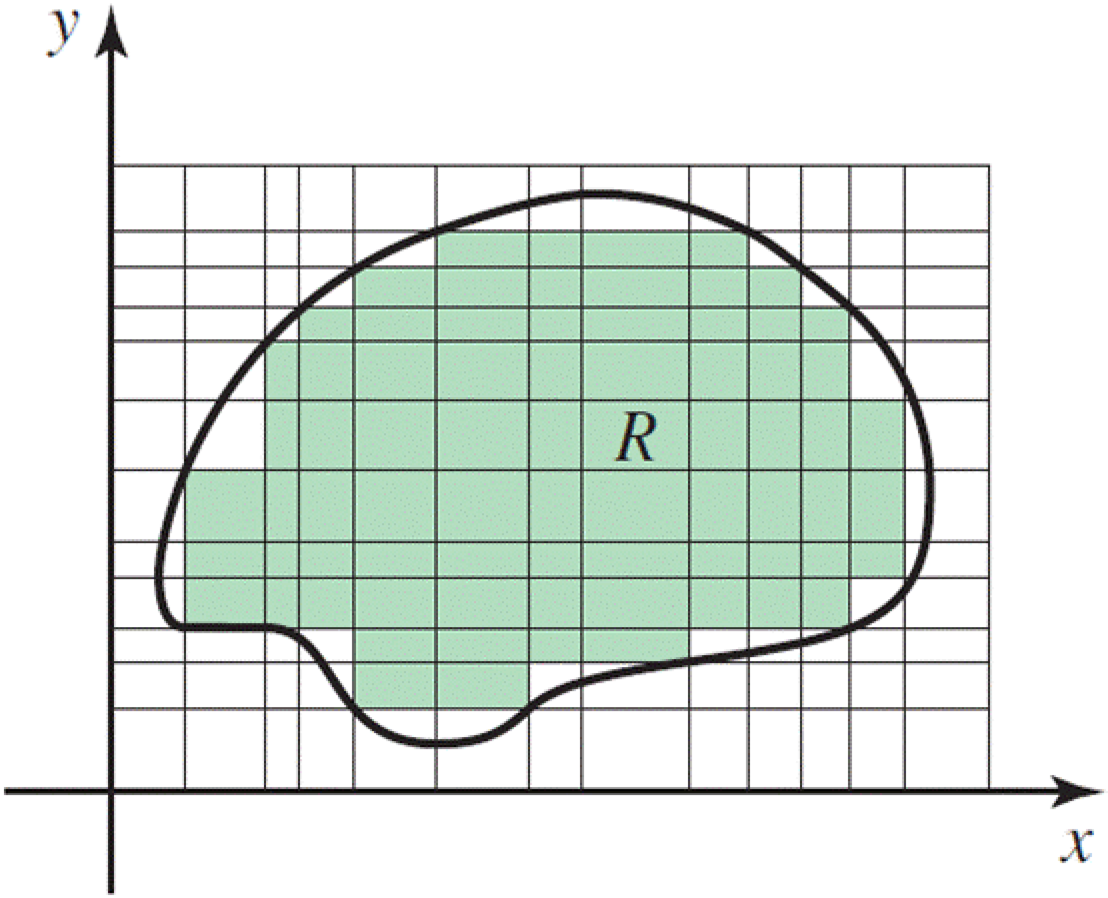

FIGURE 13.9

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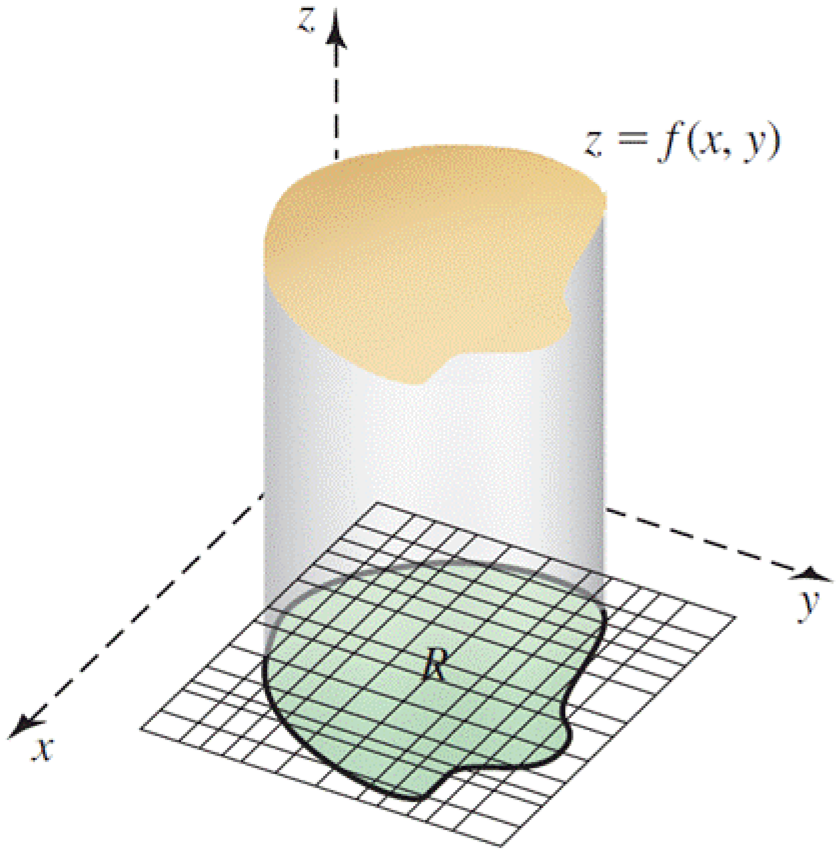

$$\text{Volume of solid} = \iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k)\, \Delta A_k$$

FIGURE 13.10

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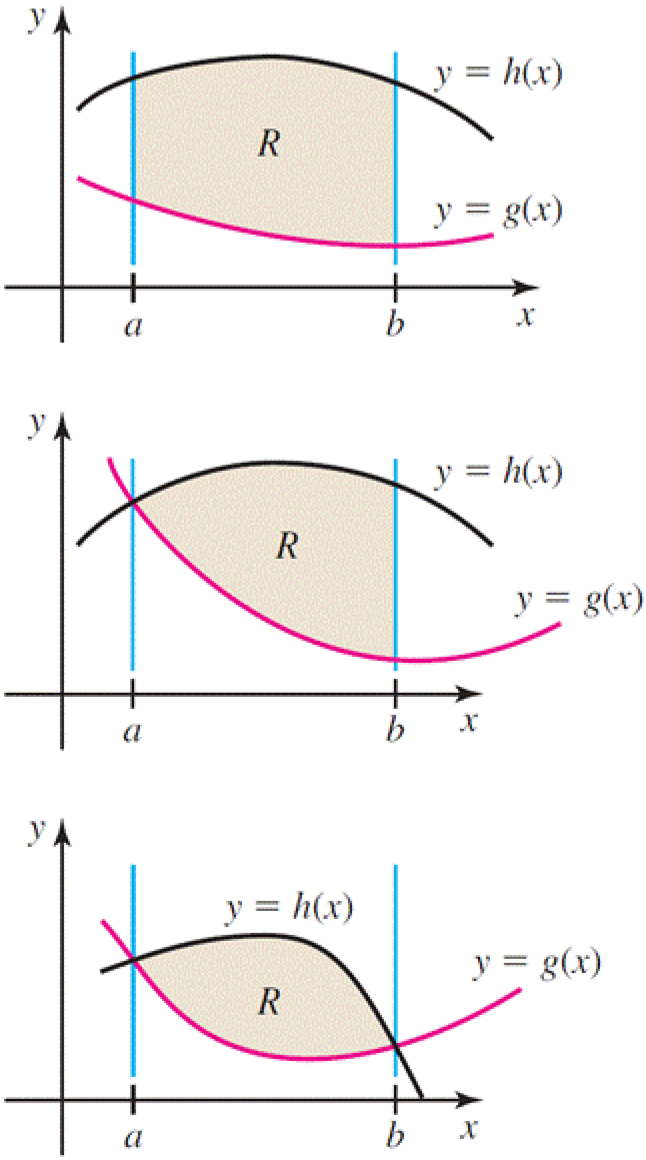

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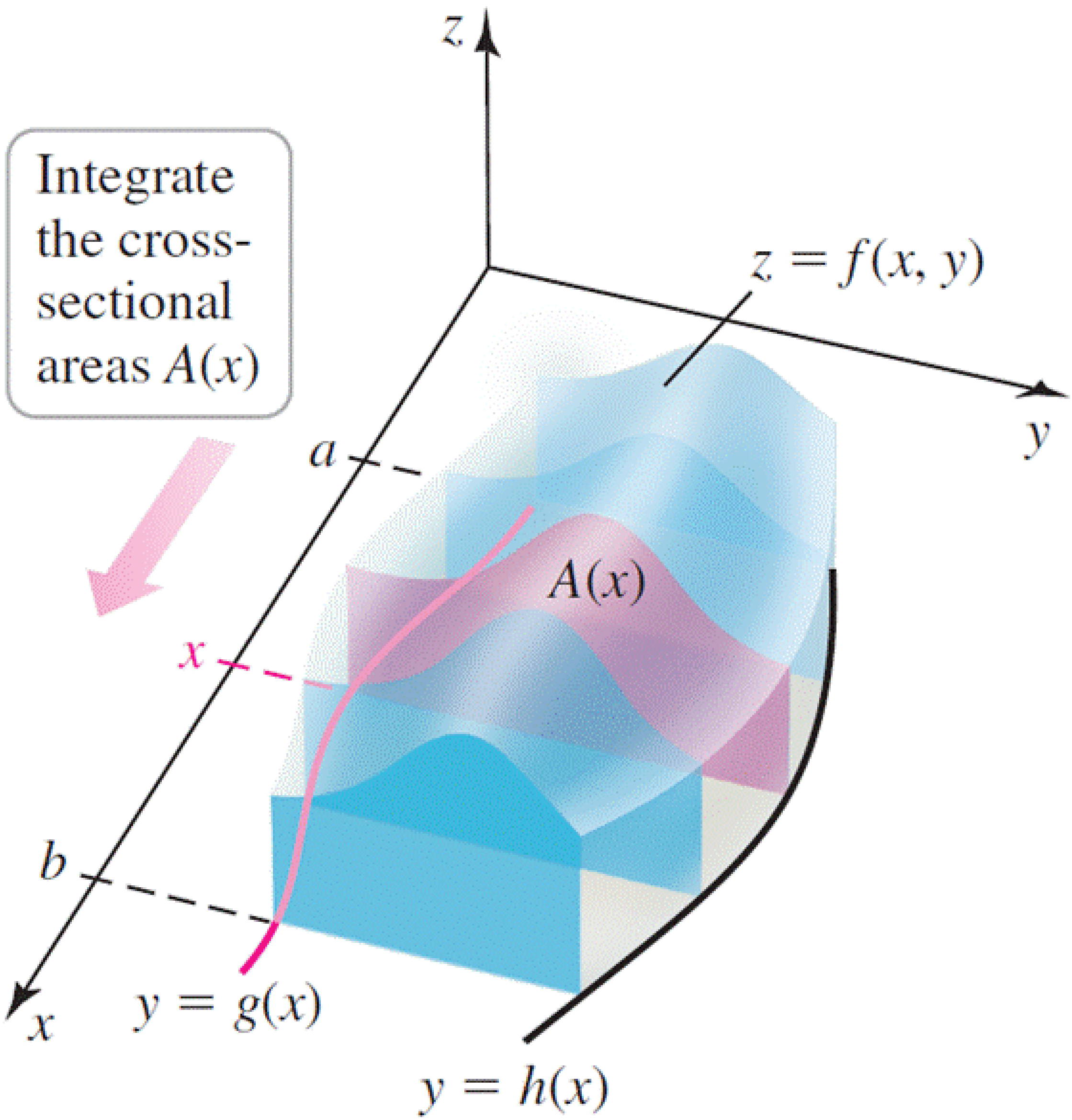

FIGURE 13.12

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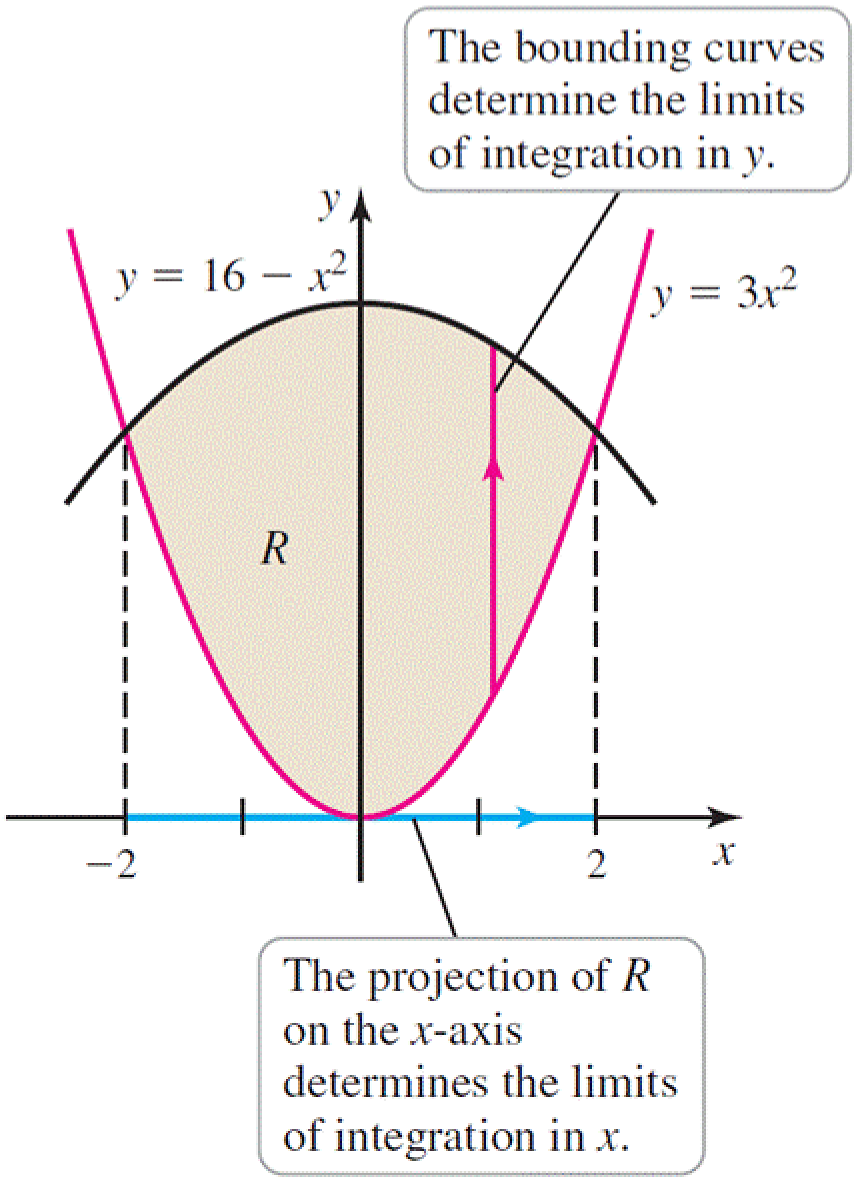

FIGURE 13.13

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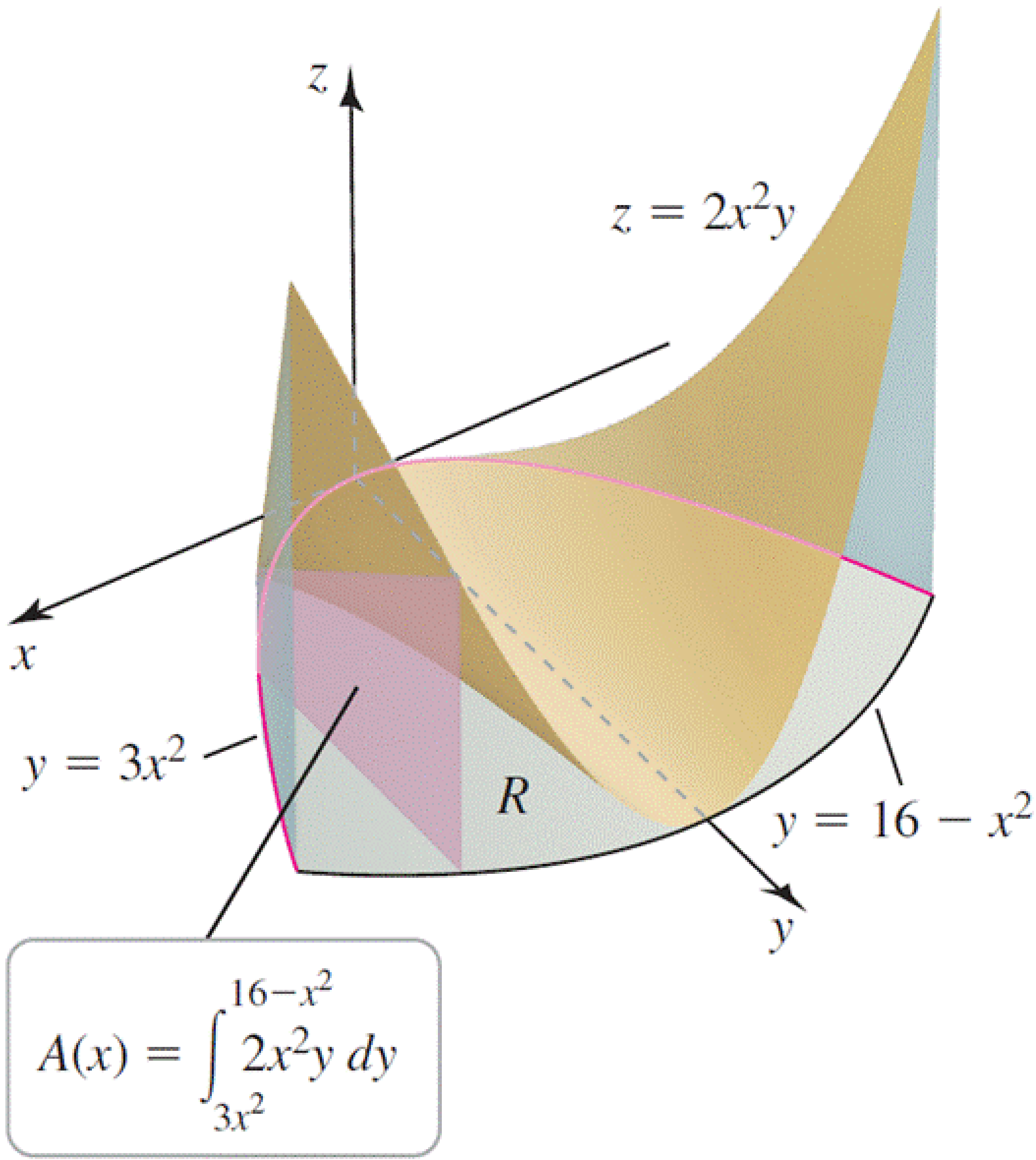

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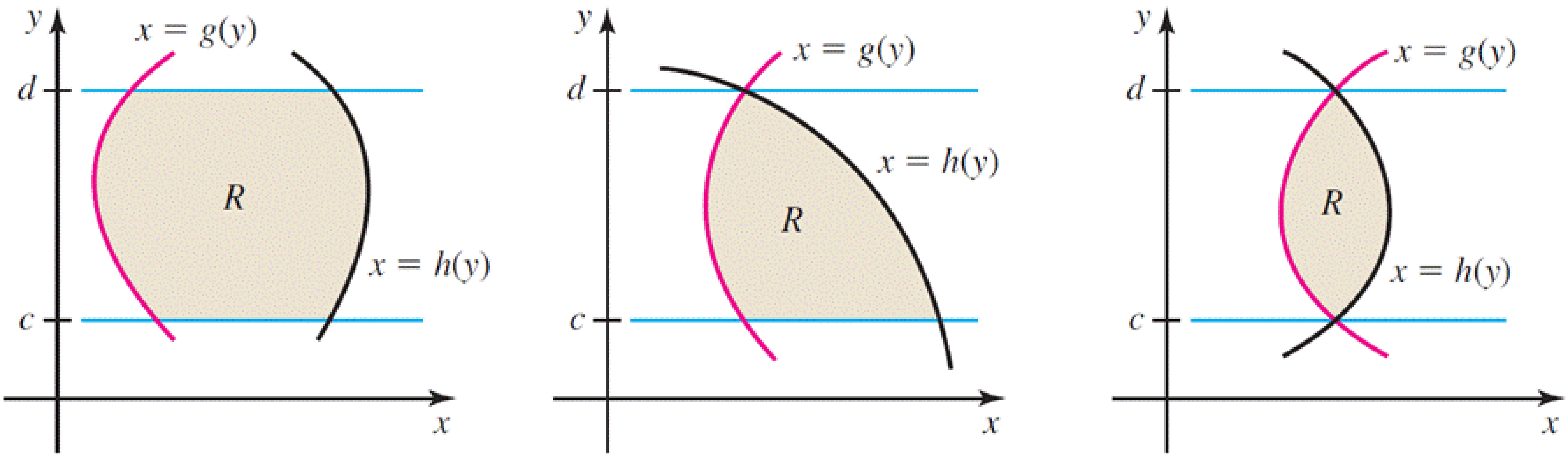

FIGURE 13.15

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THEOREM 13.2 Double Integrals over Nonrectangular Regions

Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx.$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$. If f is continuous on R, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy.$$

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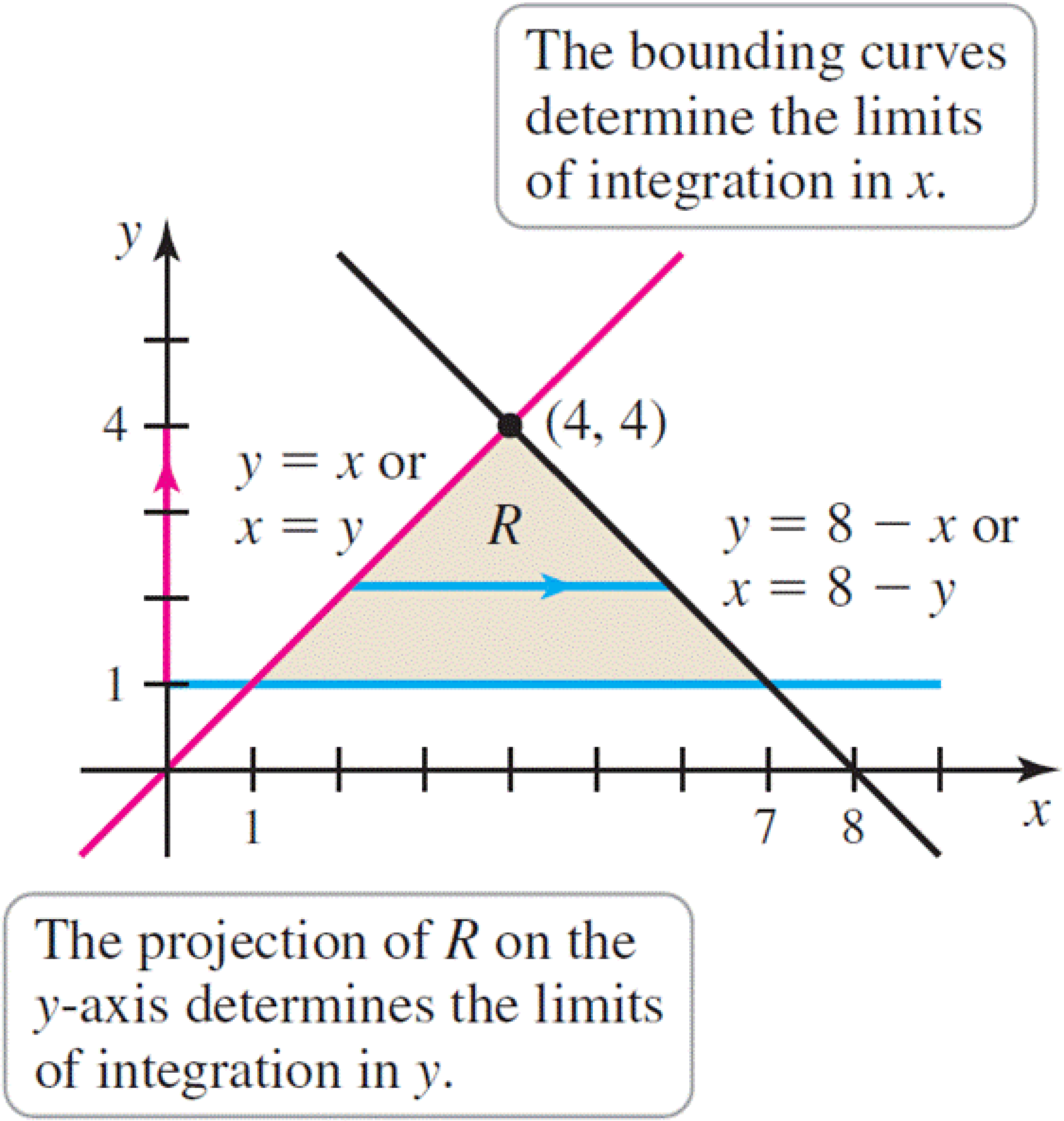

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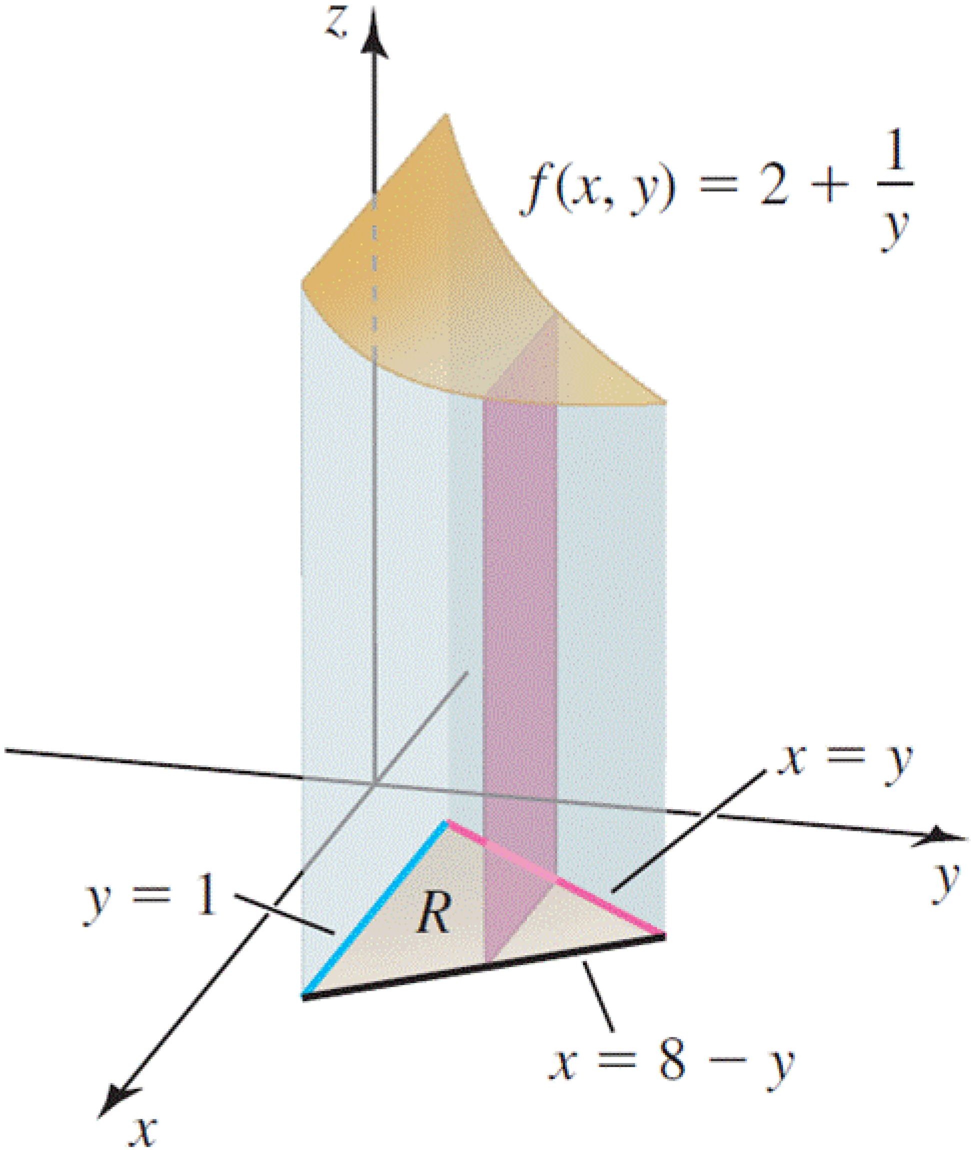

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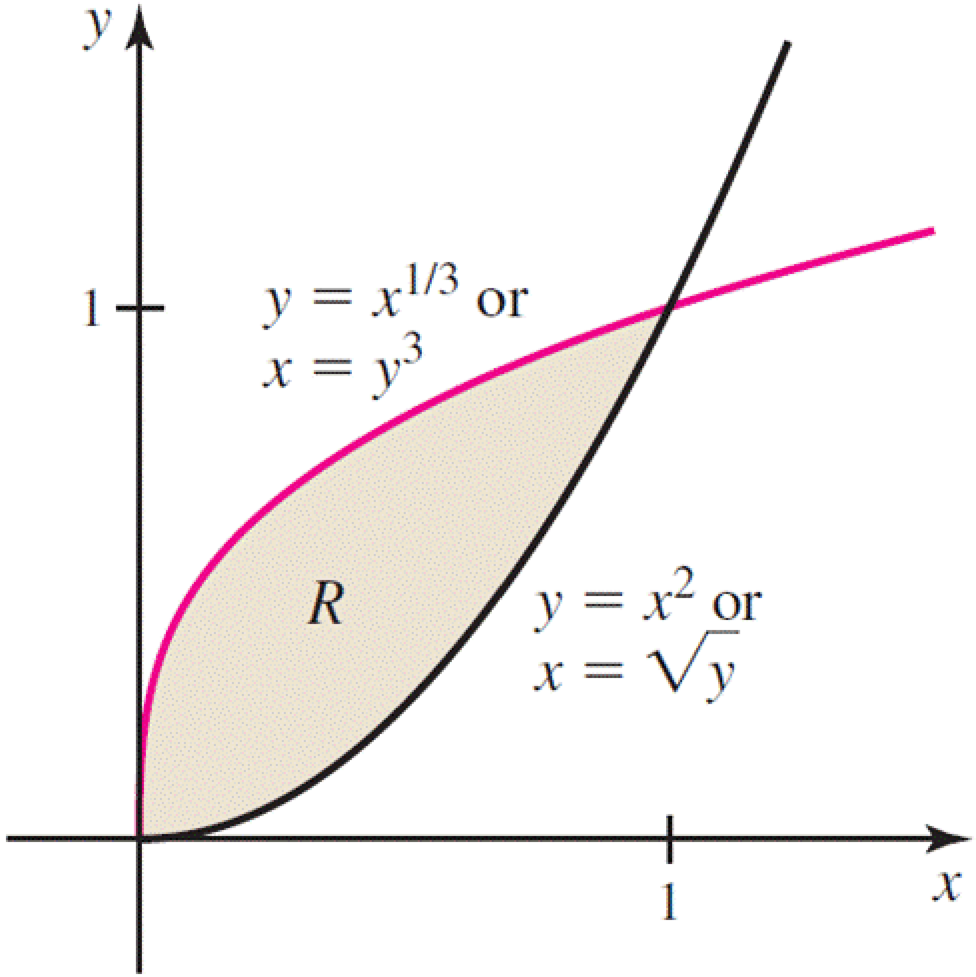

R is bounded above and below, and on the right and left by curves.

FIGURE 13.18

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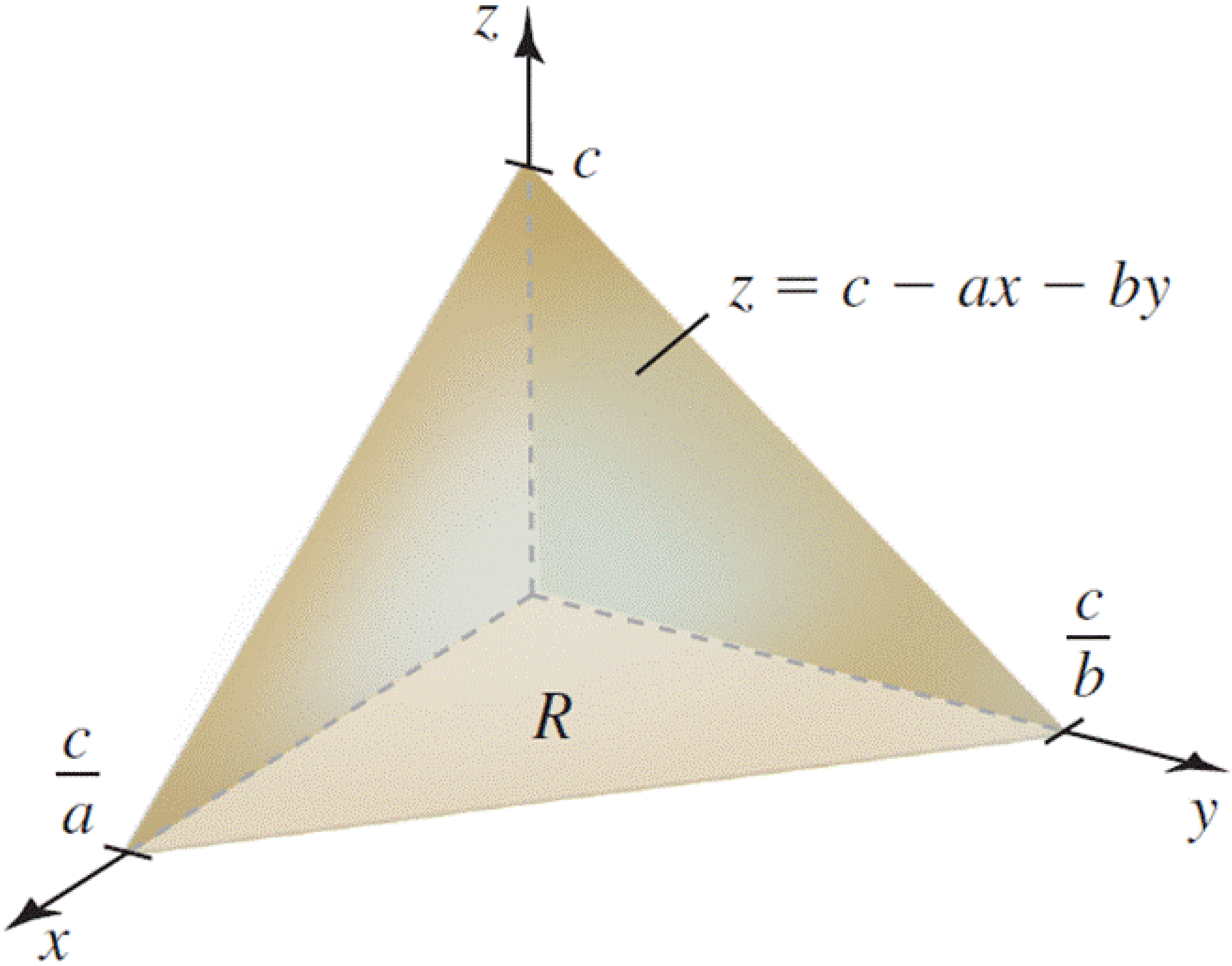

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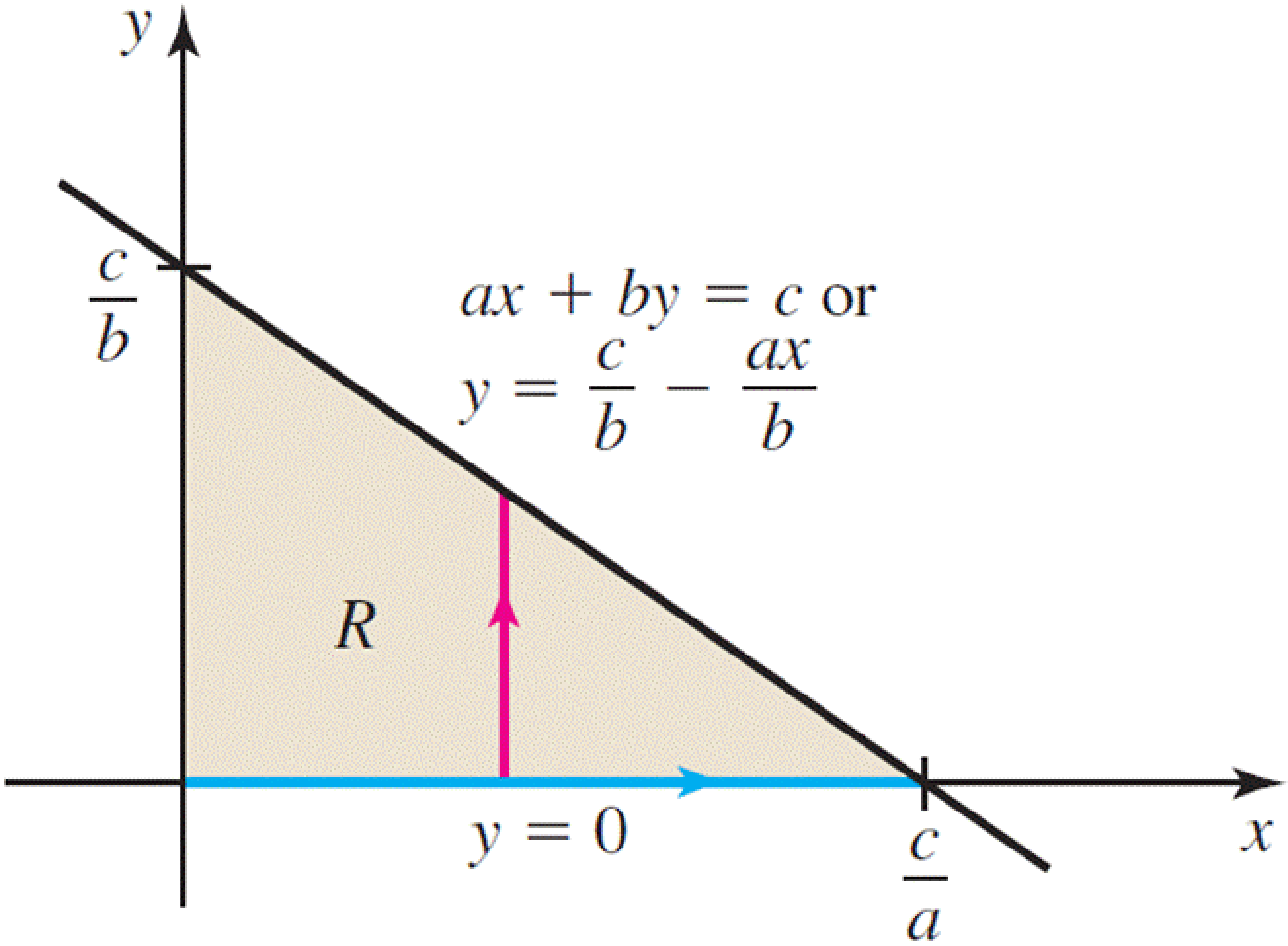

FIGURE 13.20

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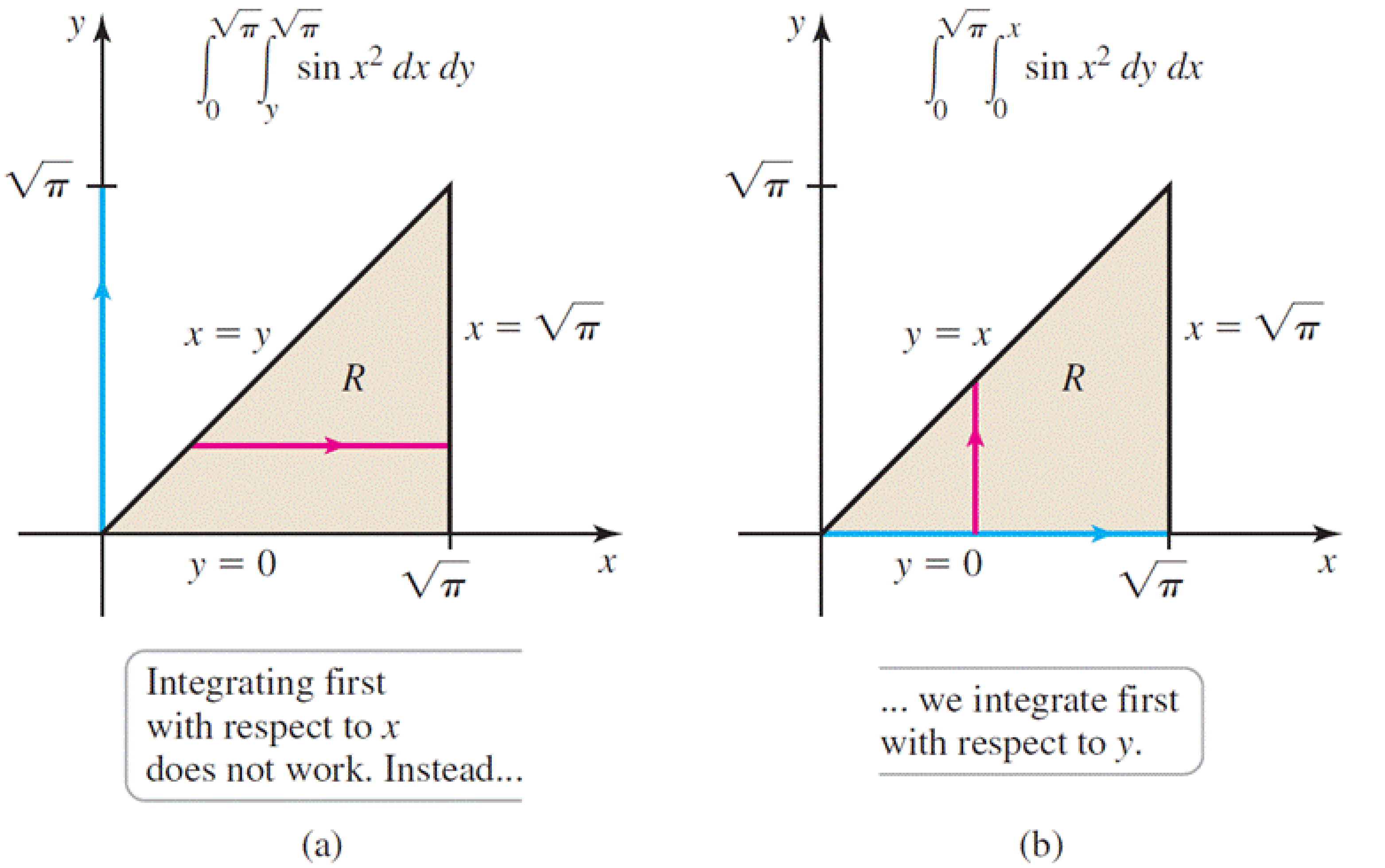
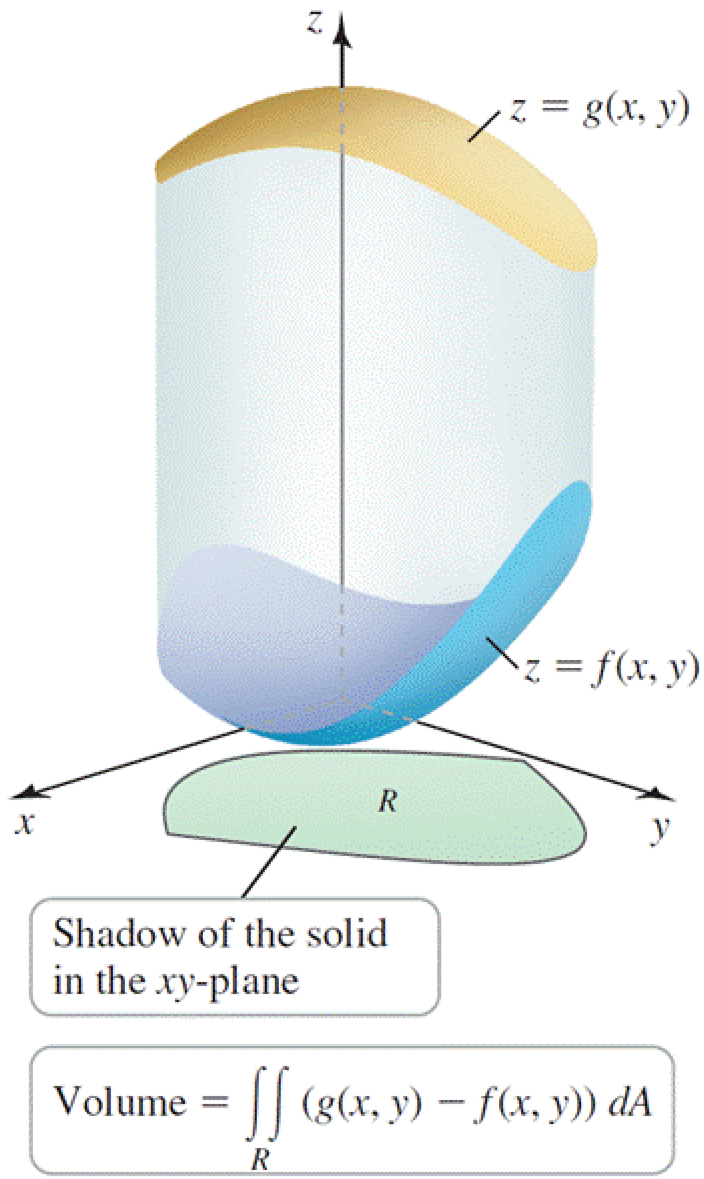

$\int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}} \sin x^2 \, dx \, dy$
y
$\sqrt{\pi}$
$x = y$
R
$x = \sqrt{\pi}$
$y = 0$
$\sqrt{\pi}$
x
Integrating first
with respect to x
does not work. Instead...
(a)
$\int_0^{\sqrt{\pi}}\int_0^{x} \sin x^2 \, dy \, dx$
y
$\sqrt{\pi}$
$y = x$
R
$x = \sqrt{\pi}$
$y = 0$
$\sqrt{\pi}$
x
... we integrate first
with respect to y.
(b)

FIGURE 13.21



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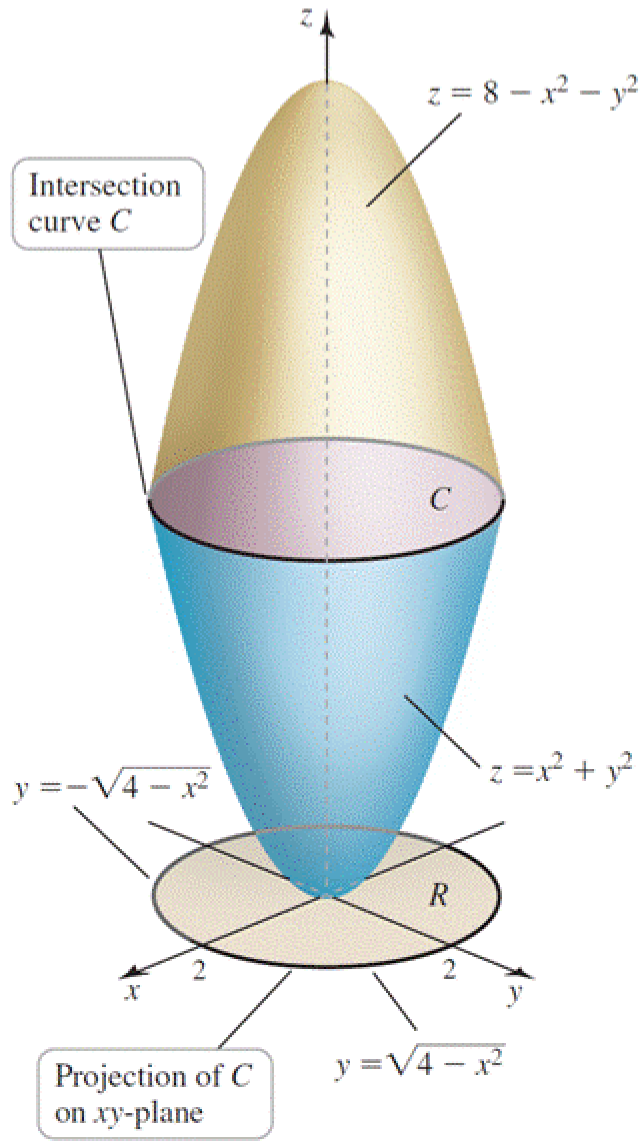

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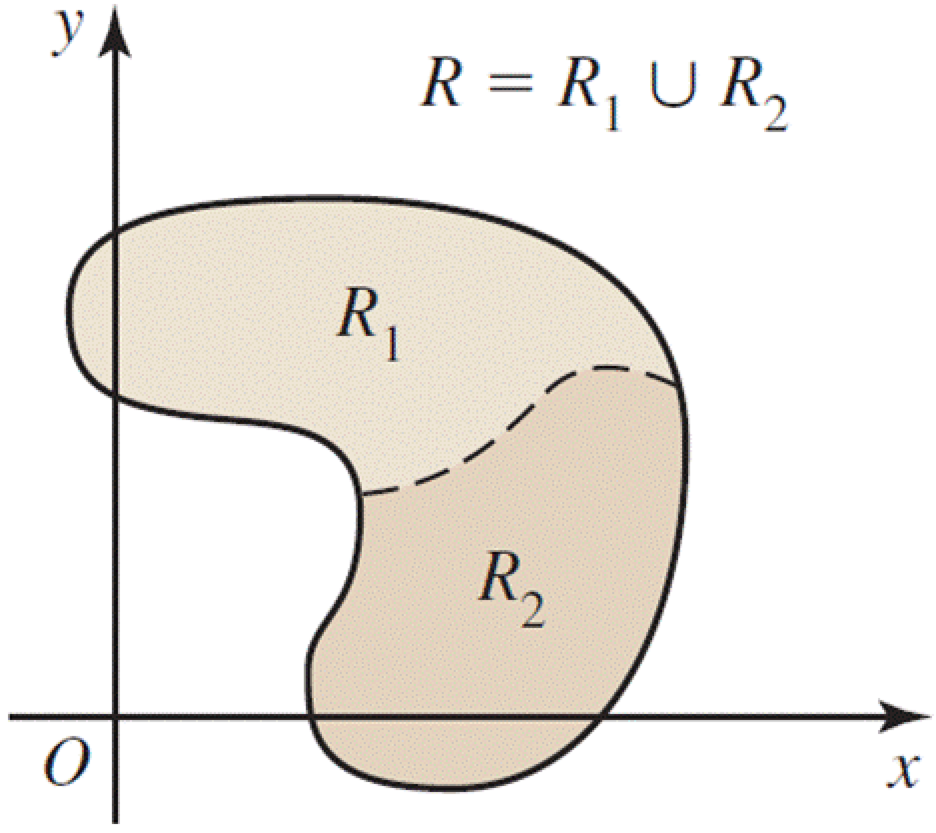

FIGURE 13.24

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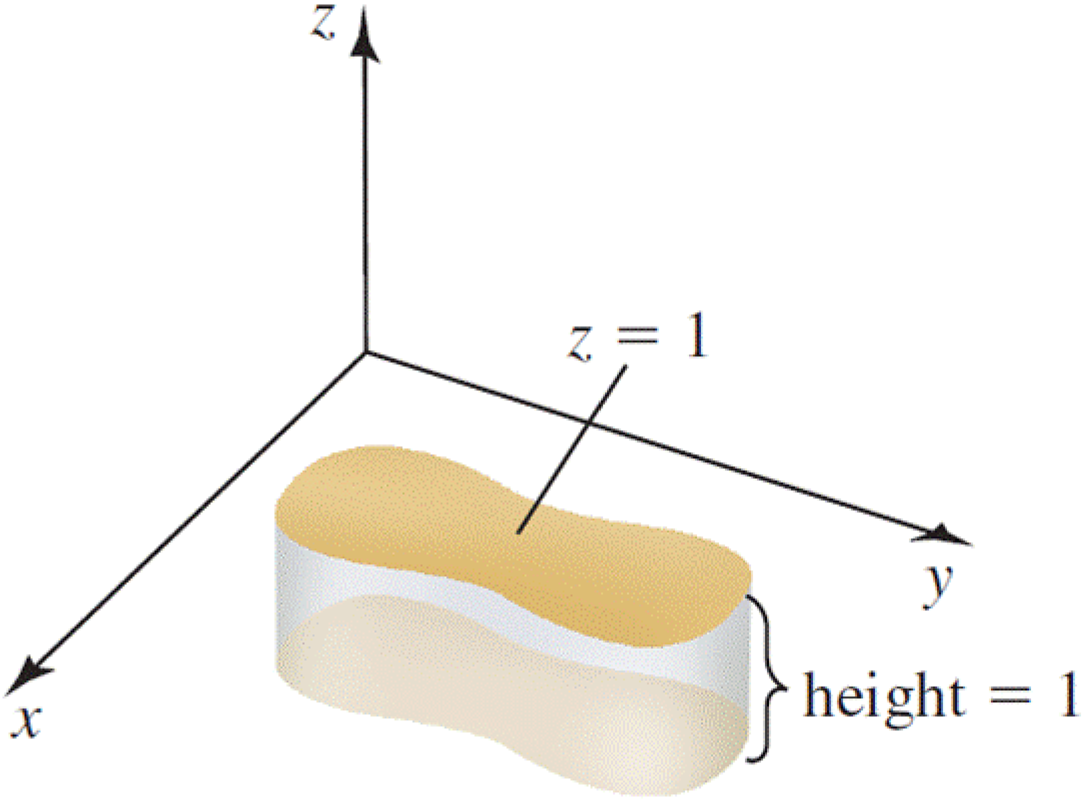

$$\text{Volume of solid} = (\text{Area of } R) \times (\text{height}) = \text{Area of } R = \iint_R 1\, dA$$

FIGURE 13.25

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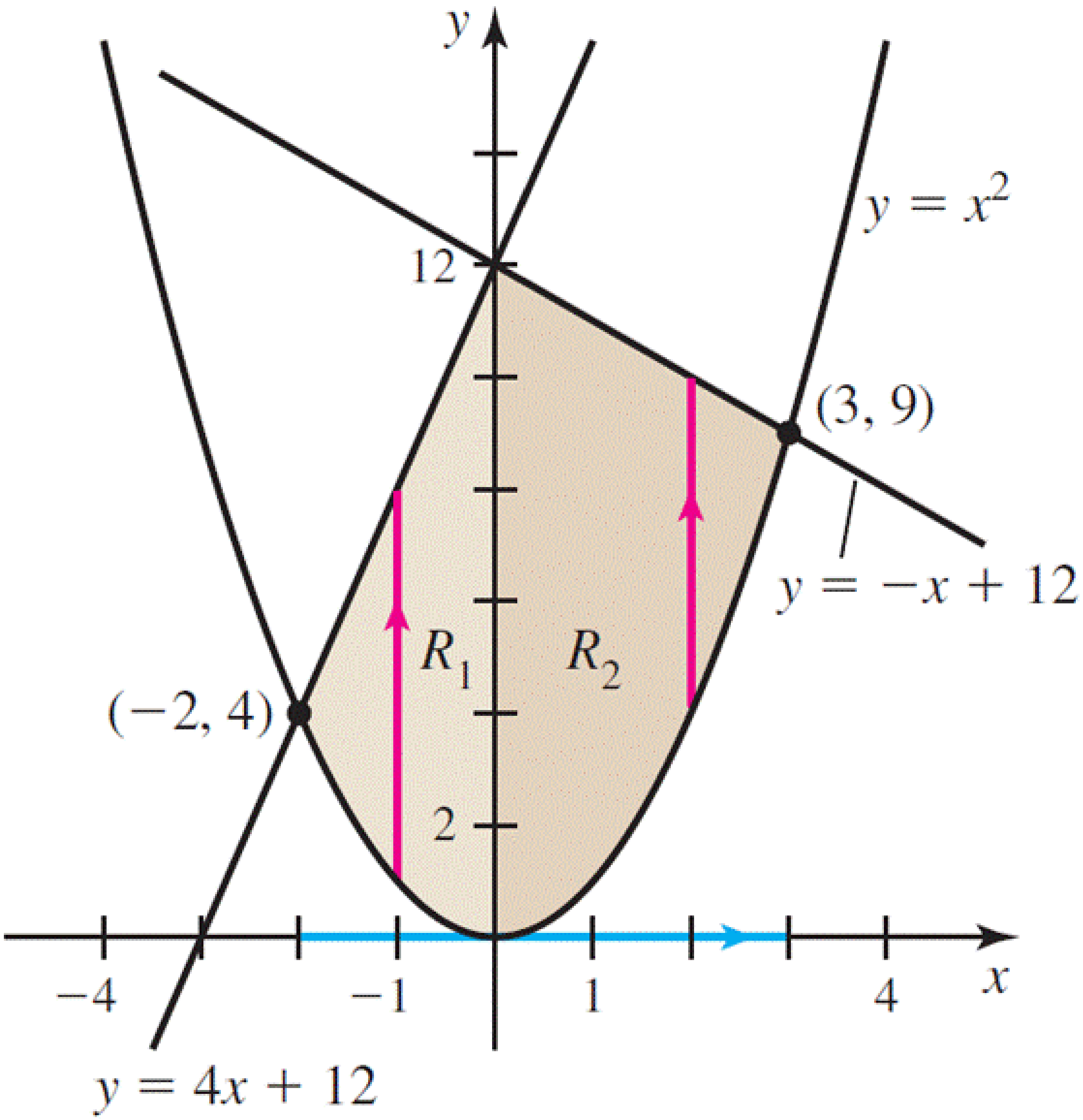

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13.3

Double Integrals in Polar Coordinates

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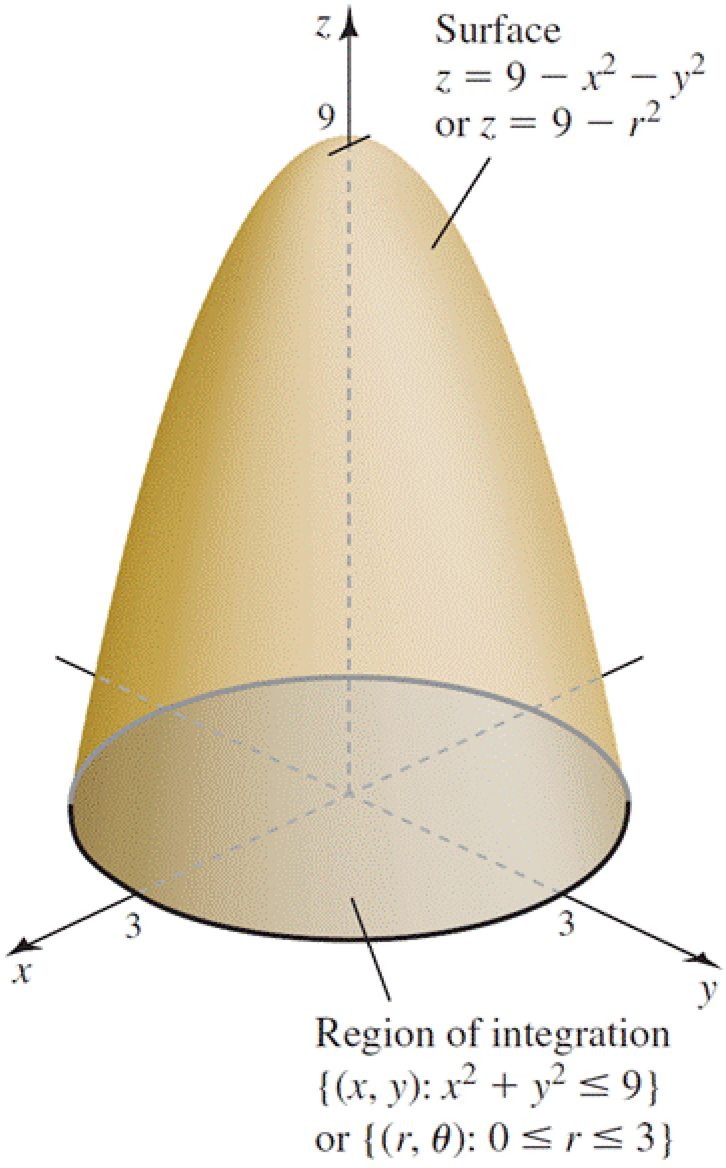

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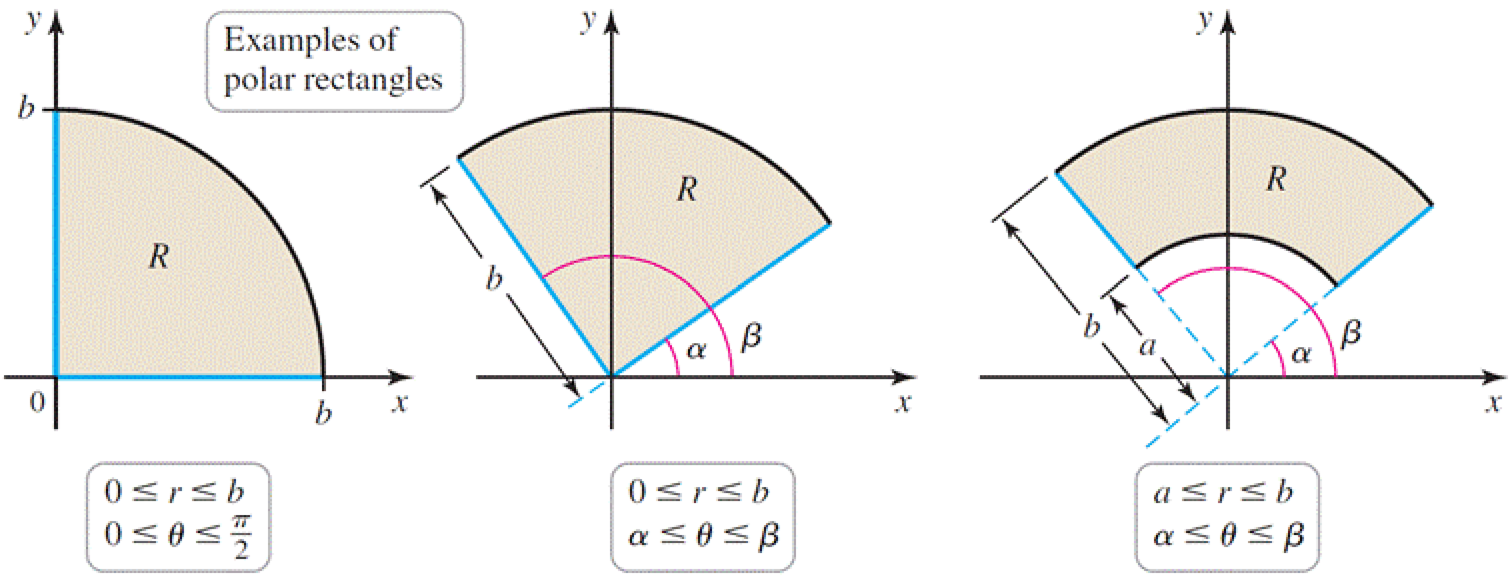
ntegration
$\cdot\ y^2 \leq 9\}$
$\leq r \leq 3\}$



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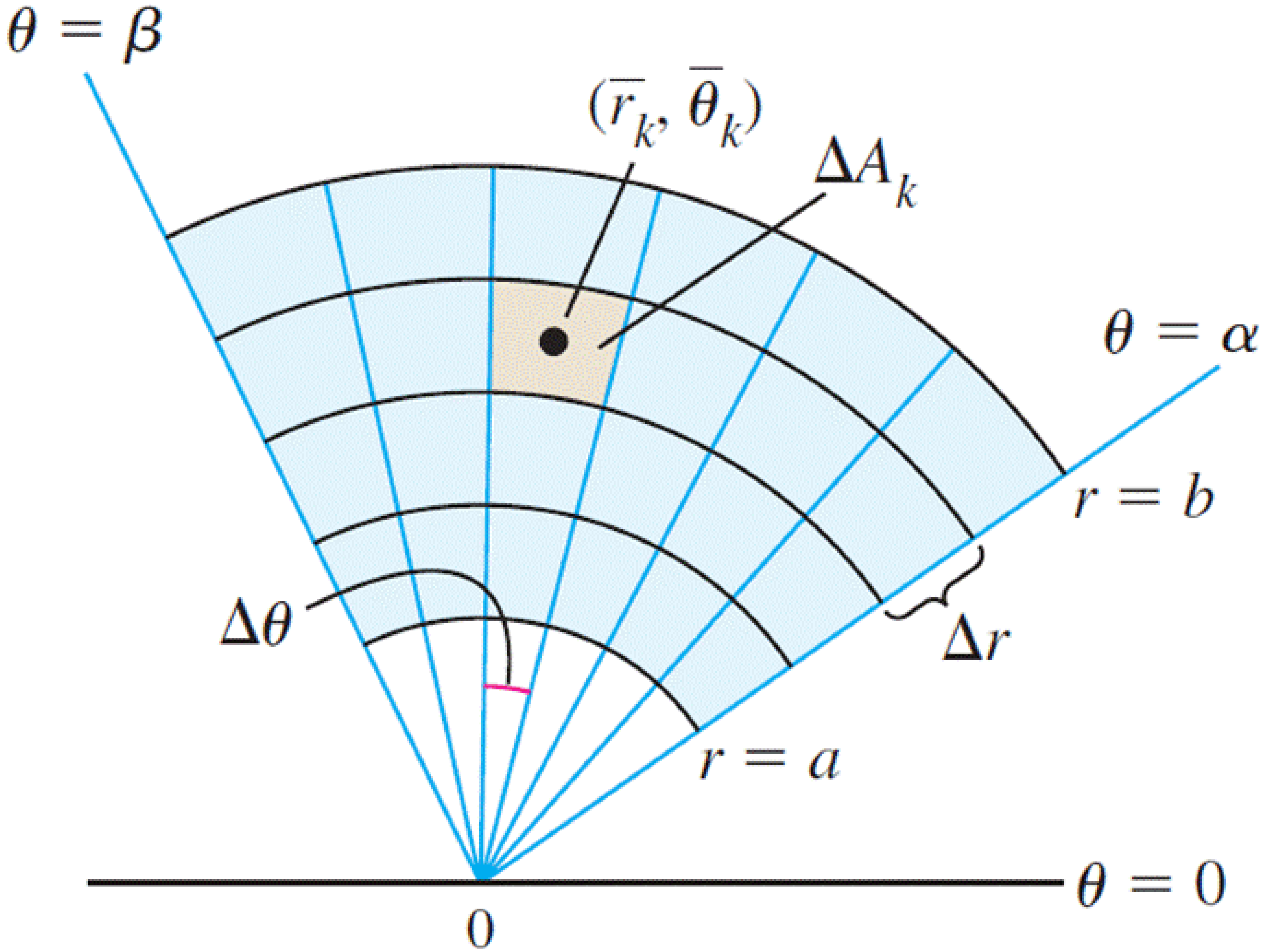
$$R = \{(r, \theta): a \le r \le b, \alpha \le \theta \le \beta\}$$



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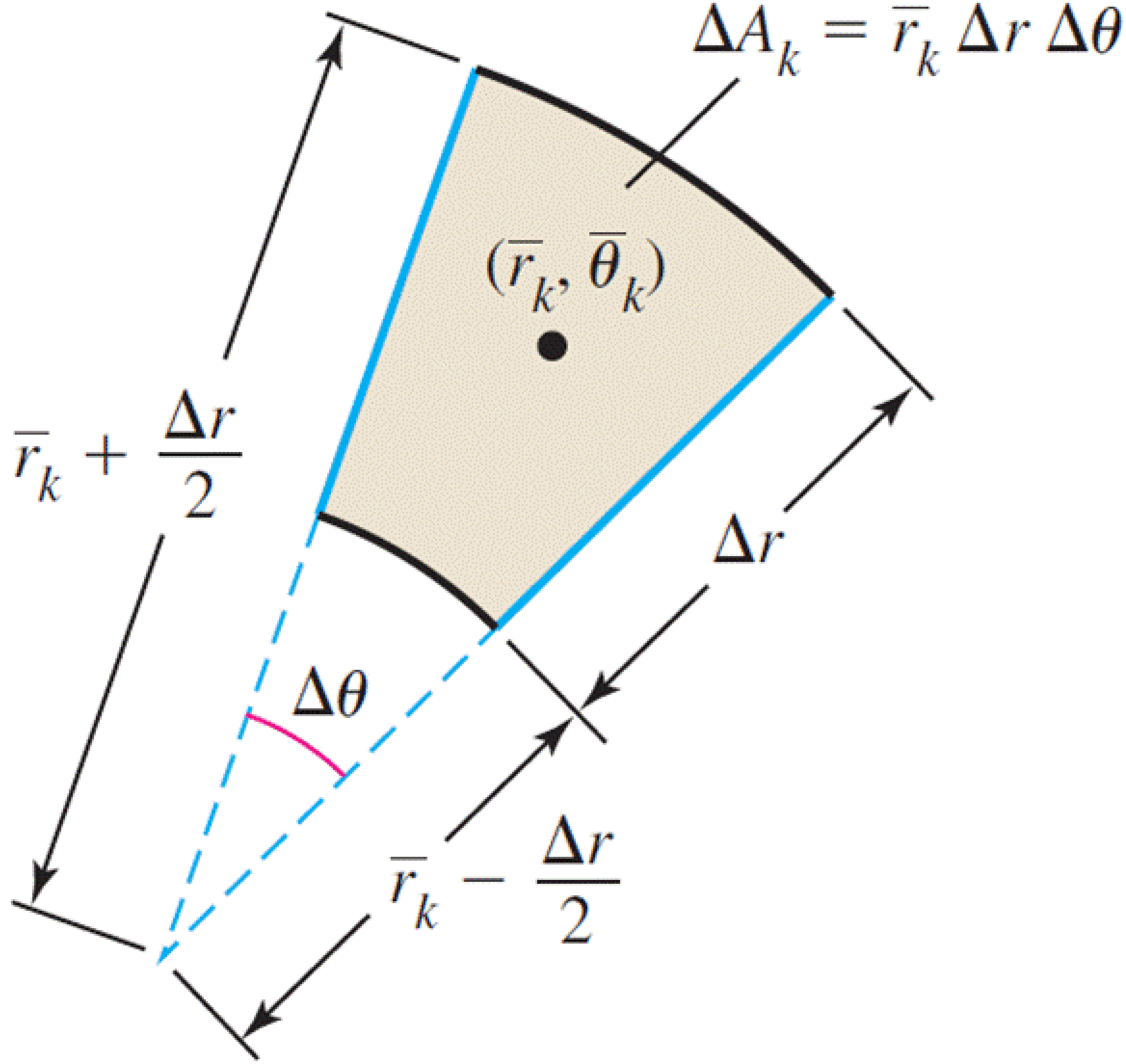

FIGURE 13.30

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THEOREM 13.3 Double Integrals over Polar Rectangular Regions

Let f be continuous on the region in the xy-plane $R = \{(r, \theta): 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$, where $\beta - \alpha \leq 2\pi$. Then

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_a^b f(r, \theta)\, r\, dr\, d\theta.$$

If f is nonnegative on R, the double integral gives the volume of the solid bounded by the surface $z = f(r, \theta)$ and R.

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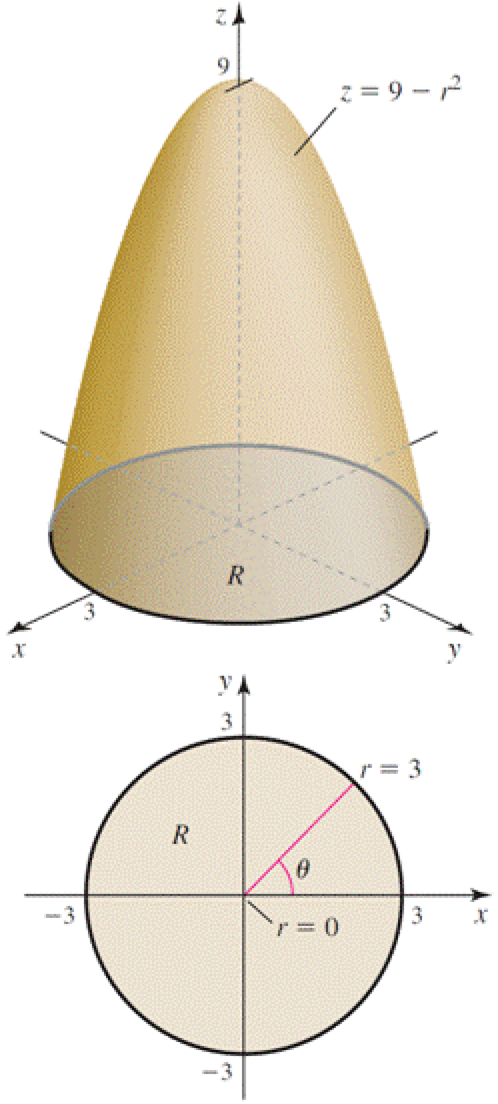

$R = \{(r, \theta): 0 \le r \le 3, 0 \le \theta \le 2\pi\}$

FIGURE 13.31

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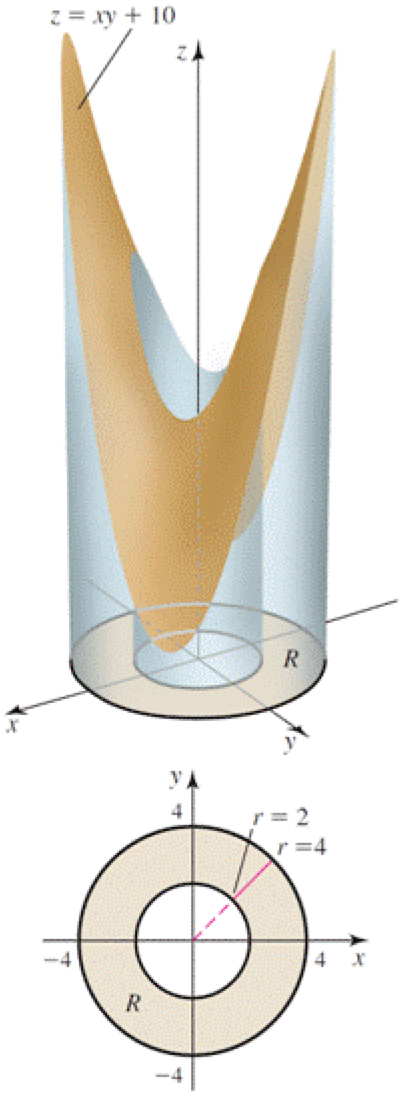

$R = \{(r, \theta) : 2 \le r \le 4, 0 \le \theta \le 2\pi\}$

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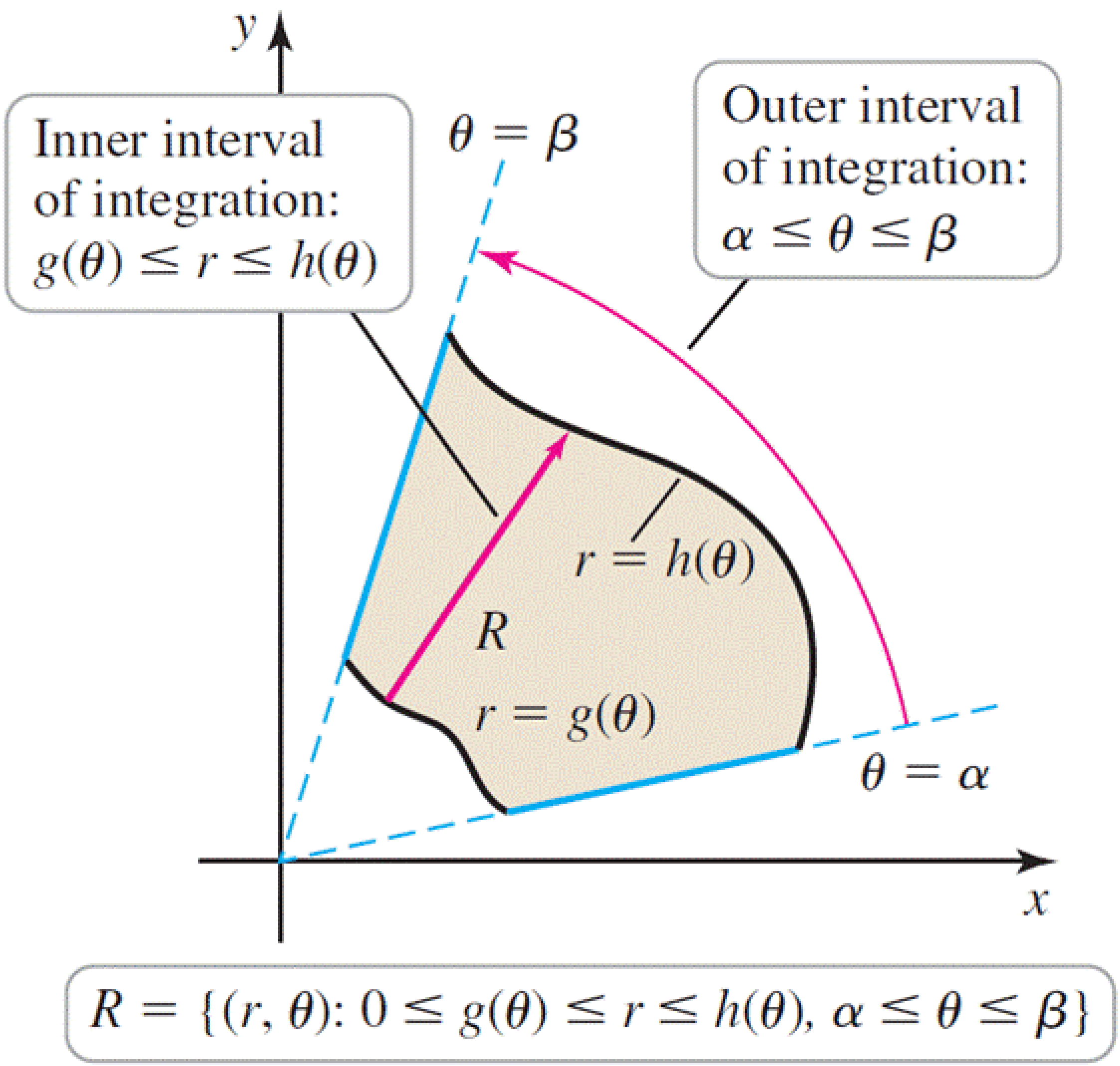

FIGURE 13.33

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THEOREM 13.4 Double Integrals over More General Polar Regions

Let f be continuous on the region in the xy-plane

$$R = \{(r, \theta): 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\},$$

where $\beta - \alpha \le 2\pi$. Then,

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

If f is nonnegative on R, the double integral gives the volume of the solid bounded by the surface $z = f(r, \theta)$ and R.

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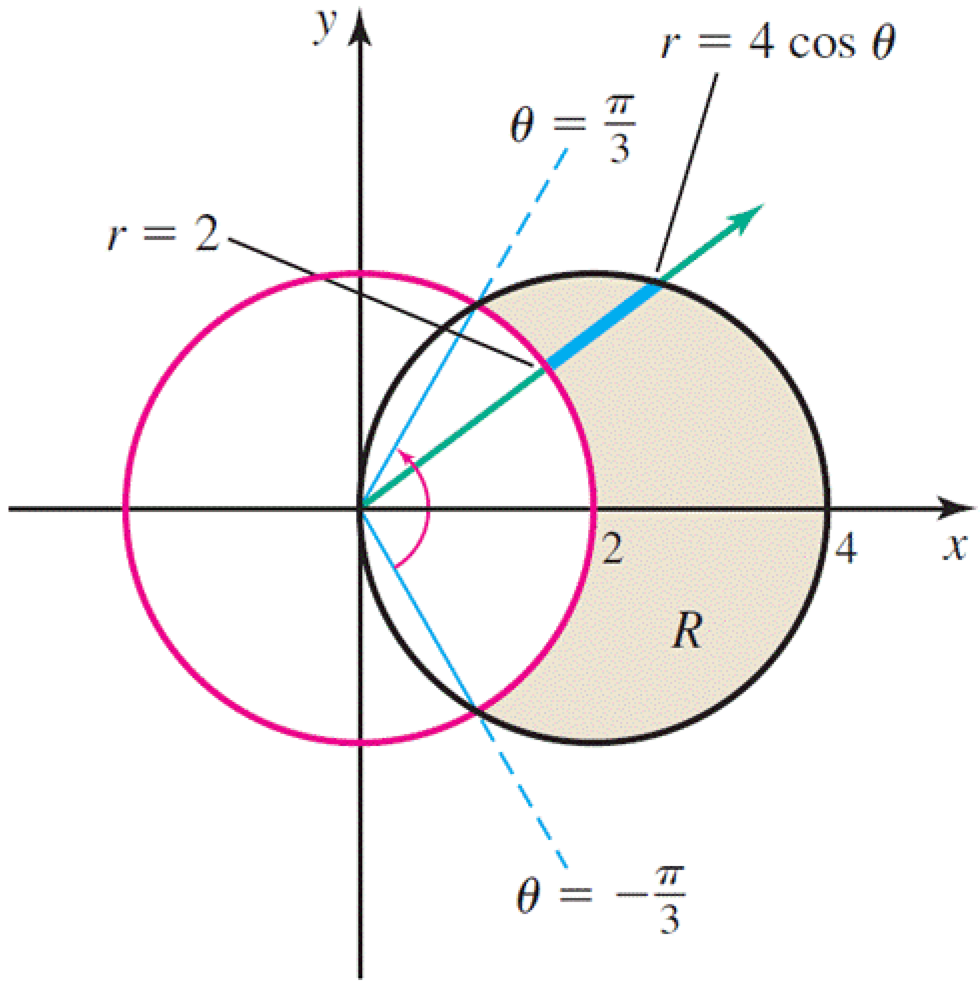

The inner and outer boundaries of R are traversed, for $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$

FIGURE 13.34

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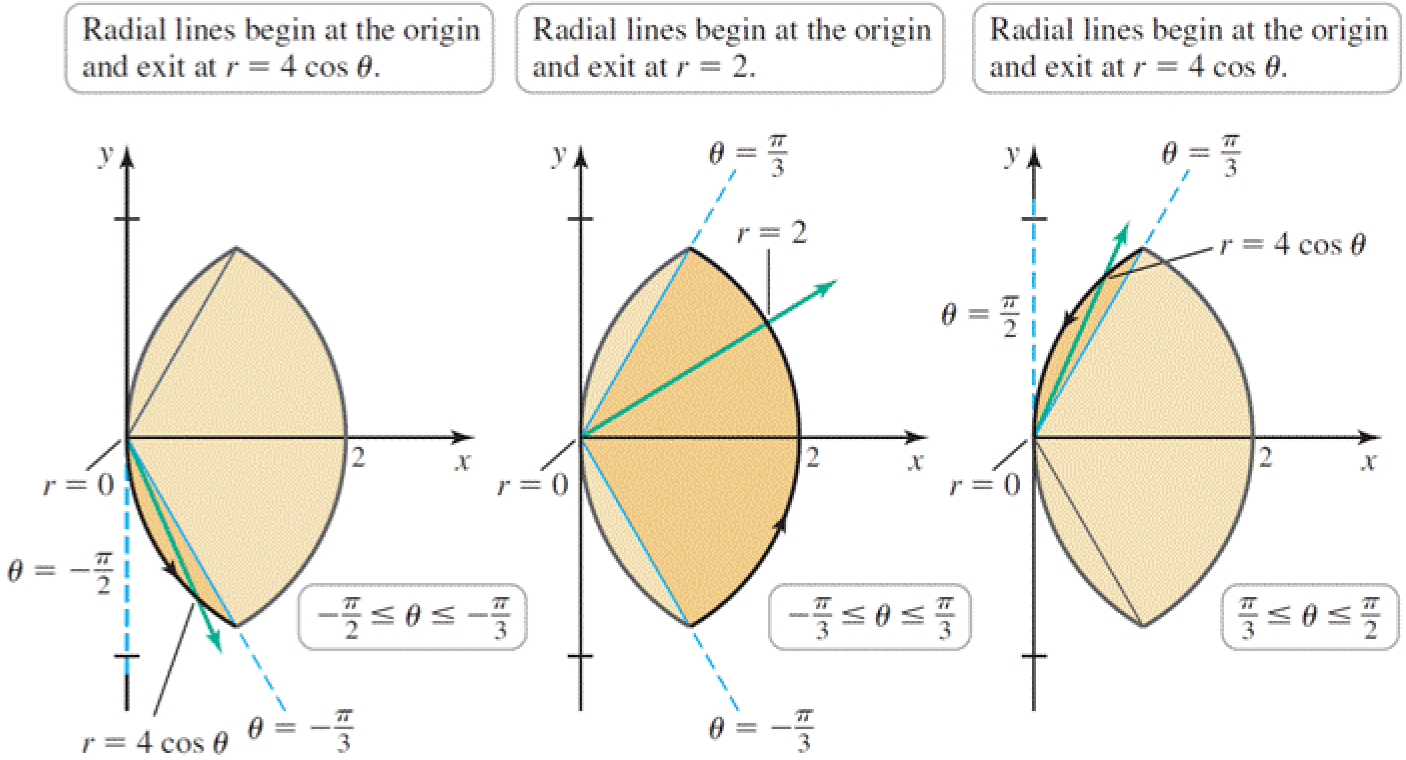

FIGURE 13.35

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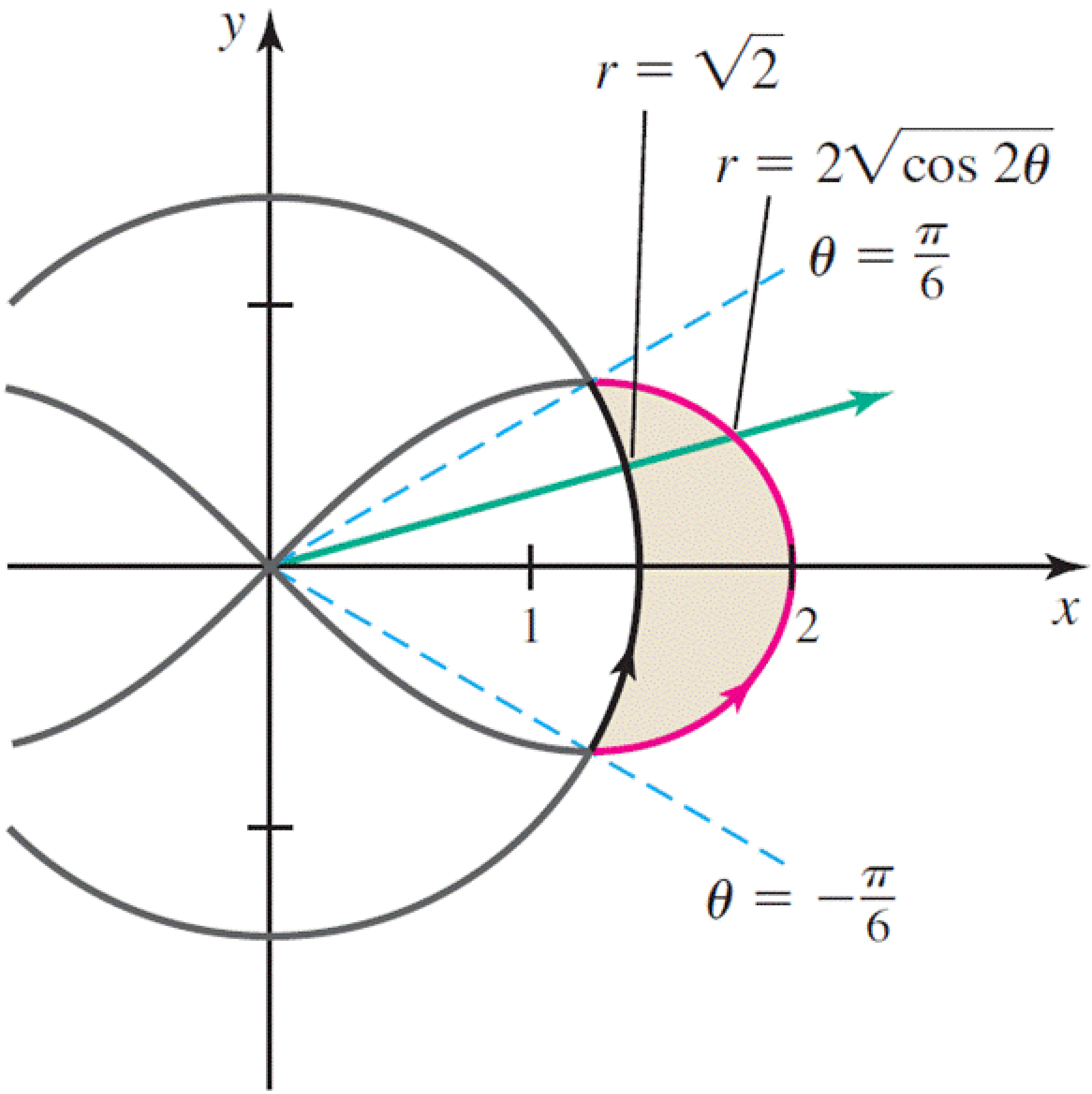

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13.4

Triple Integrals

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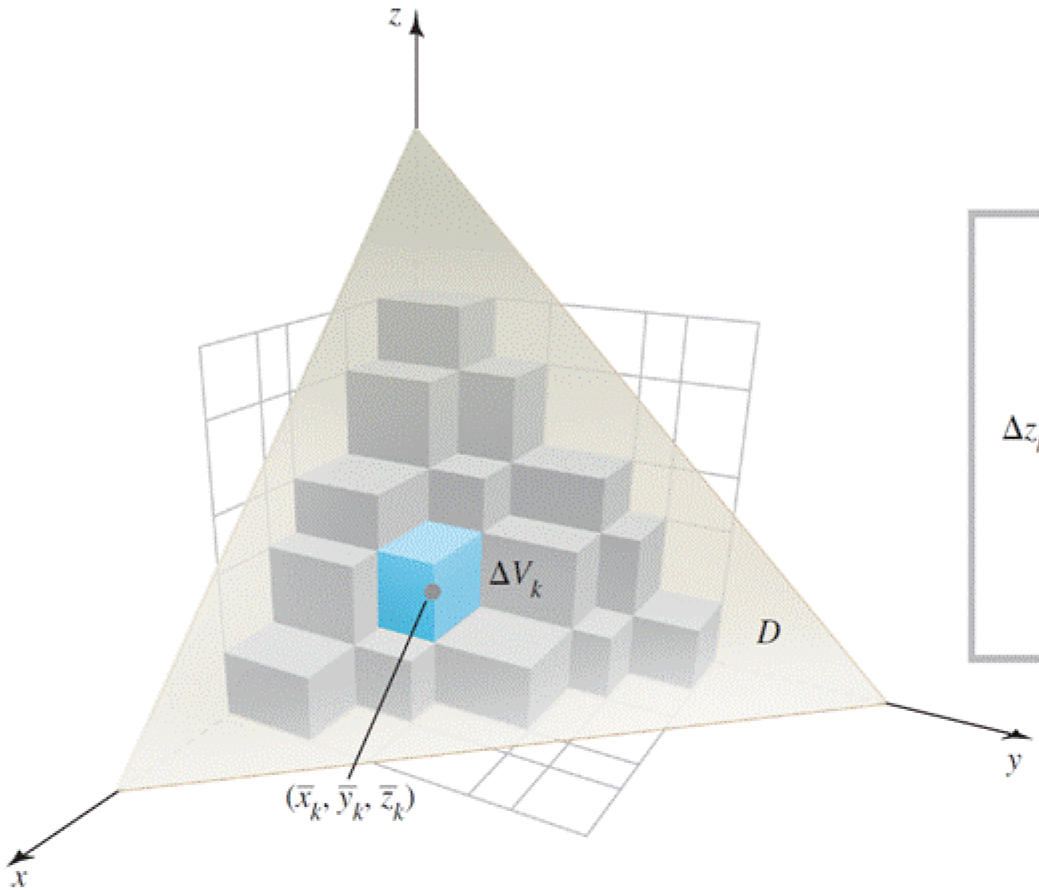

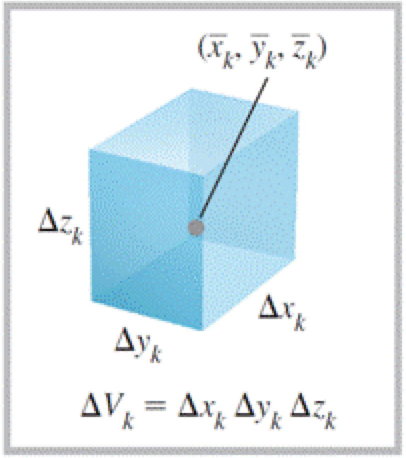

FIGURE 13.37

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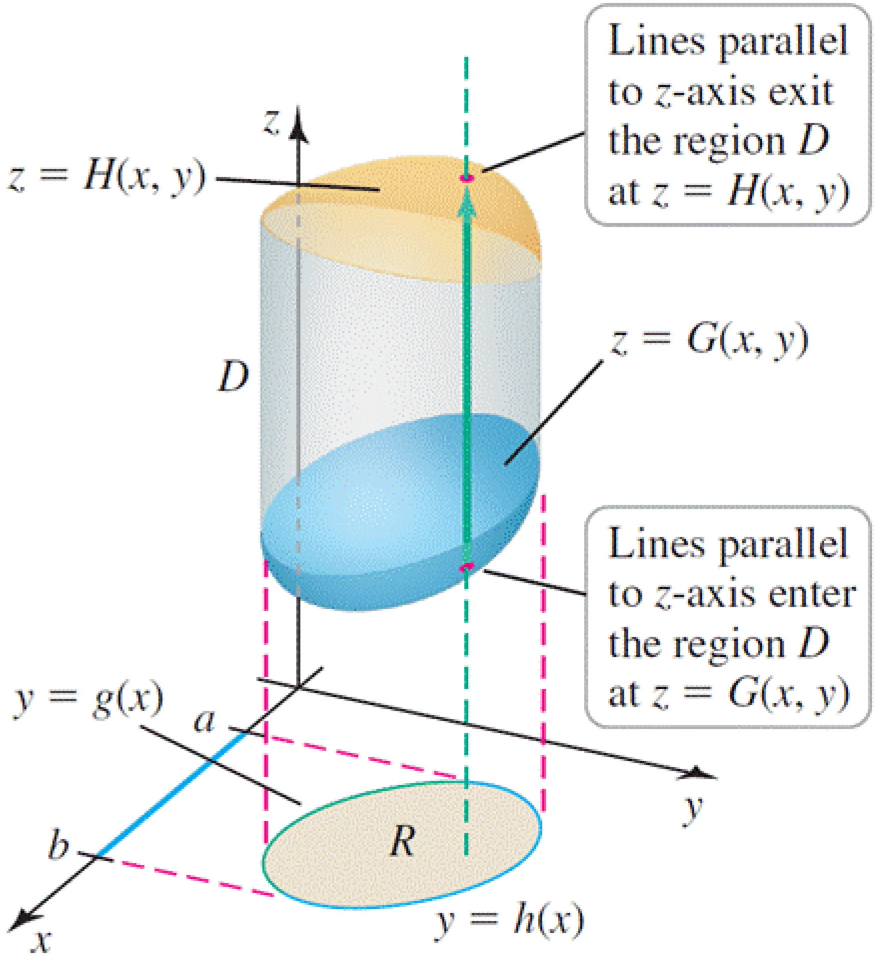

$$\iiint_D f(x, y, z)\, dV = \iint_R \left[\int_{G(x,y)}^{H(x,y)} f(x, y, z)\, dz\right] dA$$

FIGURE 13.38

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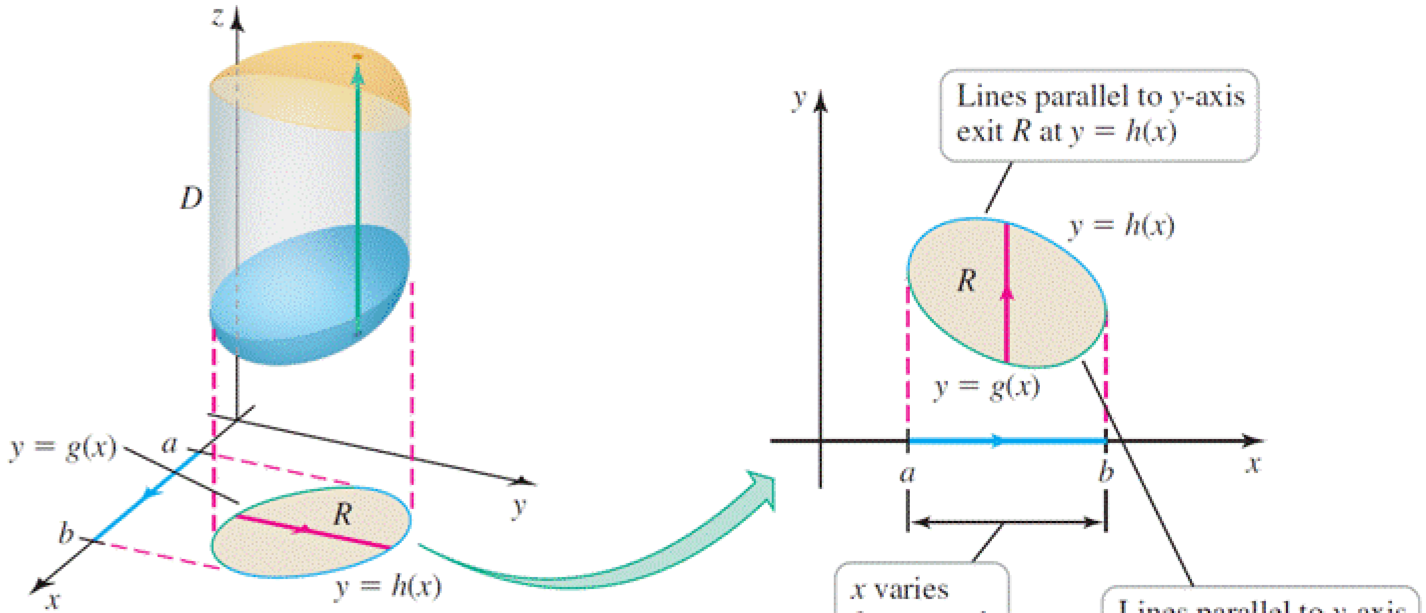

$$\iiint_R f(x, y, z) = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x, y)}^{H(x, y)} f(x, y, z)\, dz\, dy\, dx$$

FIGURE 13.39

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Table 13.1

Integral	Variable	Interval
Inner	z	$G(x, y) \leq z \leq H(x, y)$
Middle	y	$g(x) \leq y \leq h(x)$
Outer	x	$a \leq x \leq b$

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THEOREM 13.5 Triple Integrals

Let $D = \{(x, y, z): a \le x \le b, g(x) \le y \le h(x), G(x, y) \le z \le H(x, y)\}$, where g, h, G, H are continuous functions. The triple integral of a continuous function f on D is evaluated as the iterated integral

$$\iiint_D f(x, y, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

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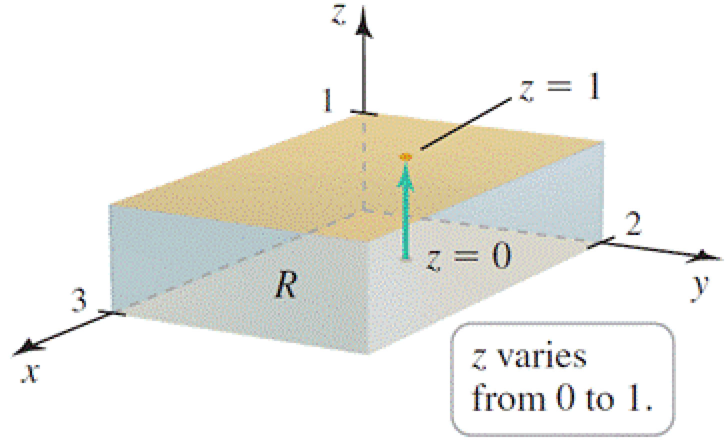

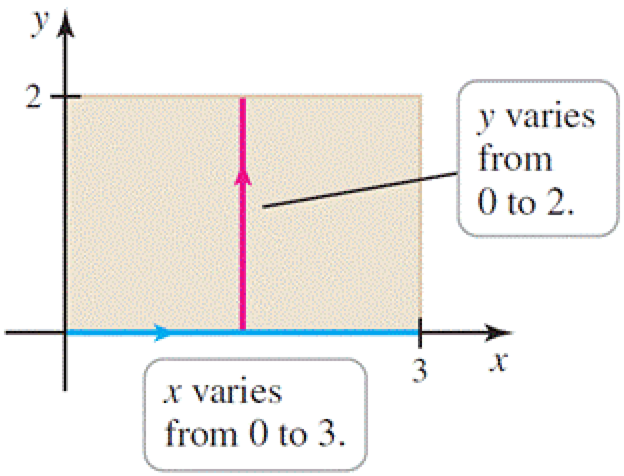

$$M = \int_0^3 \int_0^2 \int_0^1 (2 - z)\, dz\, dy\, dx$$

FIGURE 13.40

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Table 13.2

Integral	Variable	Interval
Inner	z	$0 \le z \le 1$
Middle	y	$0 \le y \le 2$
Outer	x	$0 \le x \le 3$

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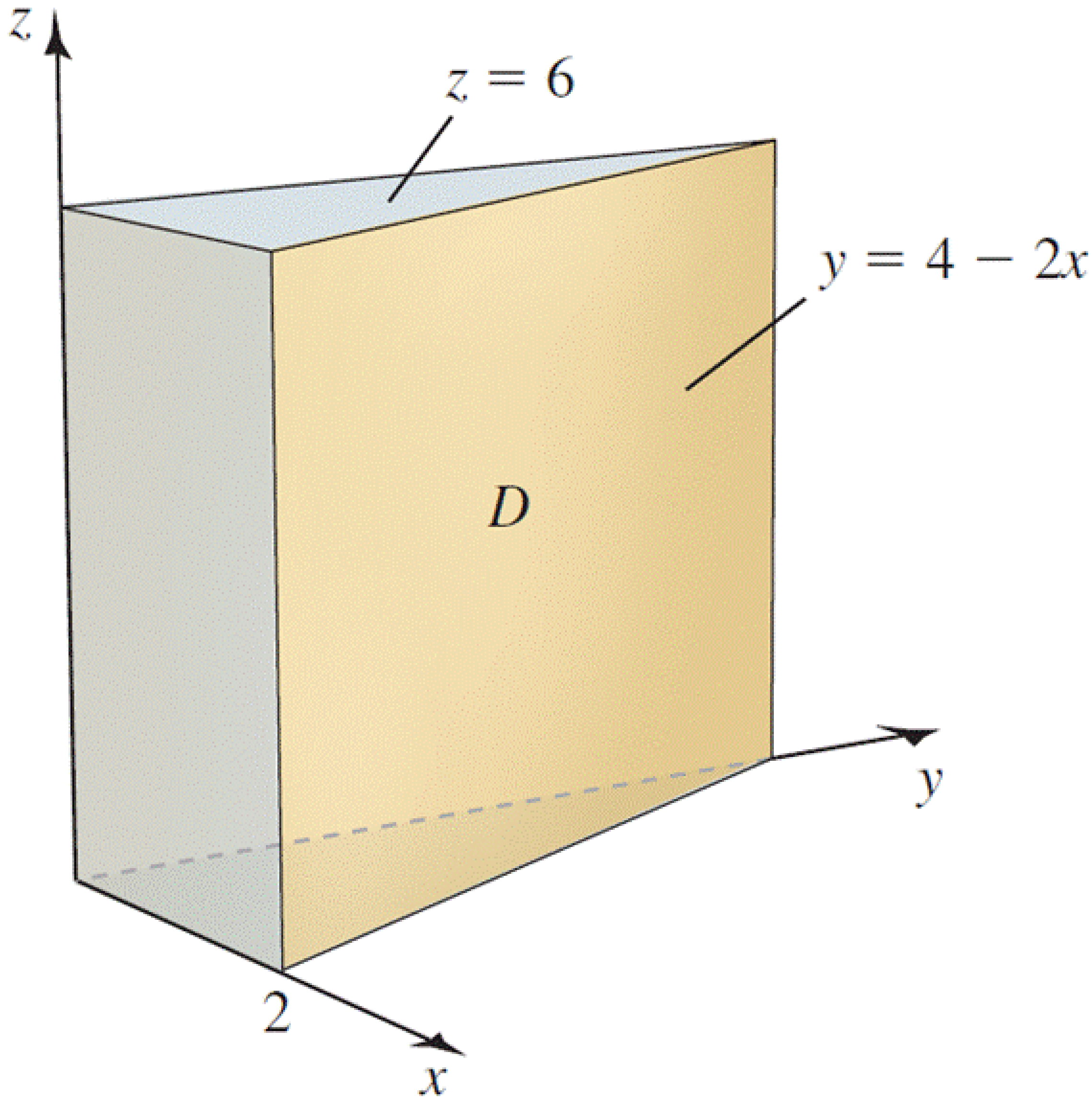

FIGURE 13.41

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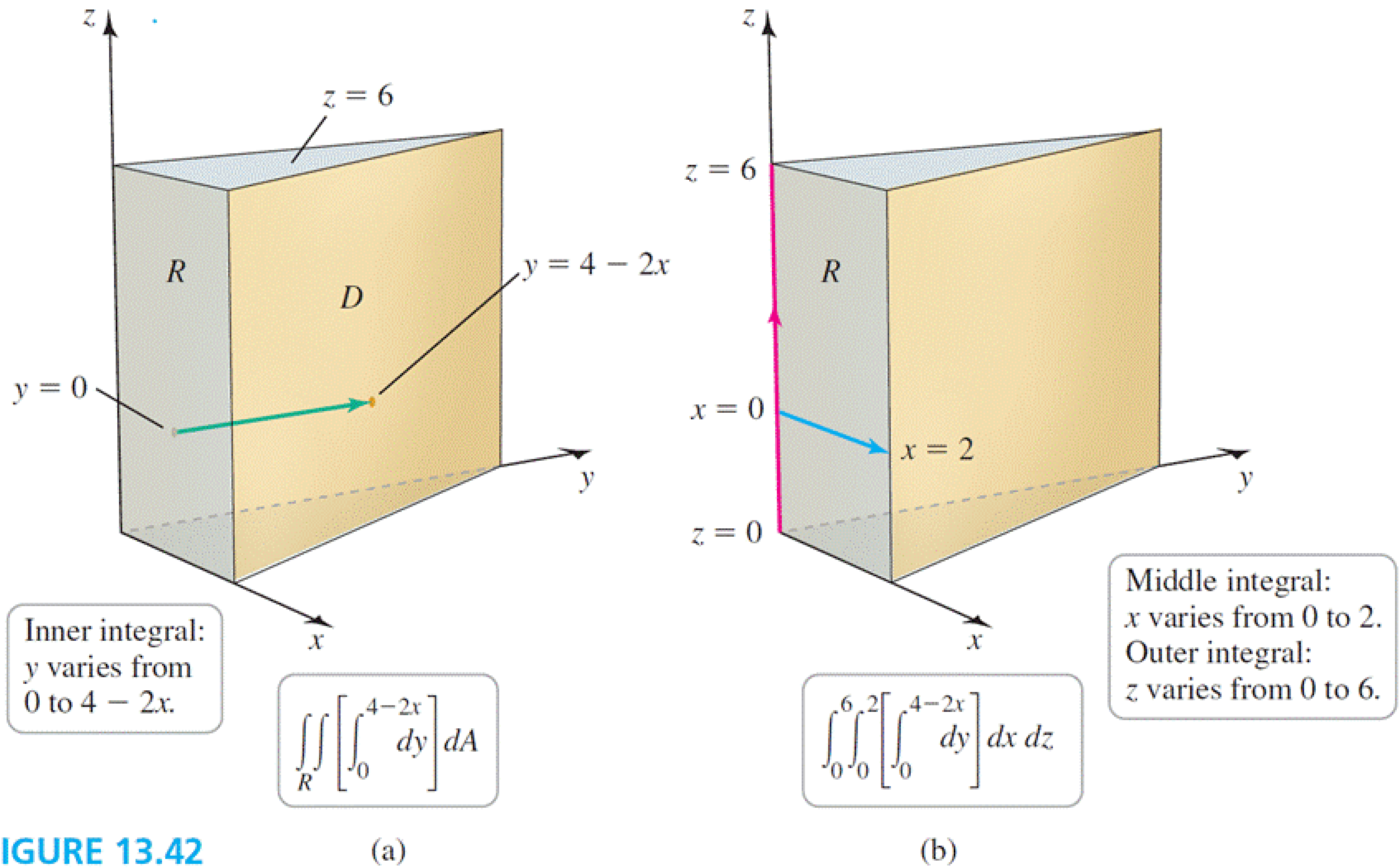

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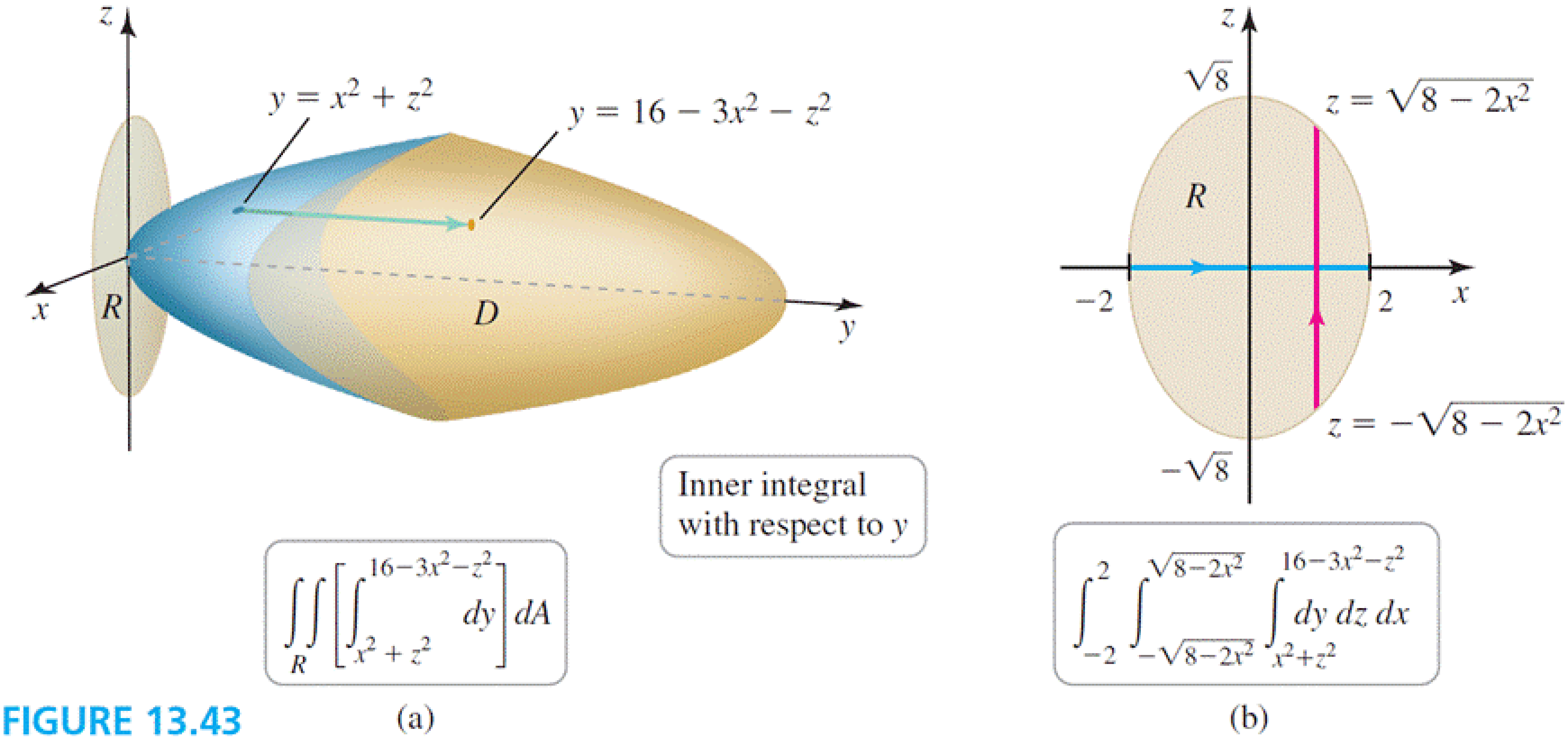

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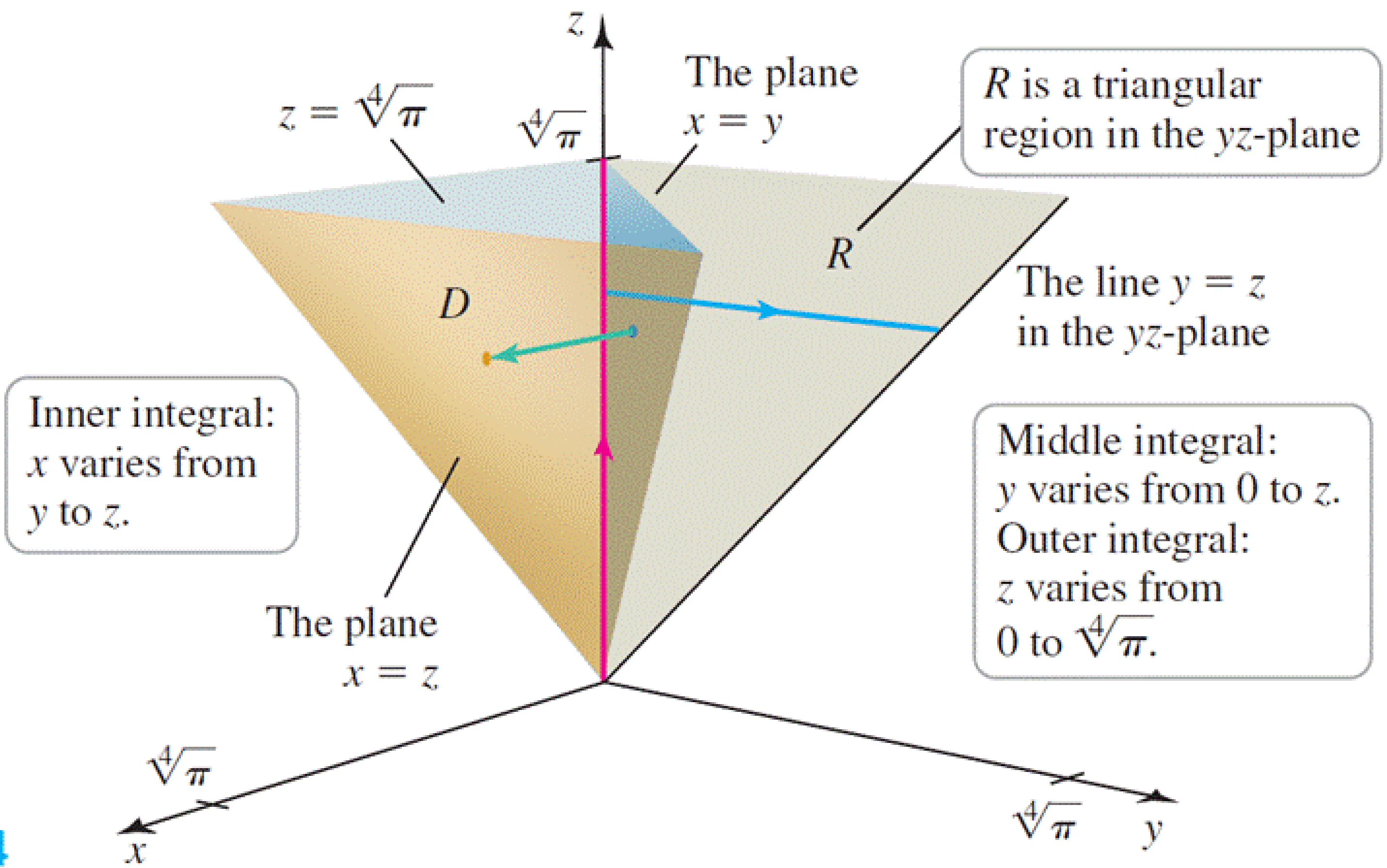

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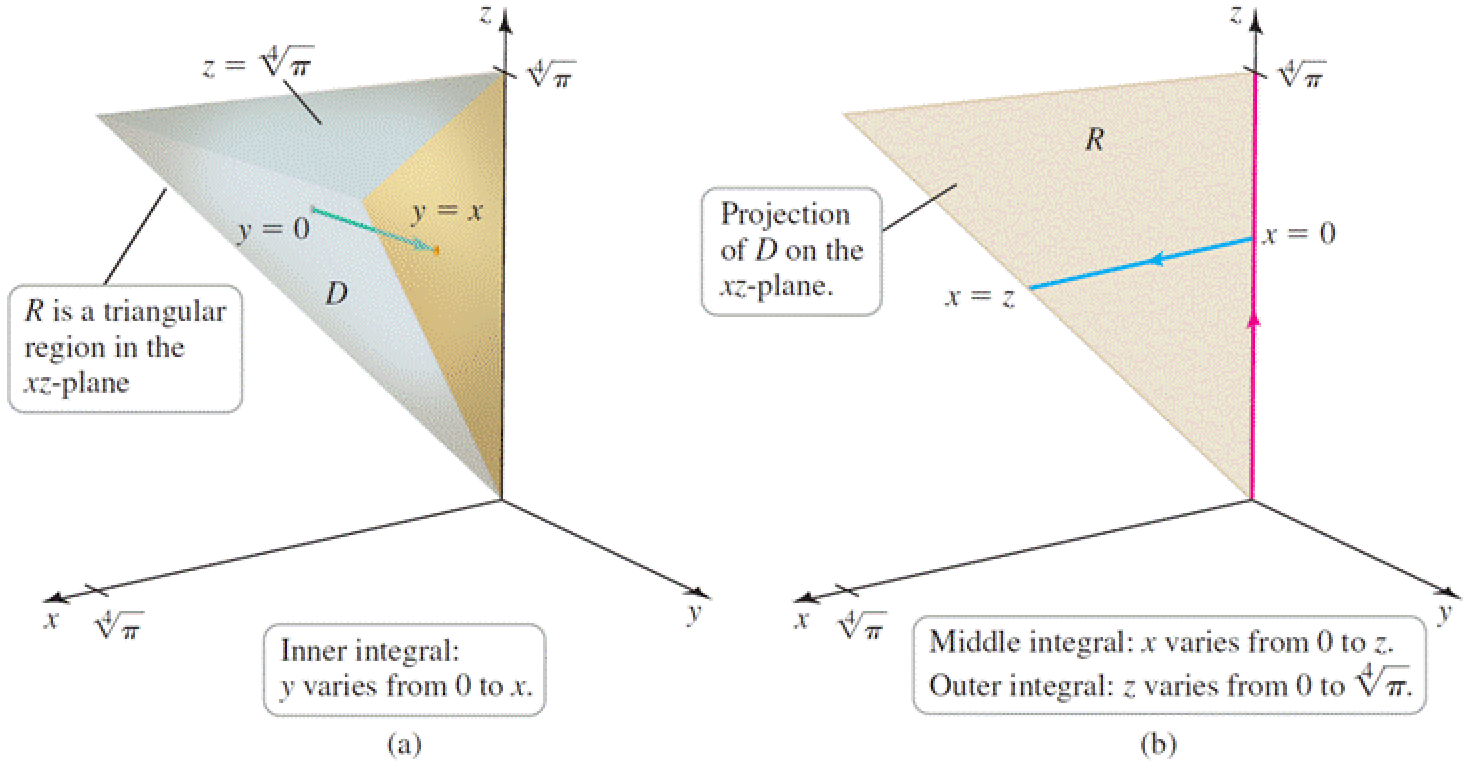

FIGURE 13.45

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DEFINITION **Average Value of a Function of Three Variables**

If f is continuous on a region D of $\mathbf{R}^3$, then the average value of f over D is

$$\bar{f} = \frac{1}{\text{volume}\,(D)} \iiint\limits_D f(x, y, z)\, dV.$$

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13.5

Triple Integrals in Cylindrical and Spherical Coordinates

PEARSON

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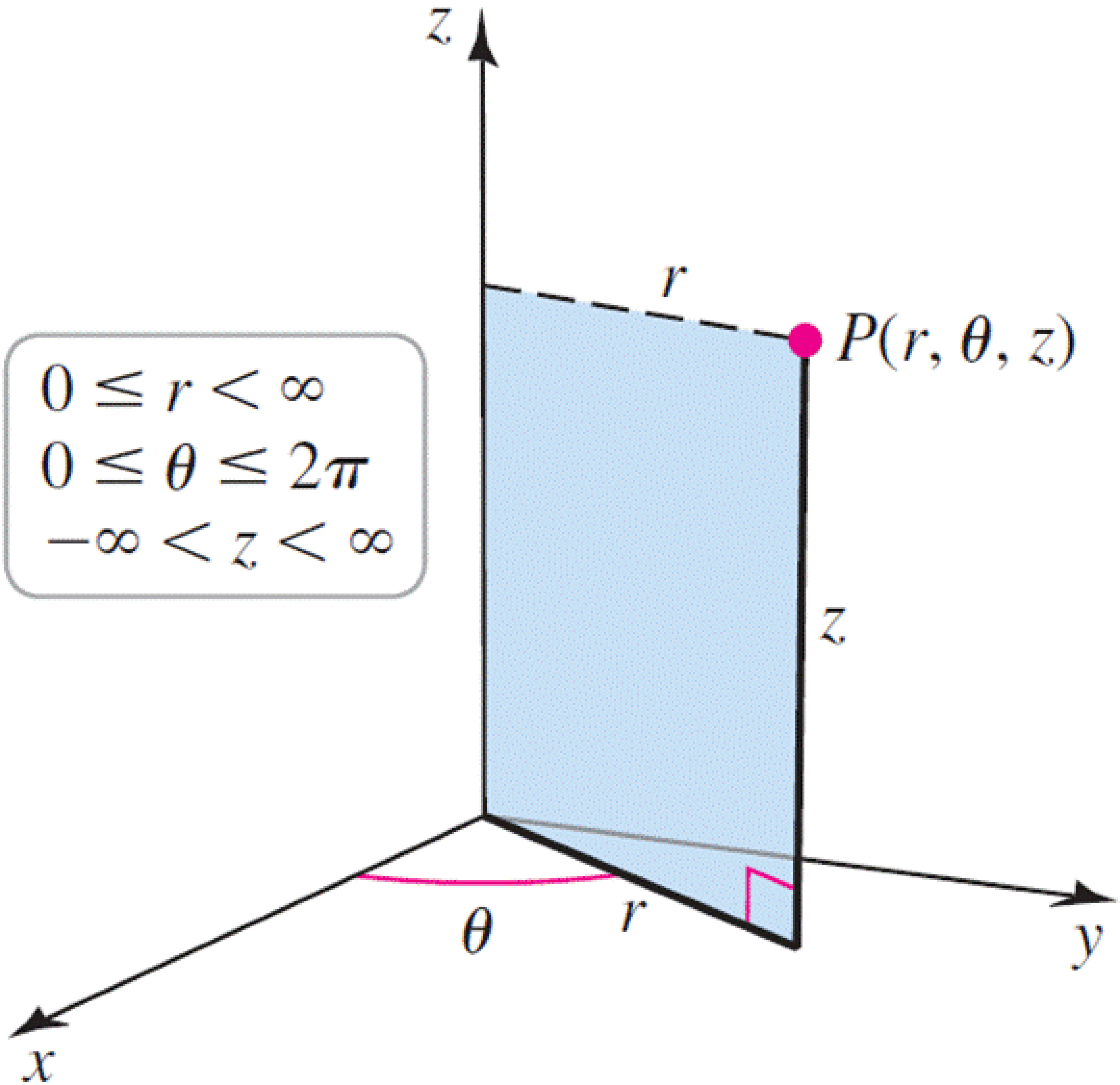

FIGURE 13.46

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Table 13.3

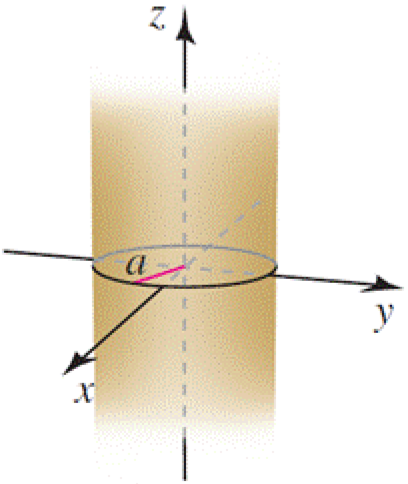
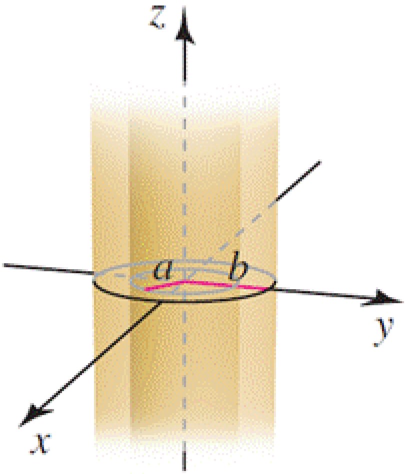
Name	Description	Example
Cylinder	$\{(r, \theta, z): r = a\}, a > 0$	
Cylindrical shell	$\{(r, \theta, z): 0 < a \leq r \leq b\}$	

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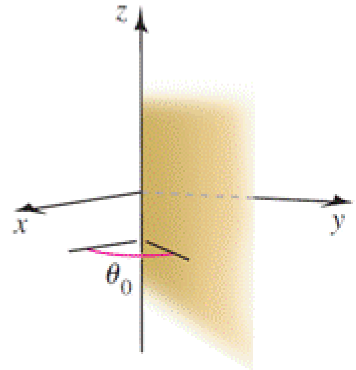
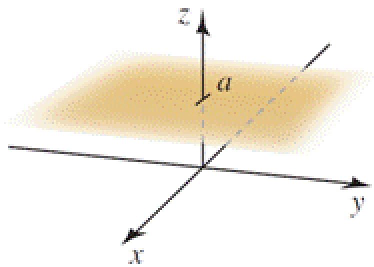
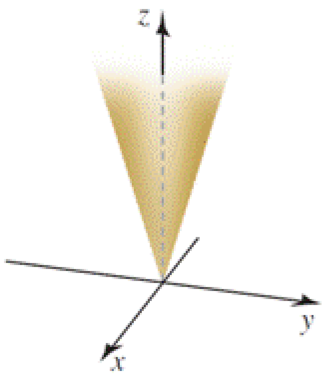
Table 13.3 **(Continued)**

Name	Description	Example
Vertical half plane	$\{(r, \theta, z): \theta = \theta_0\}$	
Horizontal plane	$\{(r, \theta, z): z = a\}$	
Cone	$\{(r, \theta, z): z = ar\}, a \neq 0$	

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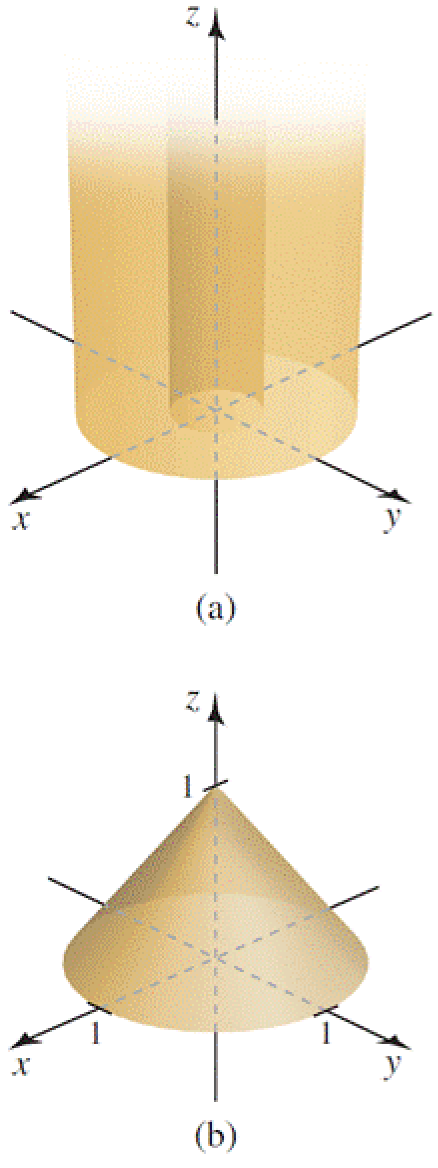

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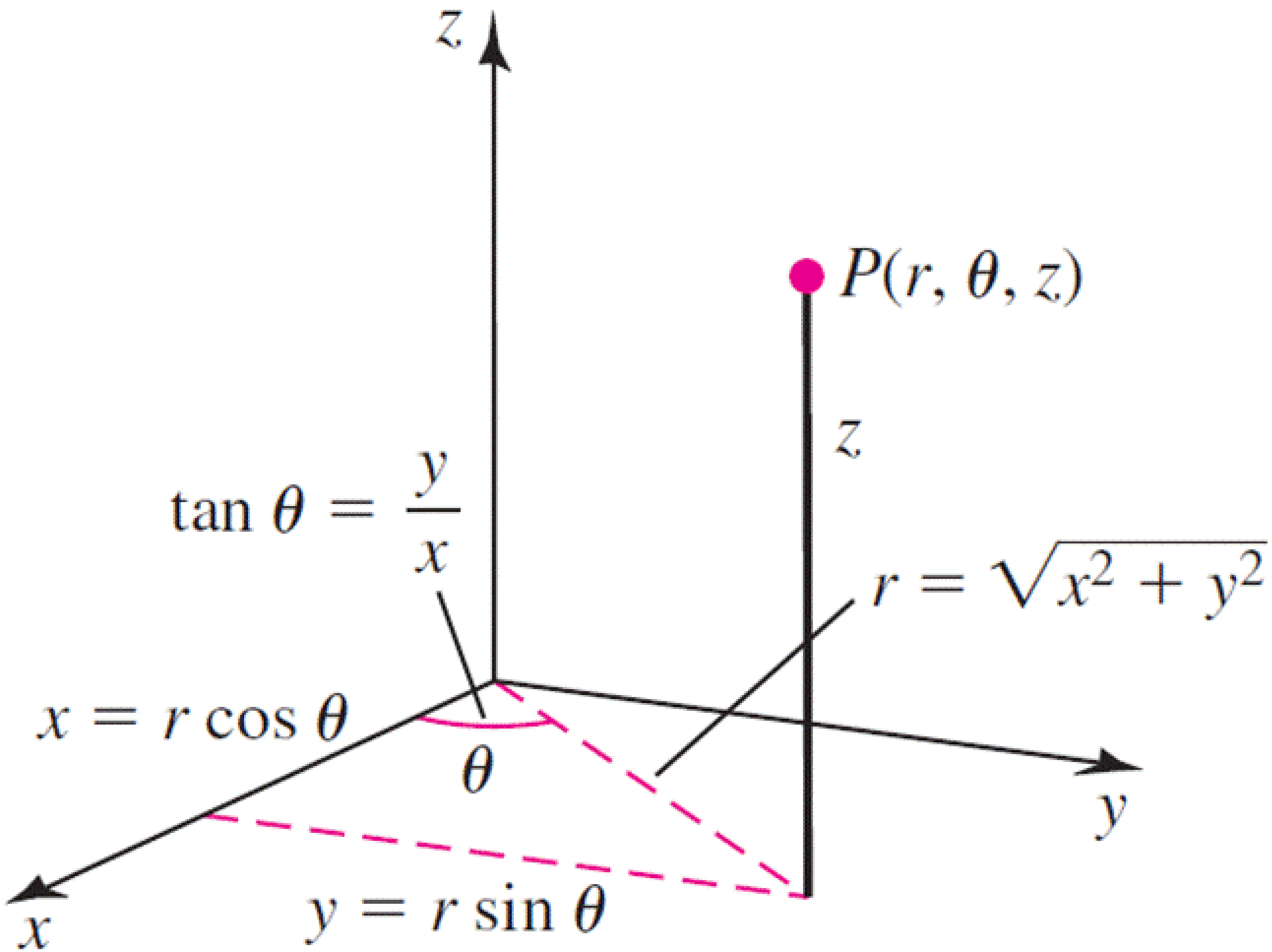

FIGURE 13.48

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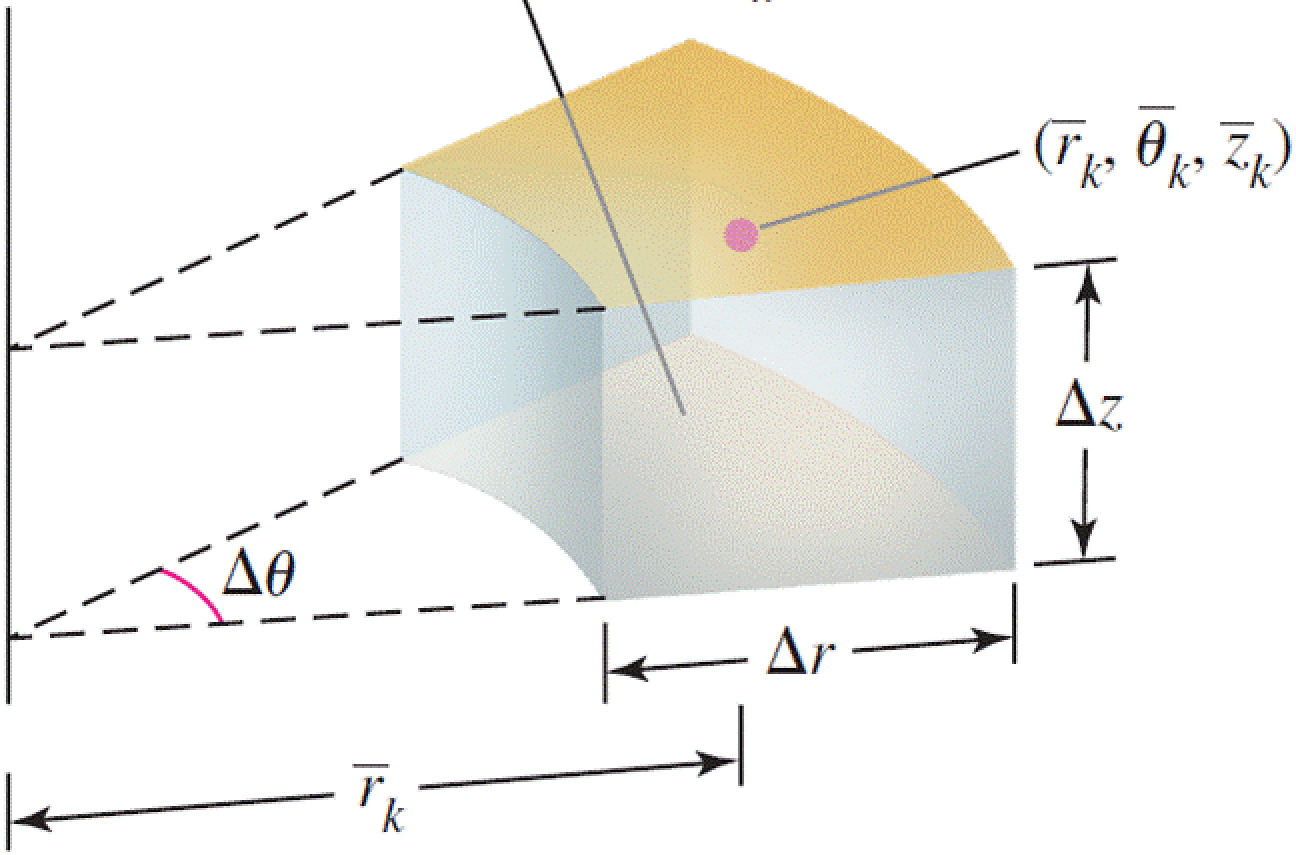

Approximate volume $\Delta V_k \approx \bar{r}_k \, \Delta r \, \Delta\theta \, \Delta z$

FIGURE 13.49

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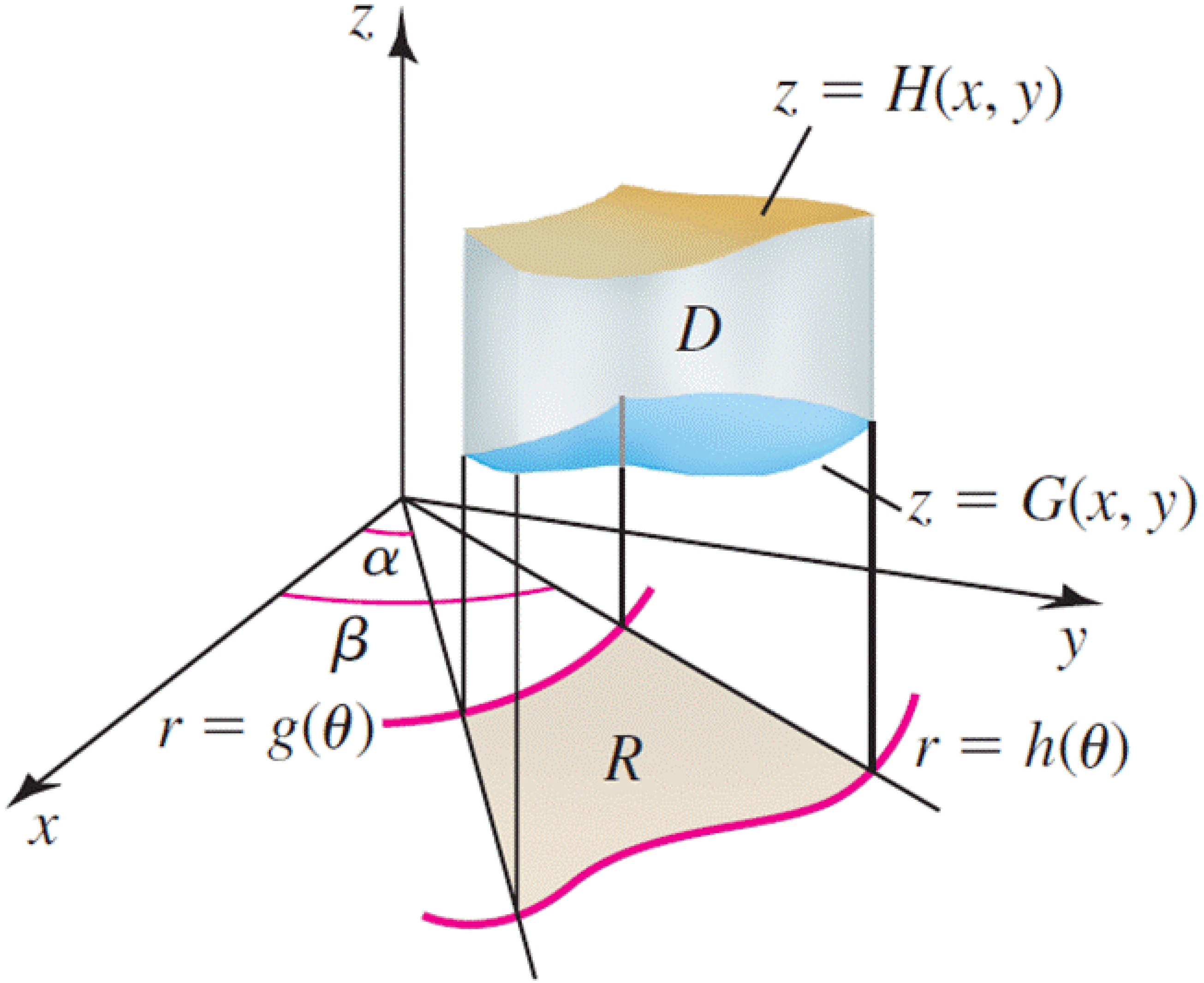

FIGURE 13.50

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THEOREM 13.6 Triple Integrals in Cylindrical Coordinates

Let f be continuous over the region

$$D = \{r, \theta, z): g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta, G(x, y) \le z \le H(x, y)\}.$$

Then f is integrable over D and the triple integral of f over D in cylindrical coordinates is

$$\iiint_D f(r, \theta, z)\, dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

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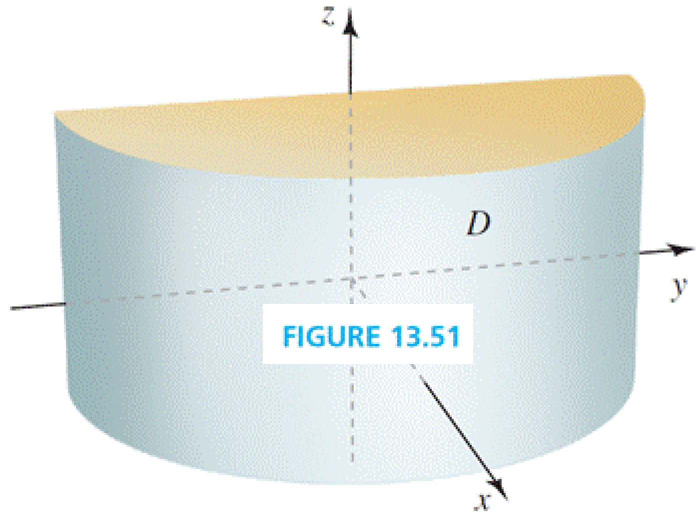

(a)

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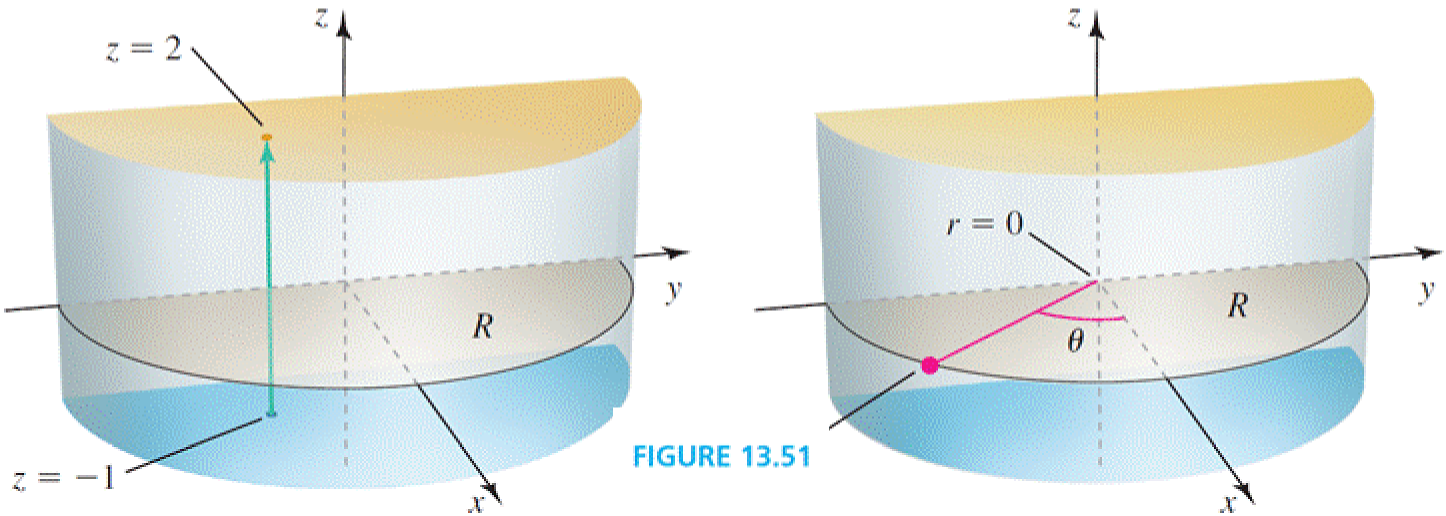

FIGURE 13.51

$$\iint_R \int_{-1}^{2} \sqrt{1 + r^2}\, dz\, dA$$

In cylindrical coordinates, integrate in z with $-1 \le z \le 2$;...

(b)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\sqrt{2}} \int_{-1}^{2} \sqrt{1 + r^2}\, dz\, r\, dr\, d\theta$$

... then integrate over R with $0 \le r \le 2\sqrt{2}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

(c)

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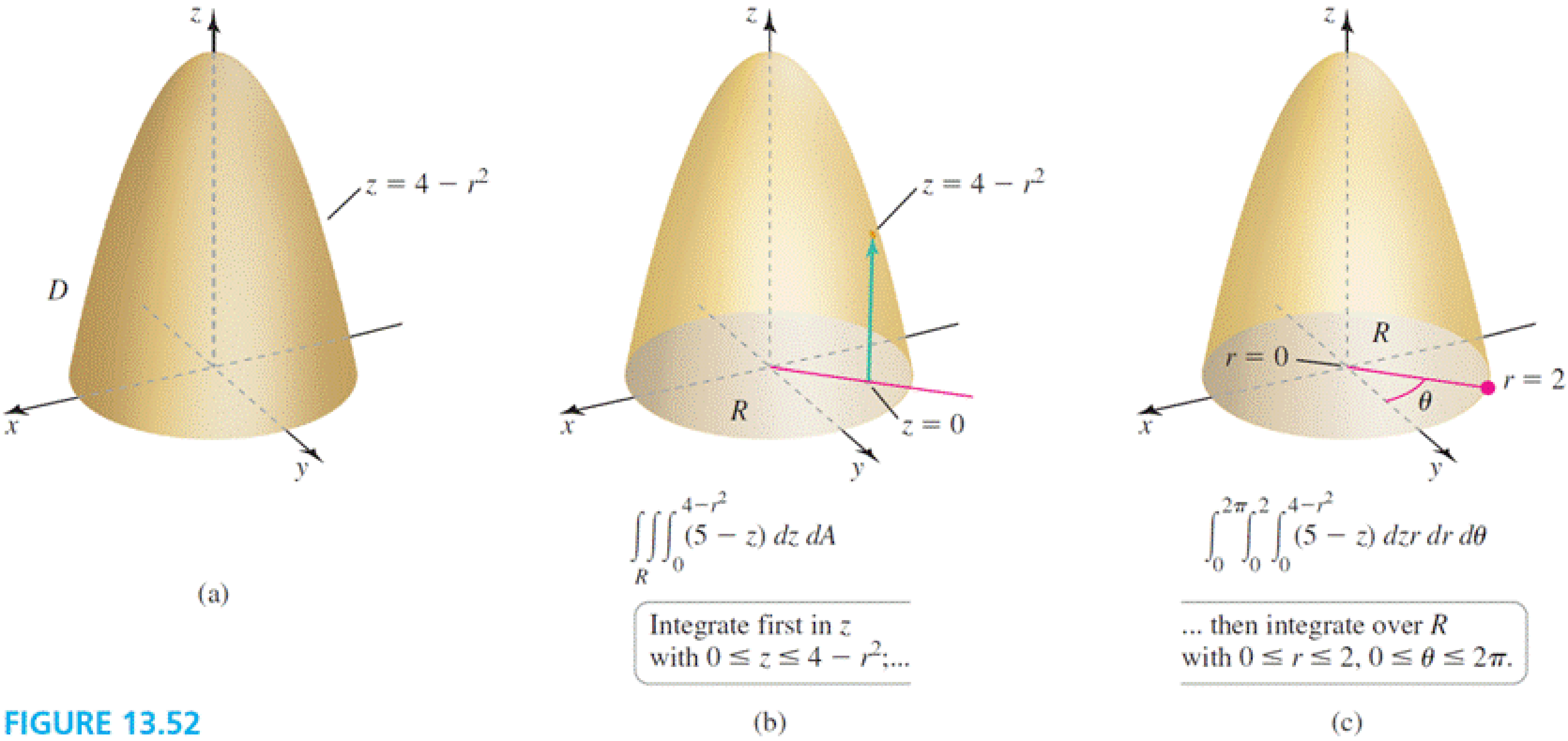

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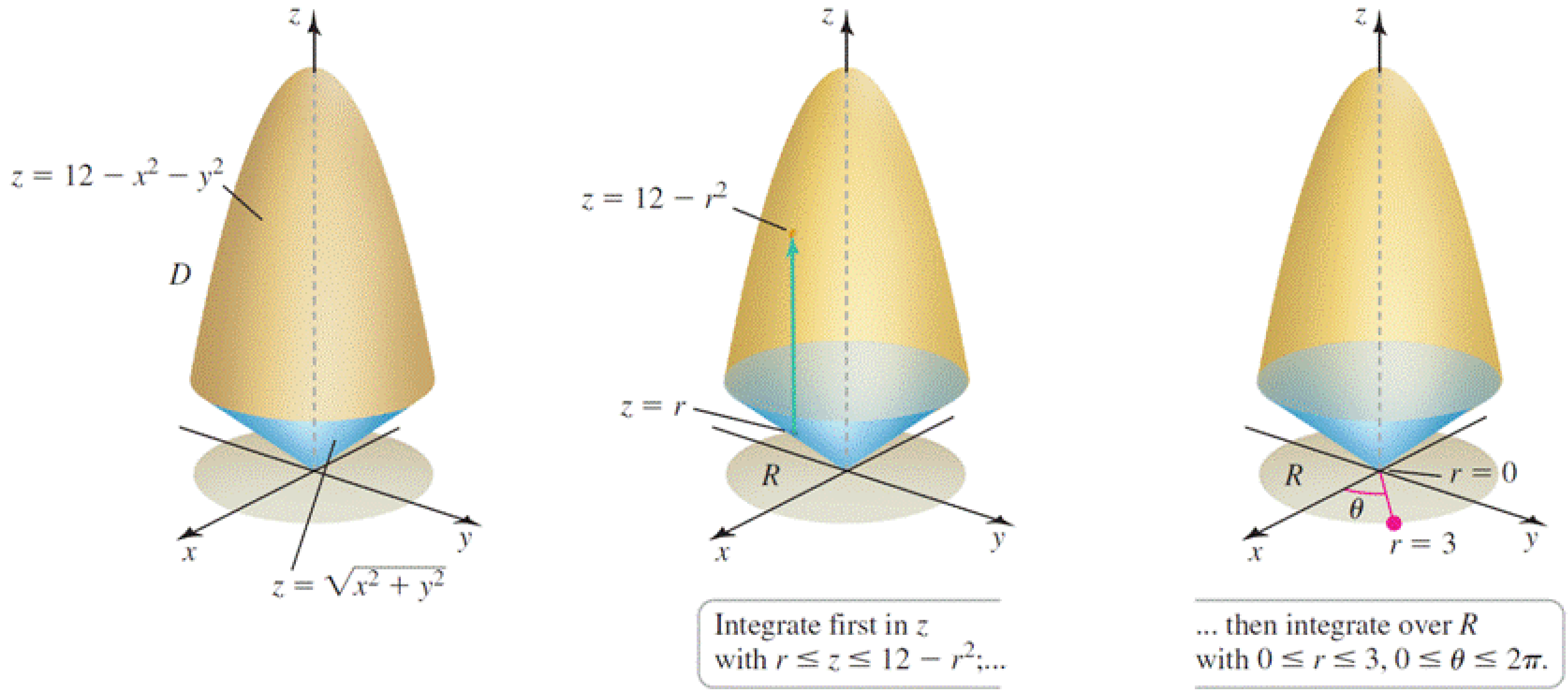

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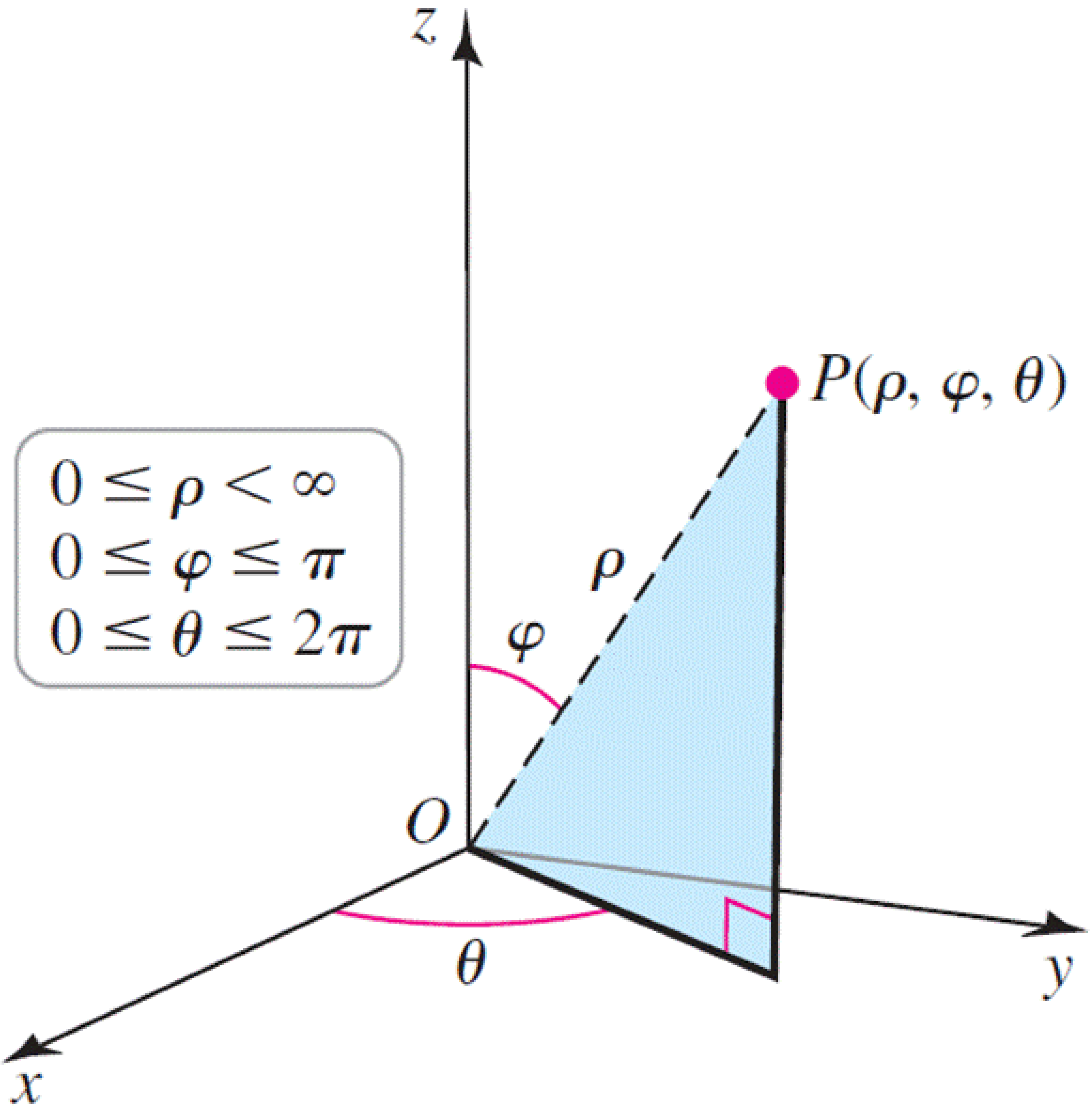

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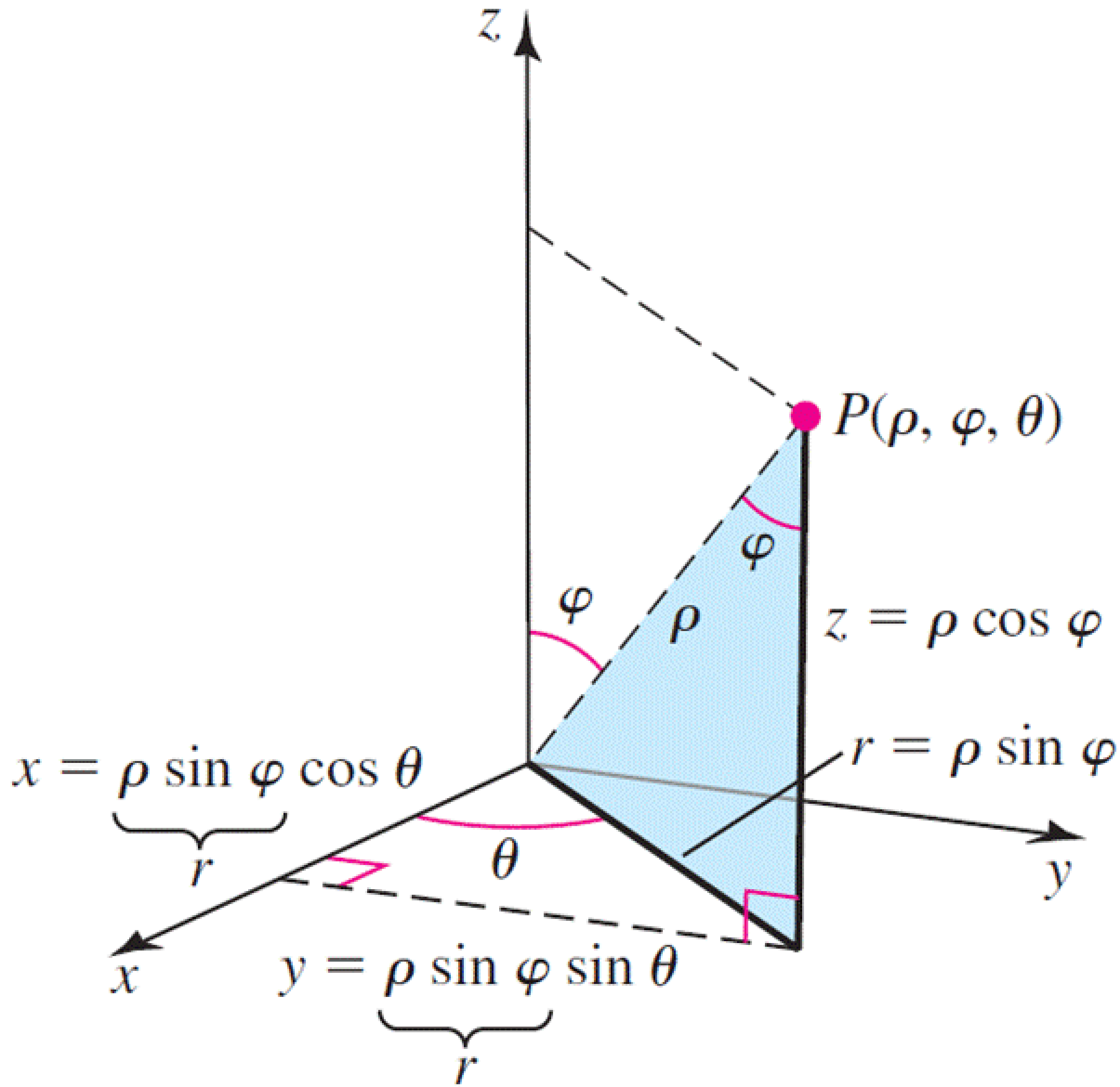

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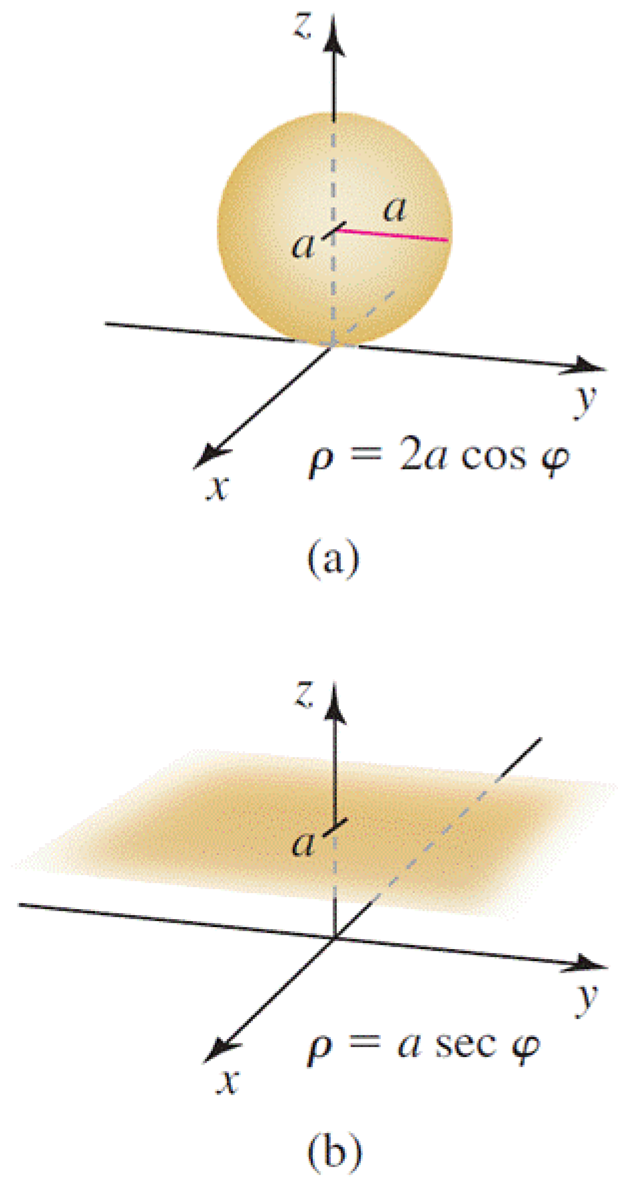

FIGURE 13.56

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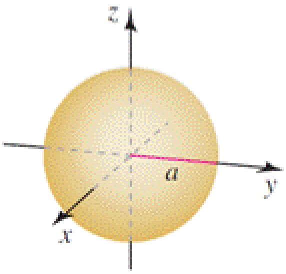
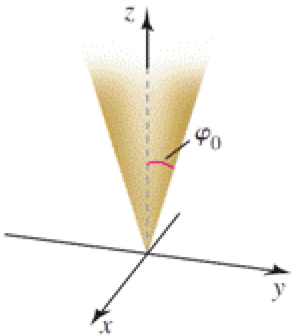
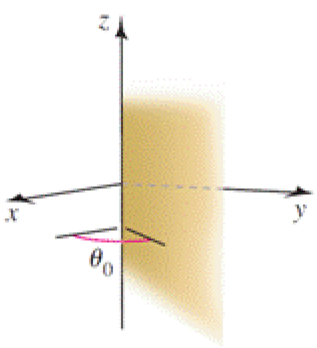
Table 13.4

Name	Description	Example
Sphere, radius a, center $(0, 0, 0)$	$\{(\rho, \varphi, \theta)\colon \rho = a\}, a > 0$	
Cone	$\{(\rho, \varphi, \theta)\colon \varphi = \varphi_0\}, \varphi_0 \neq 0, \pi/2, \pi$	
Vertical half plane	$\{(\rho, \varphi, \theta)\colon \theta = \theta_0\}$	

(*Continued*)

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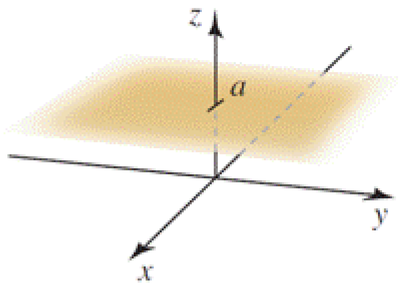
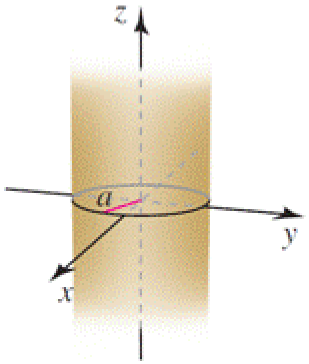
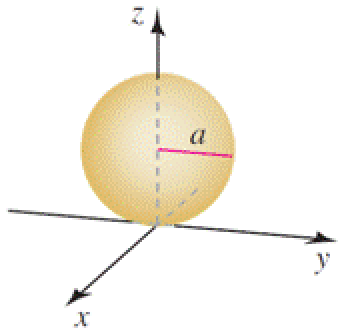
Table 13.4 (Continued)

Name	Description	Example
Horizontal plane, $z = a$	$\{(\rho, \varphi, \theta): \rho = a \sec \varphi, 0 \le \varphi < \pi/2\}$	
Cylinder, radius $a > 0$	$\{(\rho, \varphi, \theta): \rho = a \csc \varphi, 0 < \varphi < \pi\}$	
Sphere, radius $a > 0$, center $(0, 0, a)$	$\{(\rho, \varphi, \theta): \rho = 2a \cos \varphi, 0 \le \varphi \le \pi/2\}$	

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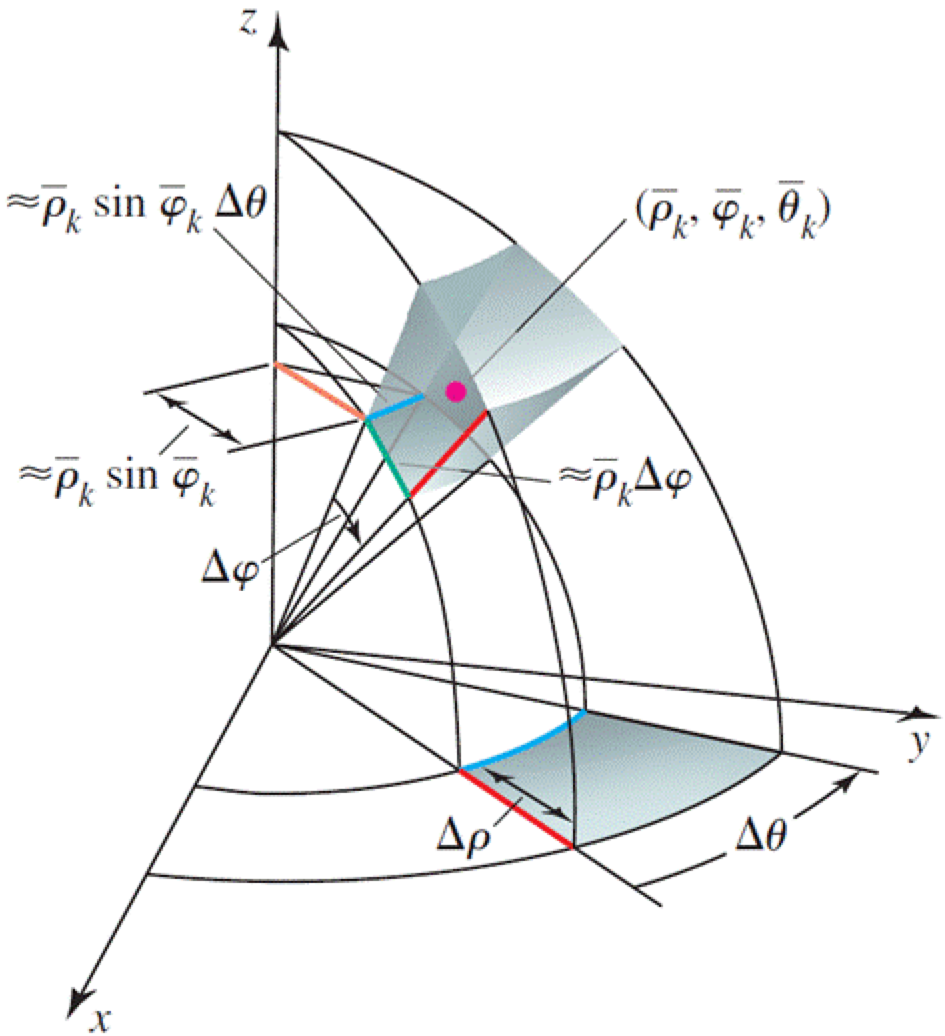

Approximate volume =
$\Delta V_k \approx \bar{\rho}_k^{\,2} \sin \bar{\varphi}_k \, \Delta\rho \, \Delta\varphi \, \Delta\theta$

FIGURE 13.57

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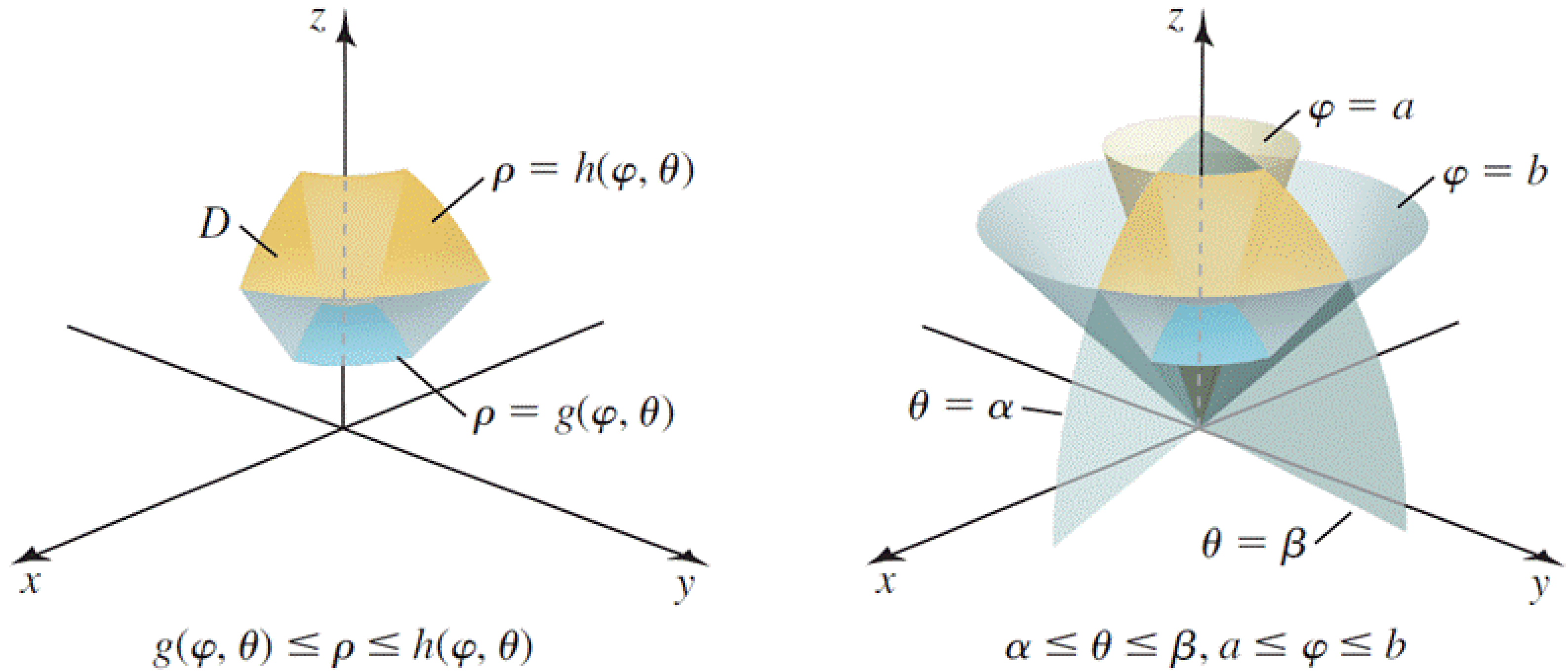

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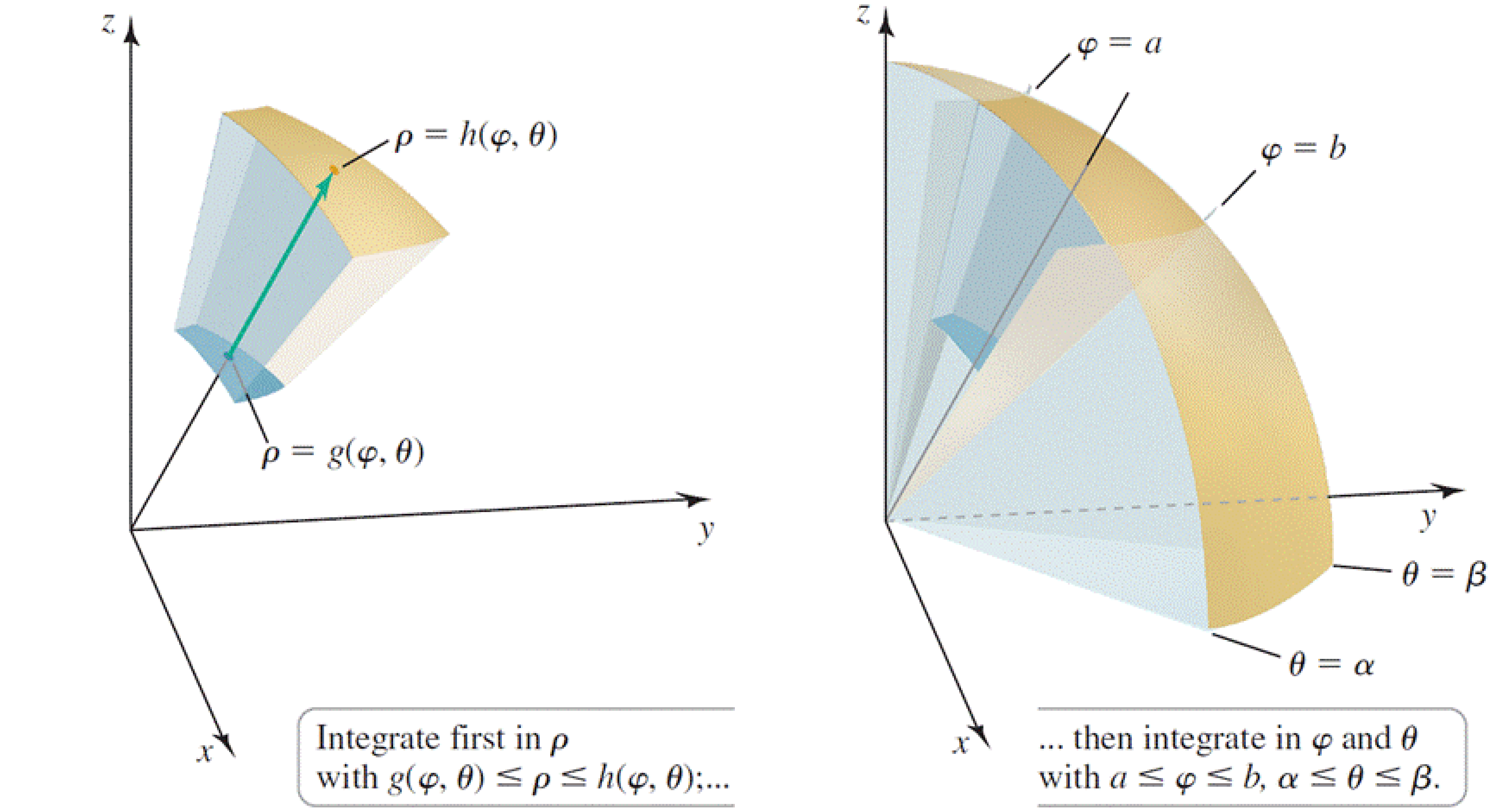

FIGURE 13.59

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THEOREM 13.7 Triple Integrals in Spherical Coordinates

Let f be continuous over the region

$$D = \{(\rho, \varphi, \theta): g(\varphi, \theta) \le \rho \le h(\varphi, \theta), a \le \varphi \le b, \alpha \le \theta \le \beta\}.$$

Then f is integrable over D and the triple integral of f over D in spherical coordinates is

$$\iiint_D f(\rho, \varphi, \theta)\, dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho, \varphi, \theta)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

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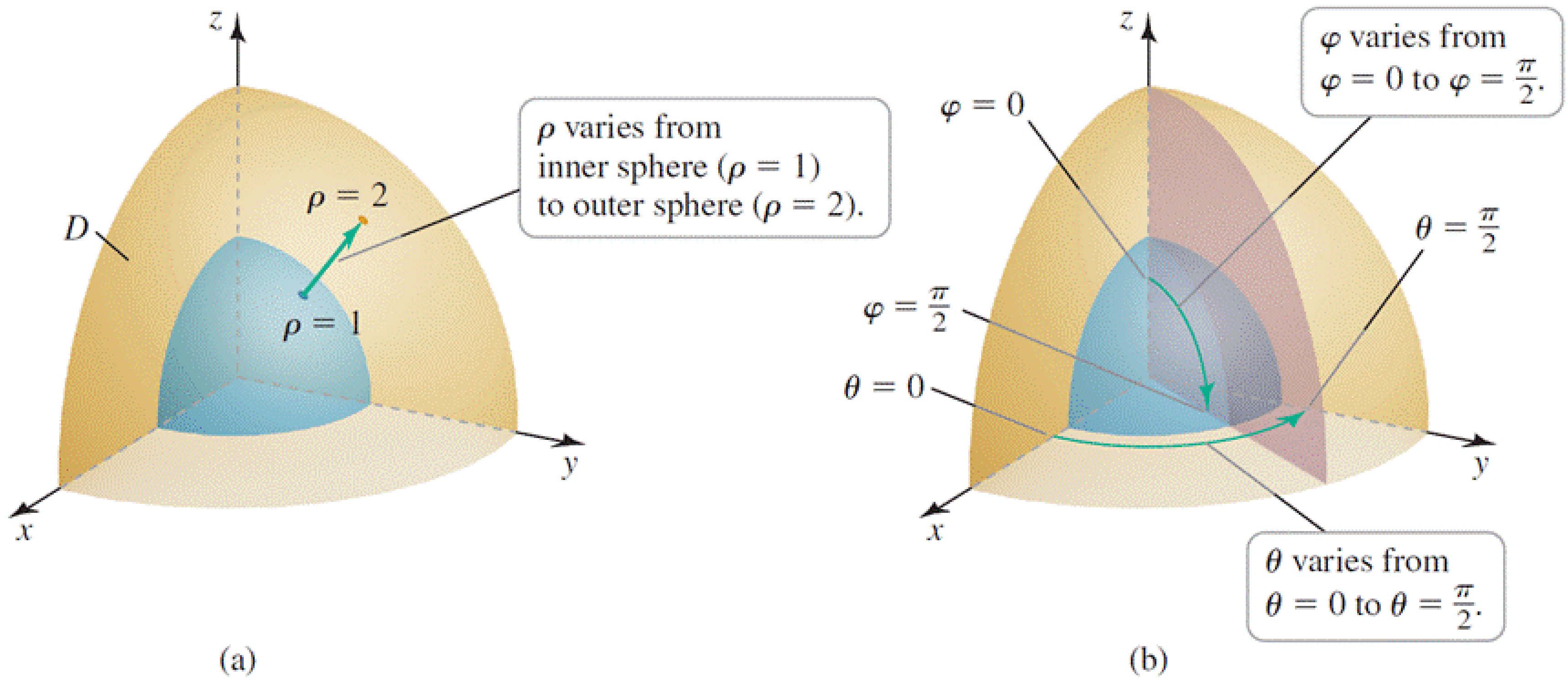

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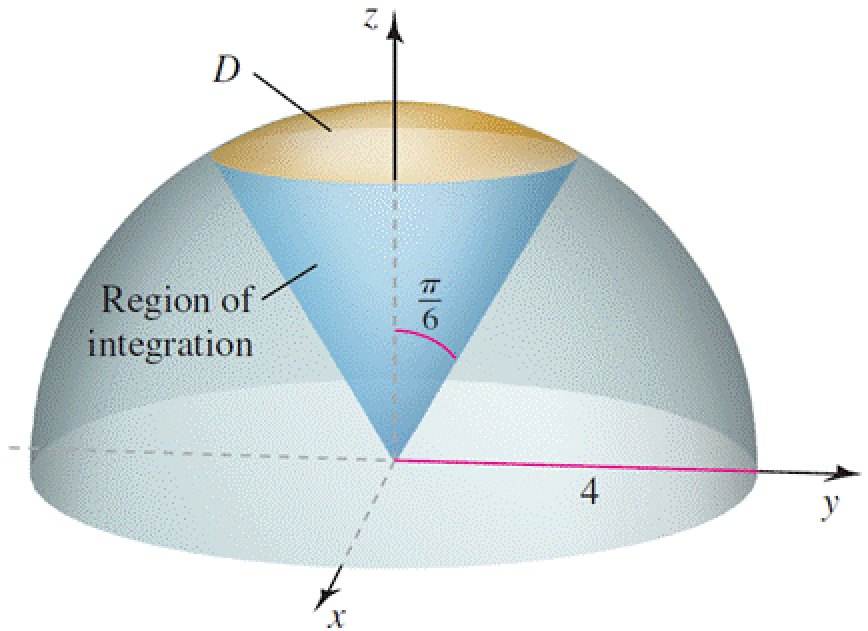

(a)

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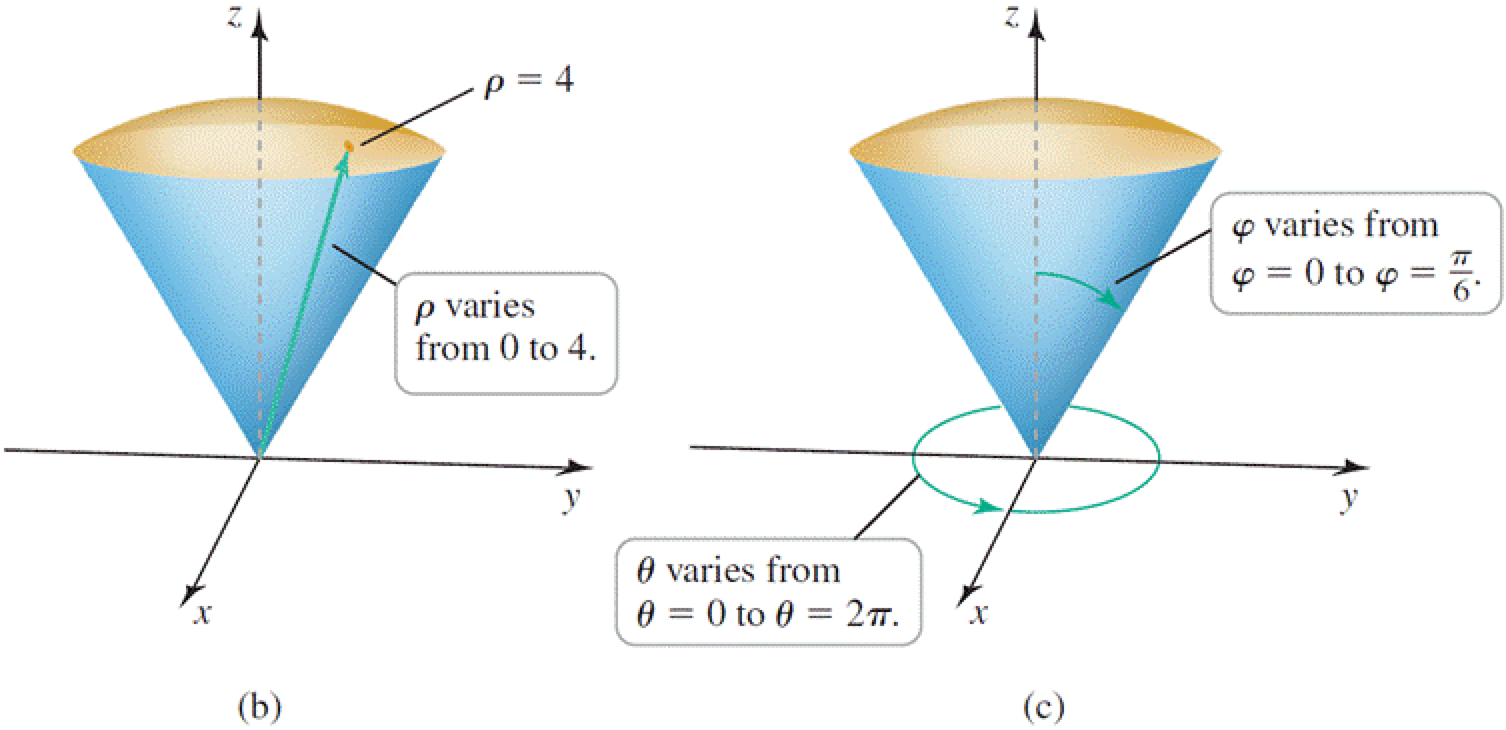

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13.6

Integrals for Mass Calculations

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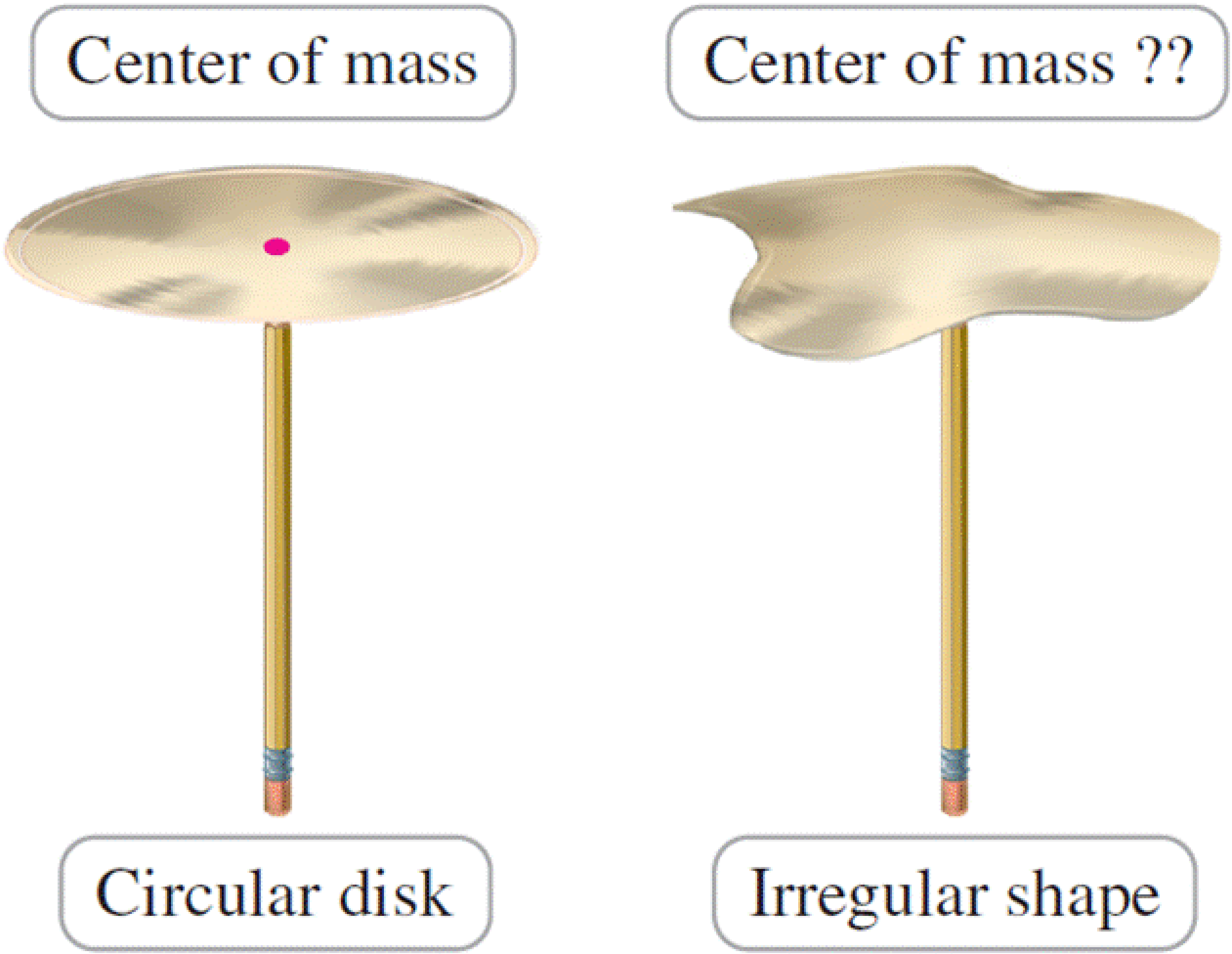

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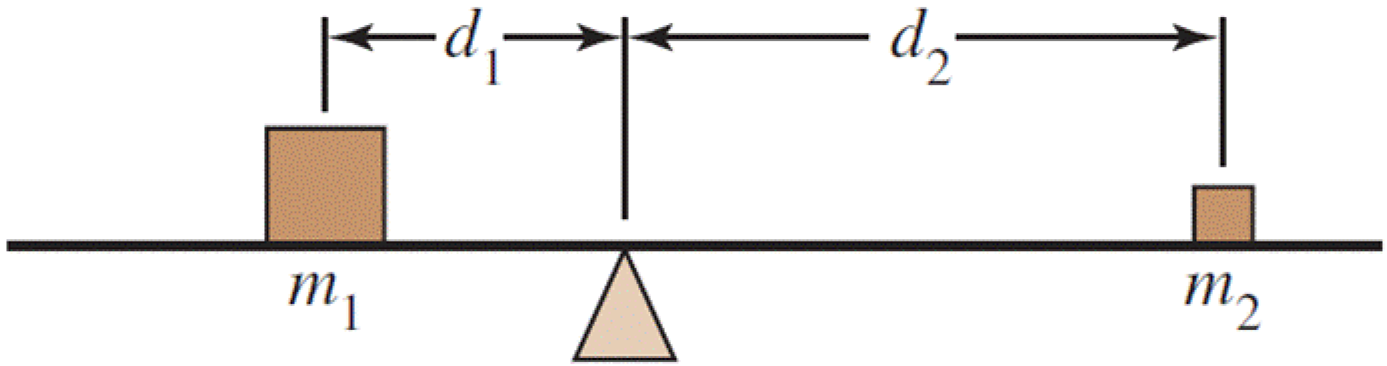

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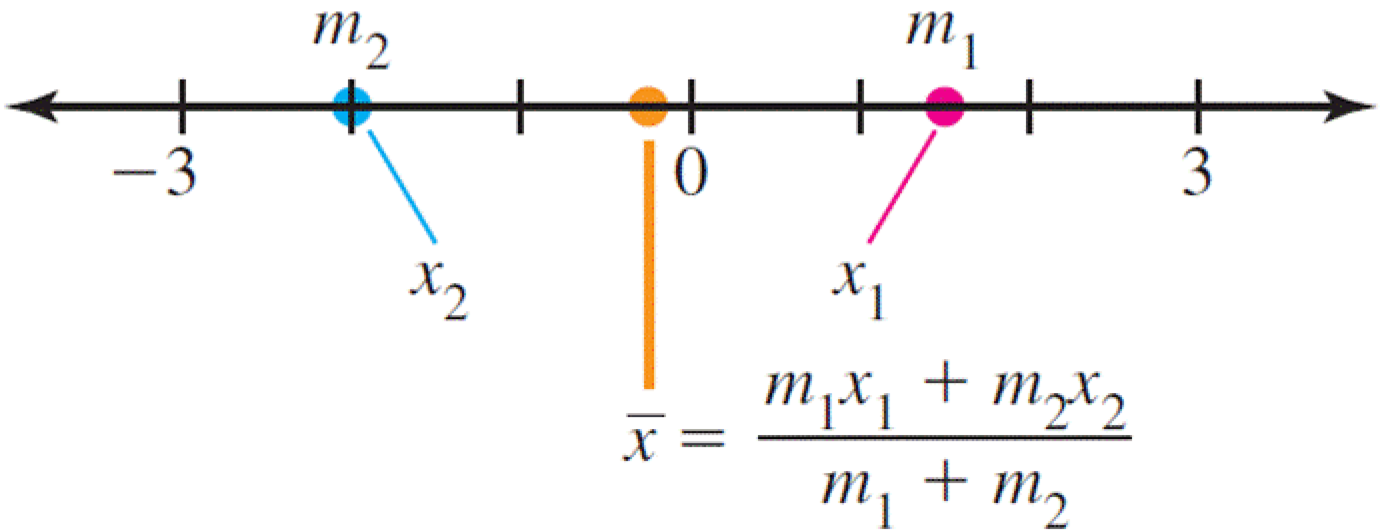

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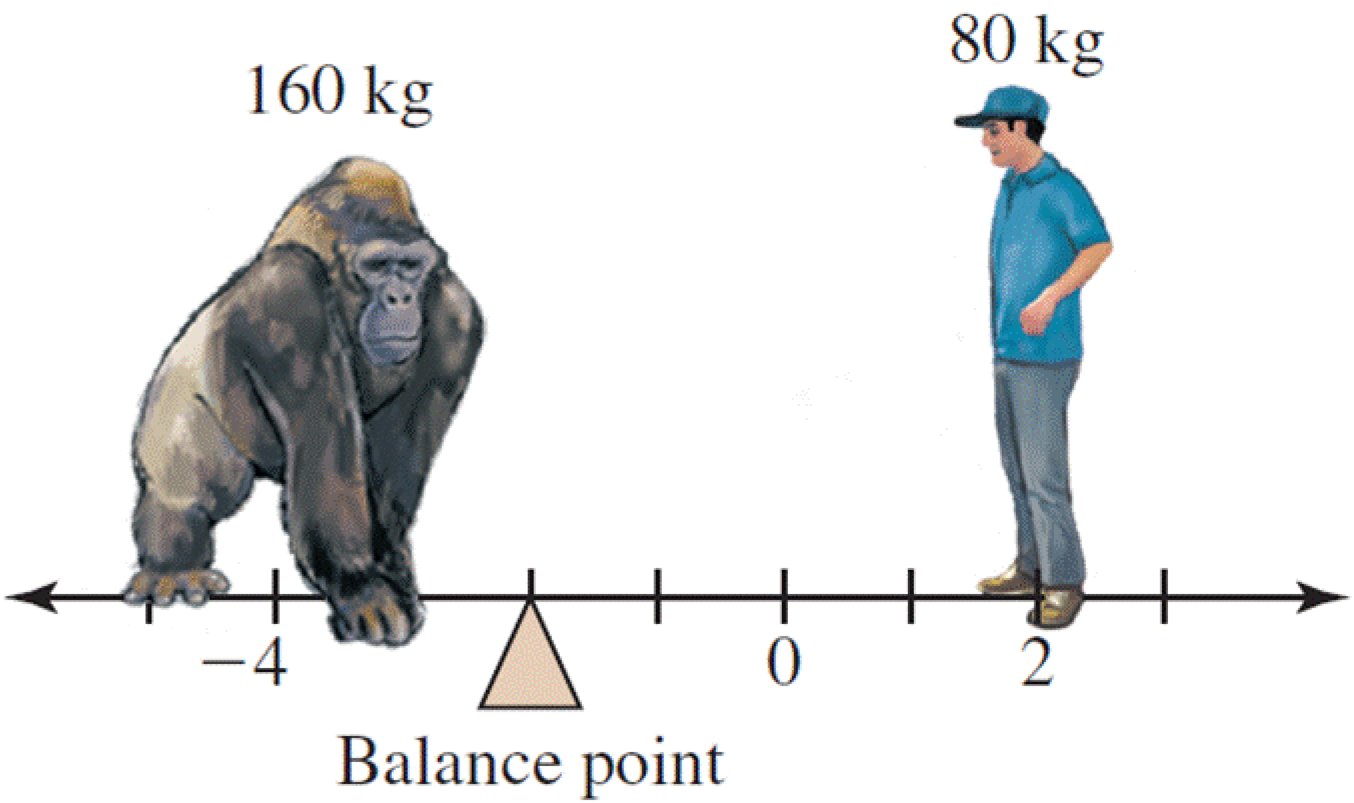

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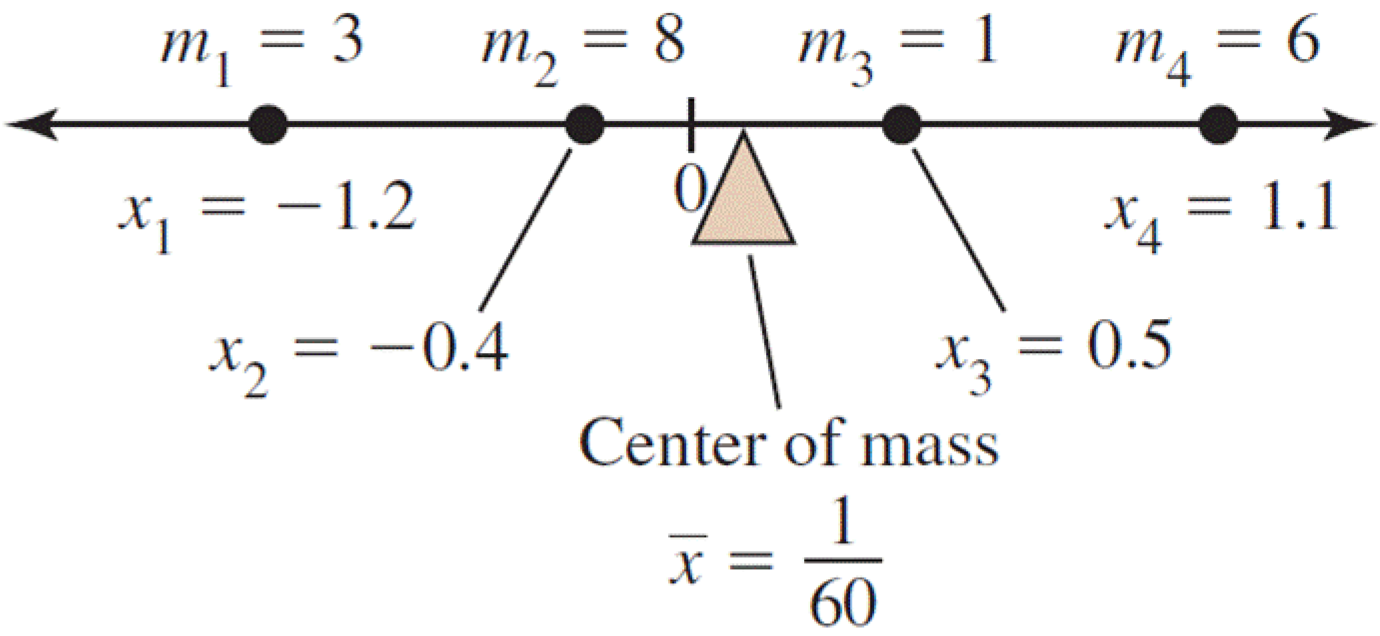

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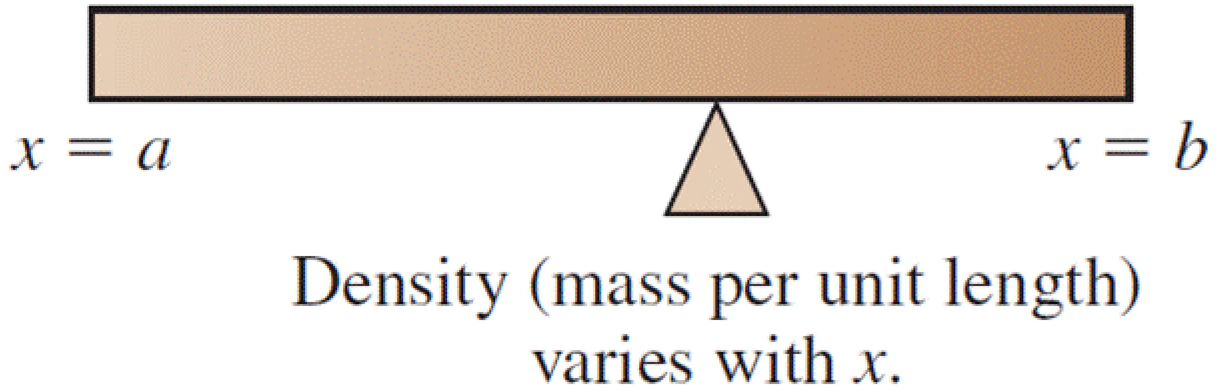

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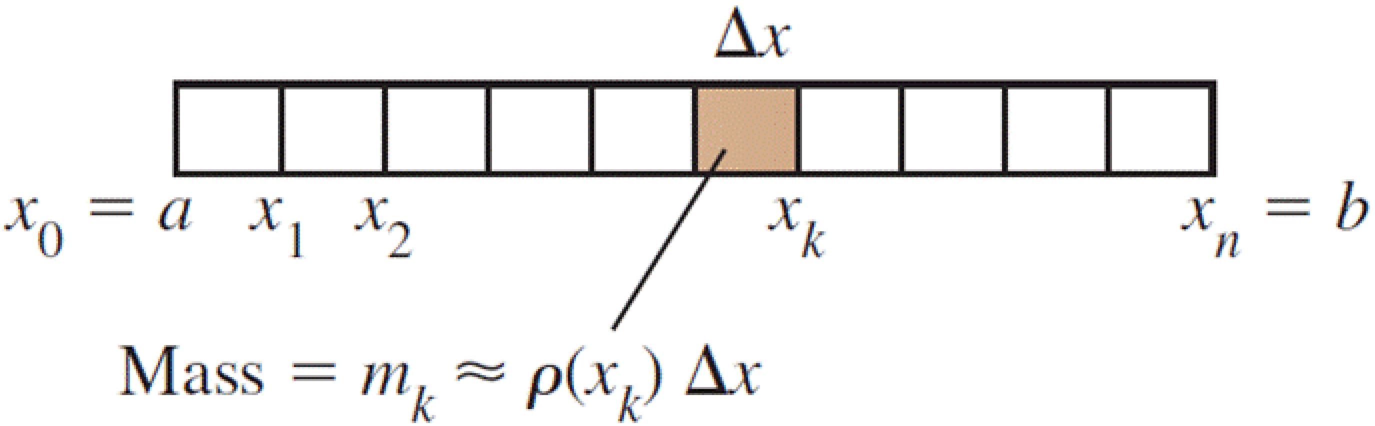

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DEFINITION **Center of Mass in One Dimension**

Let ρ be an integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The center of mass is located at the point $\overline{x} = \frac{M}{m}$, where the total moment M and mass m are

$$M = \int_a^b x\rho(x)\,dx \quad \text{and} \quad m = \int_a^b \rho(x)\,dx.$$

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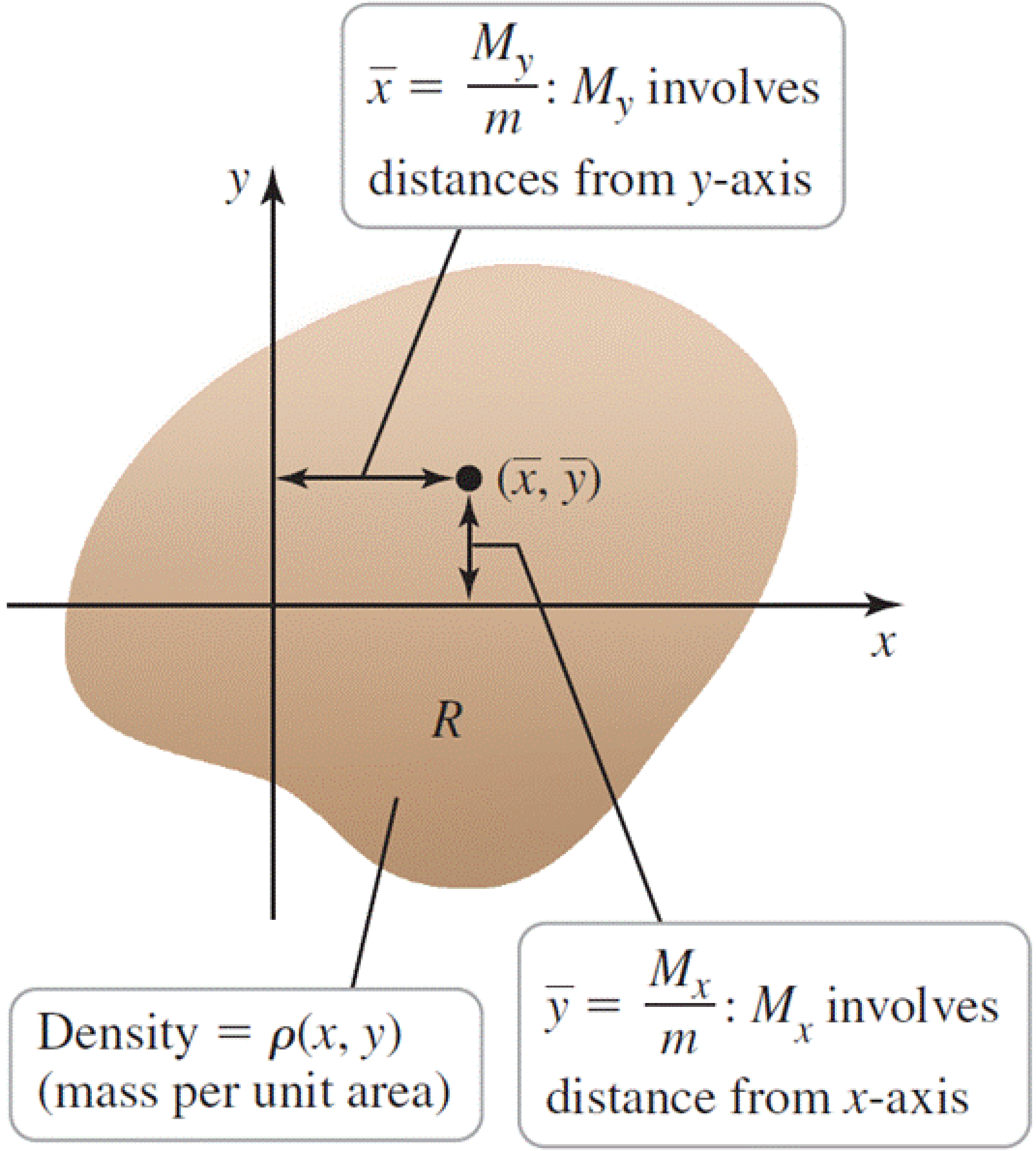

FIGURE 13.69

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DEFINITION Center of Mass in Two Dimensions

Let ρ be an integrable area density function defined over a closed bounded region R in $\mathbf{R}^2$. The coordinates of the **center of mass** of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x, y)\, dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x, y)\, dA,$$

where $m = \iint_R \rho(x, y)\, dA$ is the mass, and M_y and M_x are the moments with respect to the y-axis and x-axis, respectively. If ρ is constant, the center of mass is called the **centroid**.

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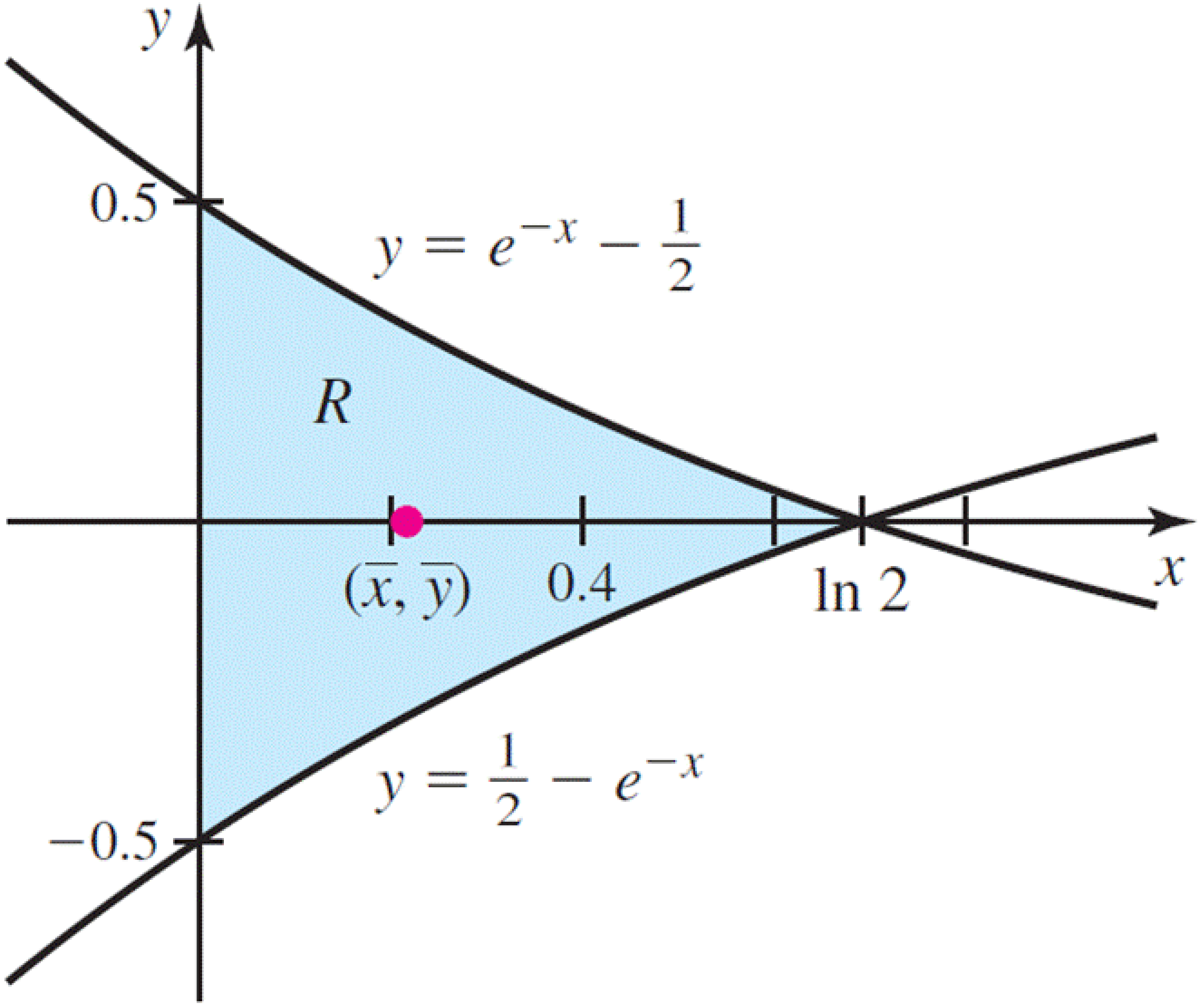

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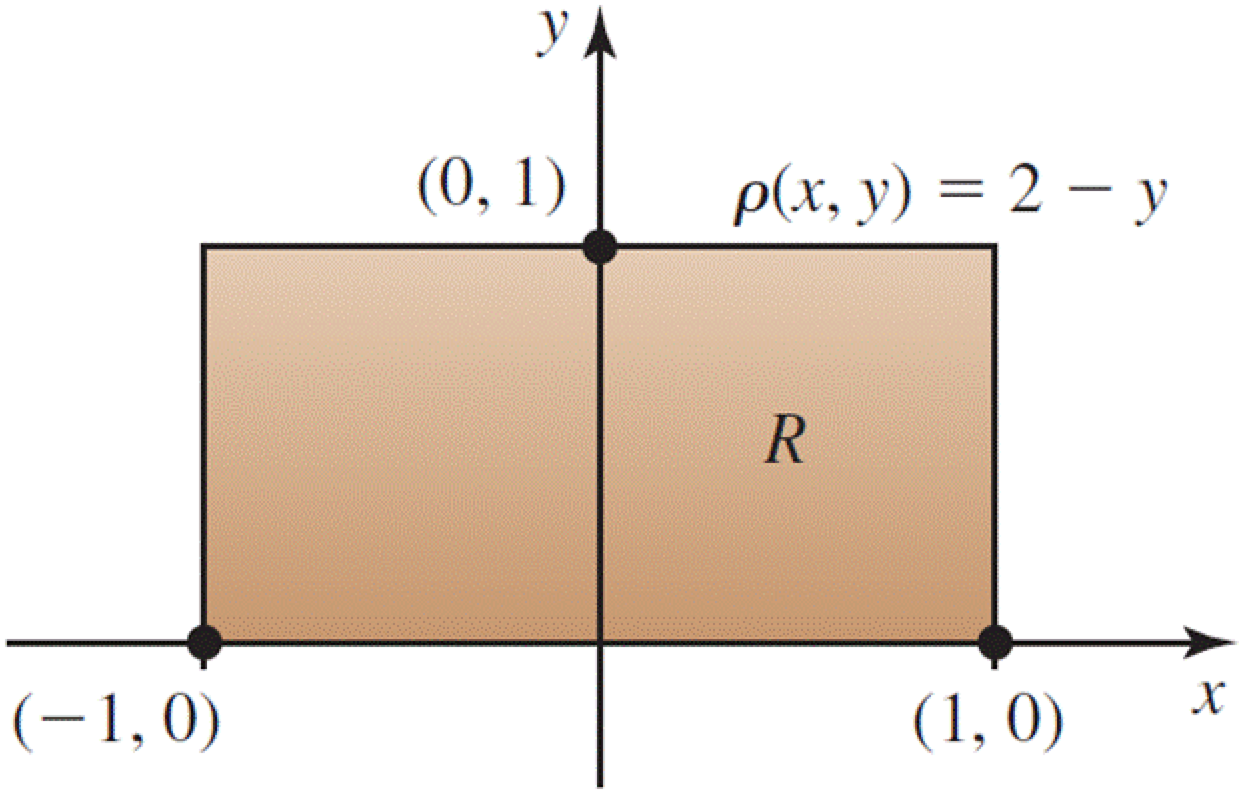

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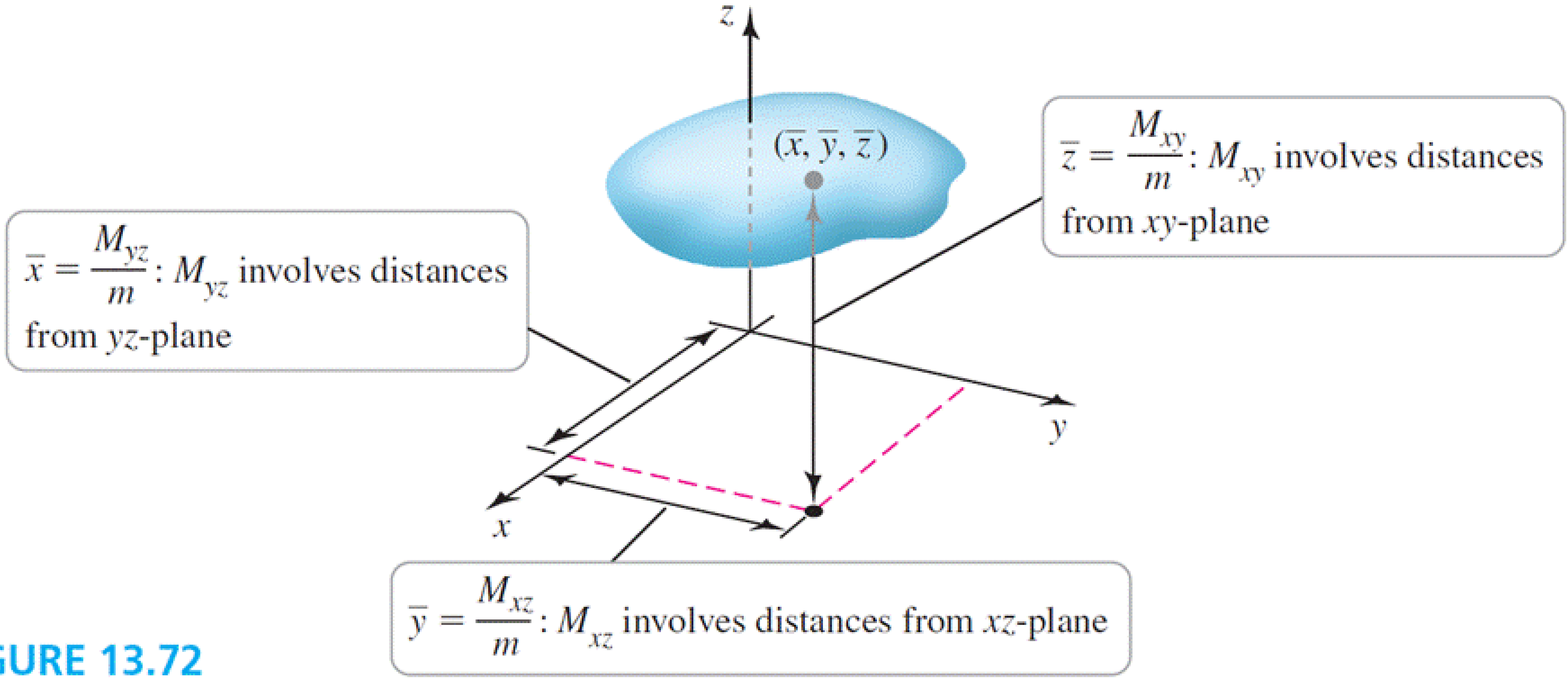

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DEFINITION **Center of Mass in Three Dimensions**

Let ρ be an integrable density function on a closed bounded region D in $\mathbf{R}^3$. The coordinates of the **center of mass** of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m} \iiint_D x\rho(x, y, z)\, dV, \quad \bar{y} = \frac{M_{xz}}{m} = \frac{1}{m} \iiint_D y\rho(x, y, z)\, dV,$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m} \iiint_D z\rho(x, y, z)\, dV,$$

where $m = \iiint_D \rho(x, y, z)\, dV$ is the mass, and M_{yz}, M_{xz}, and M_{xy} are the moments with respect to the coordinate planes.

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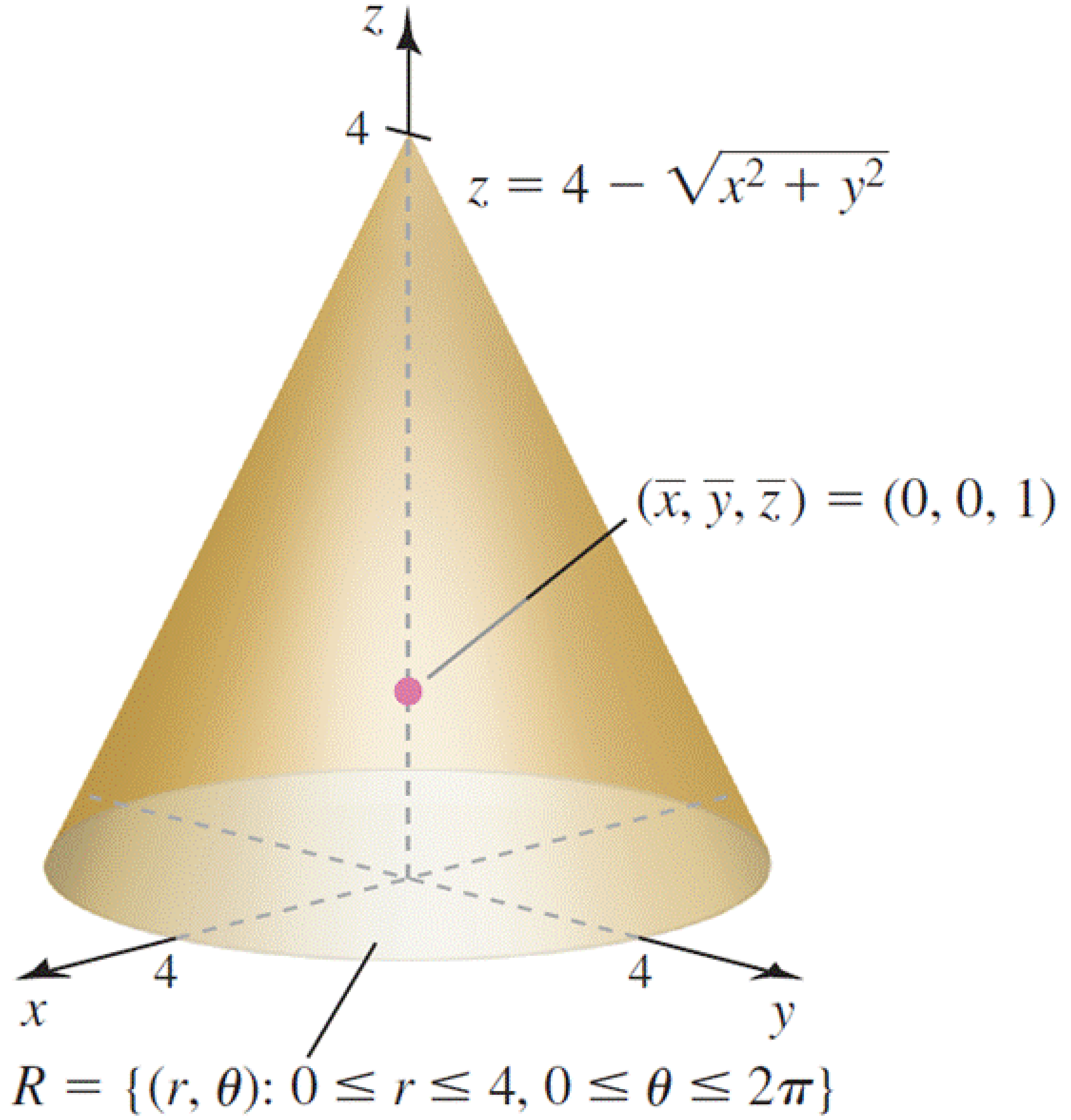

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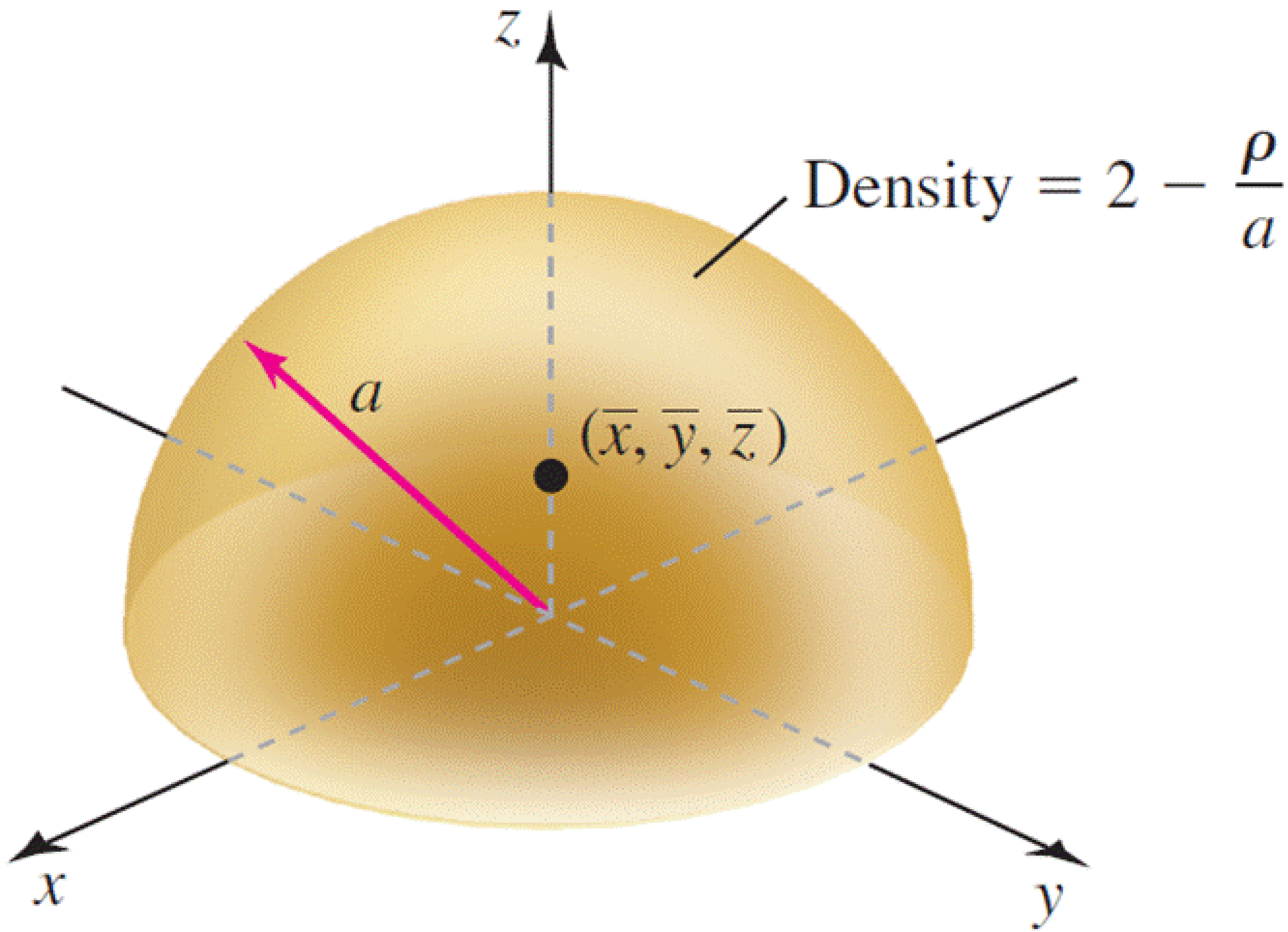

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13.7

Change of Variables in Multiple Integrals

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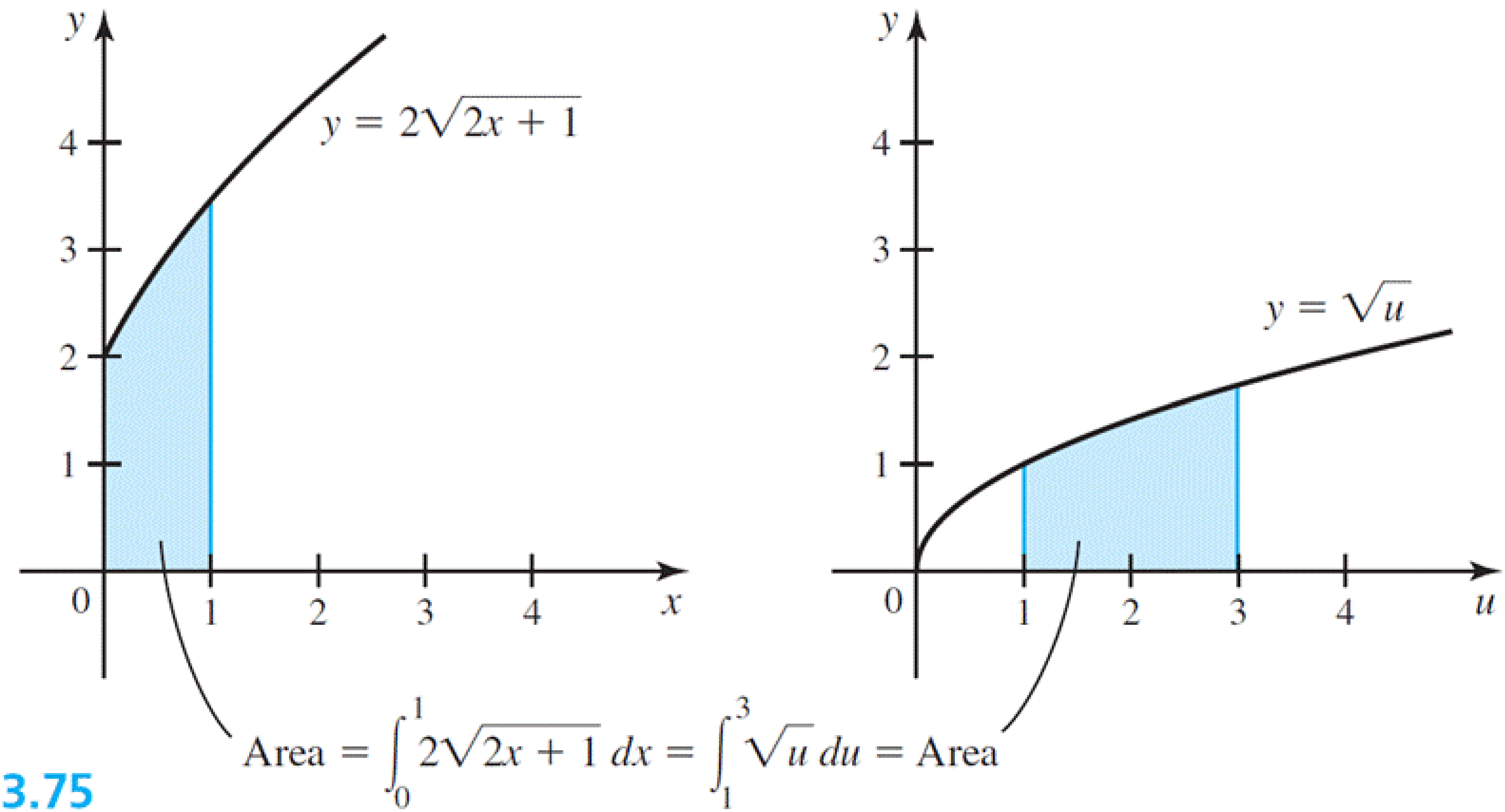

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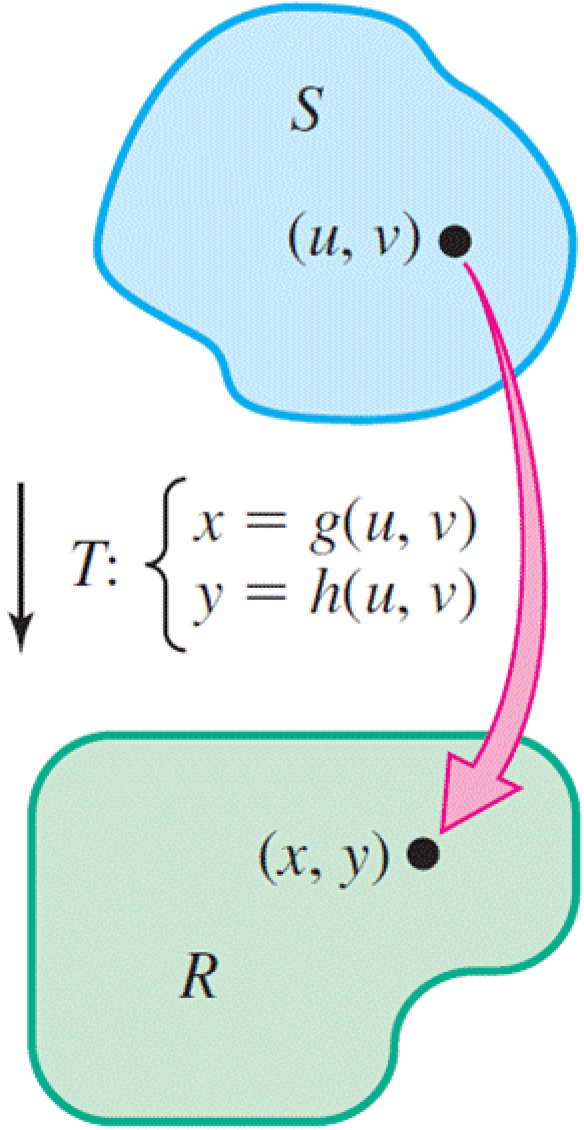

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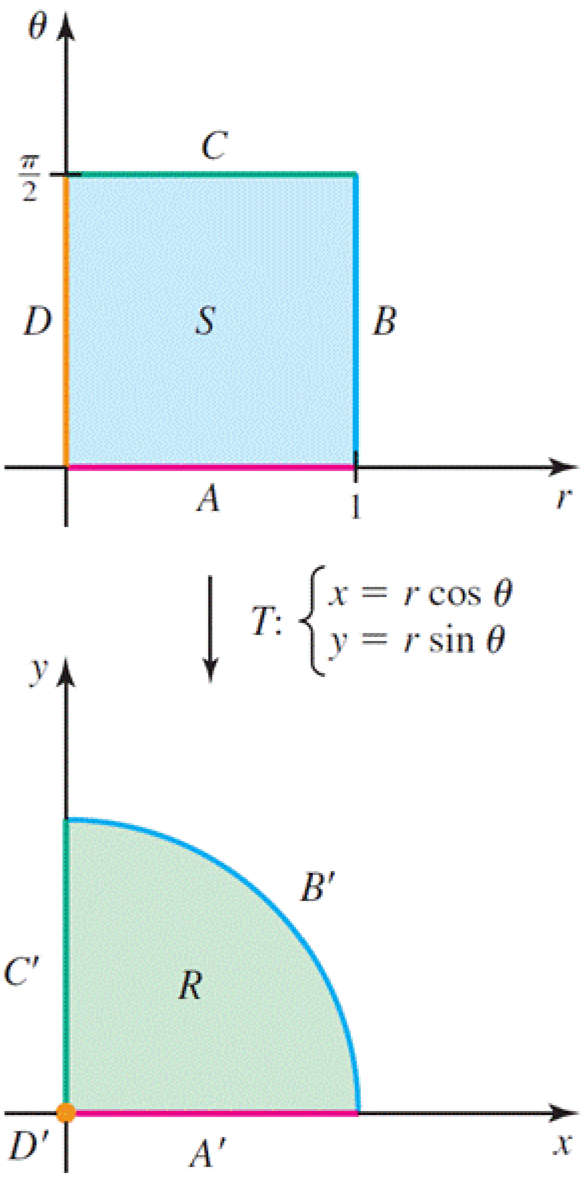

FIGURE 13.77

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Table 13.5

Boundary of S in $r\theta$-plane	Transformation equations	Boundary of R in xy-plane
A: $0 \le r \le 1, \theta = 0$	$x = r\cos\theta = r$, $y = r\sin\theta = 0$	A': $0 \le x \le 1, y = 0$
B: $r = 1, 0 \le \theta \le \pi/2$	$x = r\cos\theta = \cos\theta$, $y = r\sin\theta = \sin\theta$	B': quarter unit circle
C: $0 \le r \le 1, \theta = \pi/2$	$x = r\cos\theta = 0$, $y = r\sin\theta = r$	C': $x = 0, 0 \le y \le 1$
D: $r = 0, 0 \le \theta \le \pi/2$	$x = r\cos\theta = 0$, $y = r\sin\theta = 0$	D': single point (0, 0)

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DEFINITION **One-to-One Transformation**

A transformation T from a region S to a region R is one-to-one on S if $T(P) = T(Q)$ only when $P = Q$, where P and Q are points in S.

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DEFINITION **Jacobian Determinant of a Transformation of Two Variables**

Given a transformation $T: x = g(u, v), y = h(u, v)$, where g and h are differentiable on a region of the uv-plane, the **Jacobian determinant** (or **Jacobian**) of T is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

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THEOREM 13.8 Change of Variables for Double Integrals

Let $T: x = g(u, v), y = h(u, v)$ be a transformation that maps a closed bounded region S in the uv-plane onto a region R in the xy-plane. Assume that T is one-to-one on the interior of S and that g and h have continuous first partial derivatives there. If f is continuous on R, then

$$\iint_R f(x, y)\, dA = \iint_S f(g(u, v), h(u, v))\,|J(u, v)|\, dA.$$

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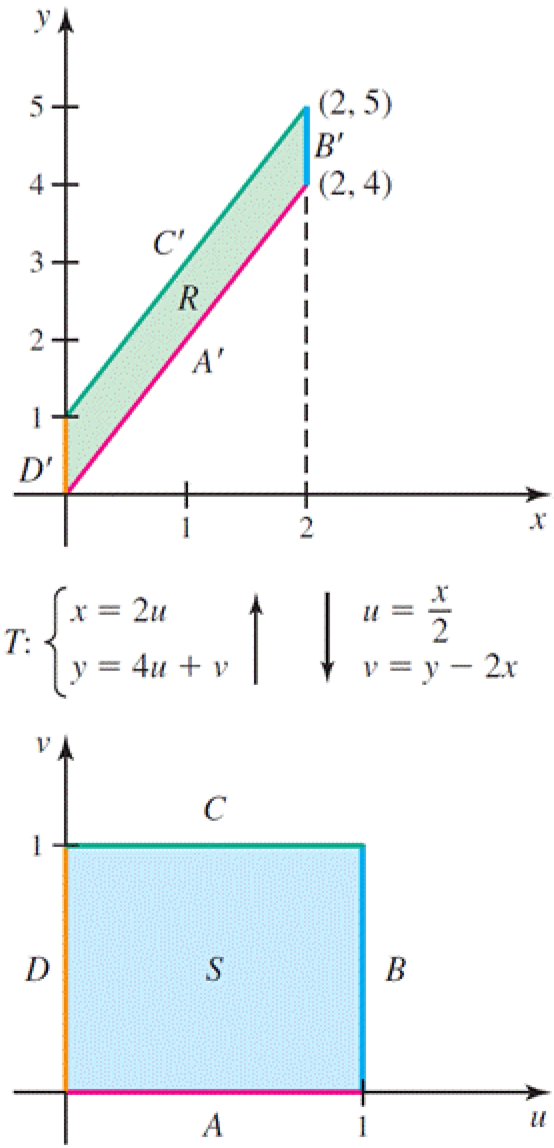

FIGURE 13.78

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Table 13.6

(x, y)	(u, v)
(0, 0)	(0, 0)
(0, 1)	(0, 1)
(2, 5)	(1, 1)
(2, 4)	(1, 0)

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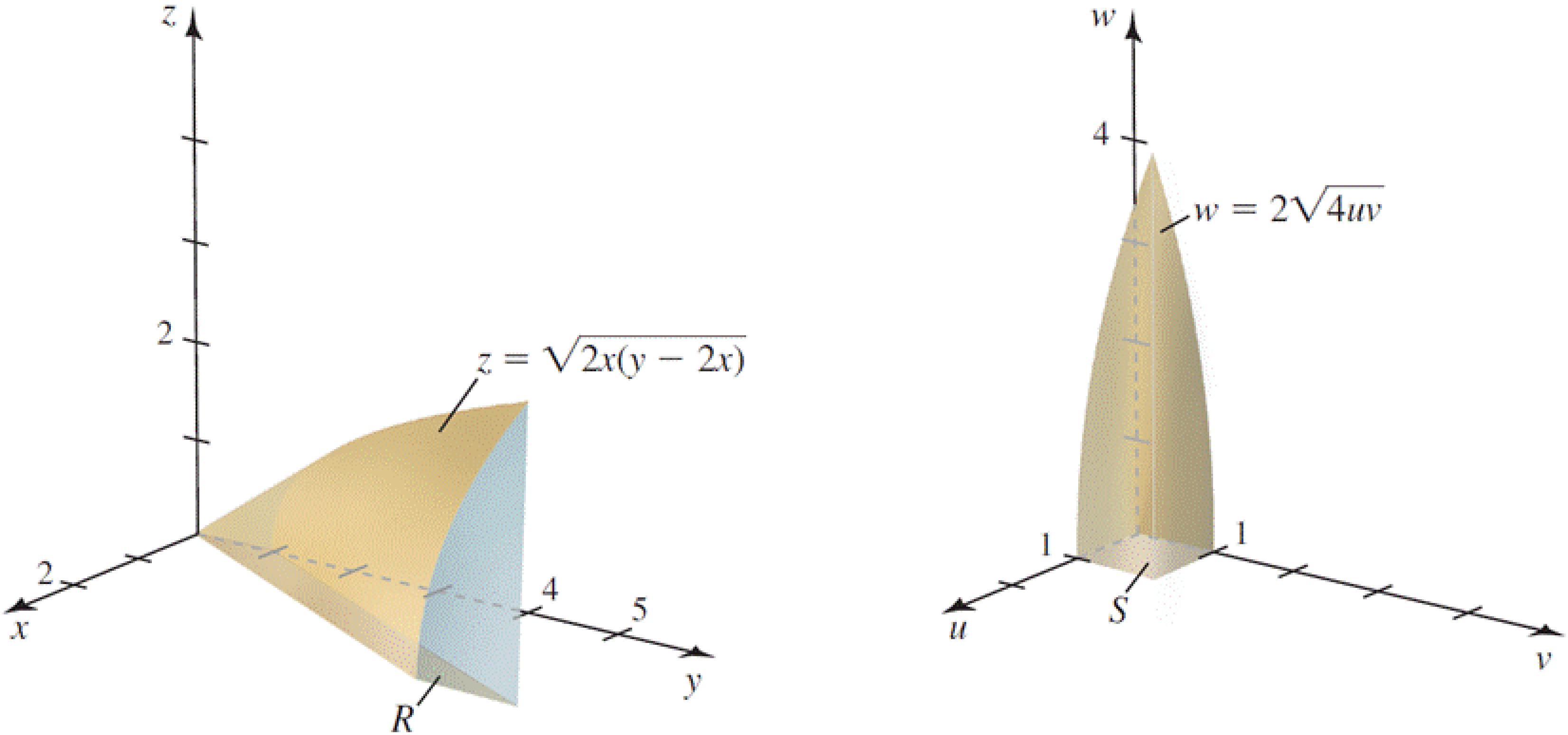

$$\iint_R \sqrt{2x(y-2x)}\, dA = \int_0^1 \int_0^1 2\sqrt{4uv}\, du\, dv$$

FIGURE 13.79

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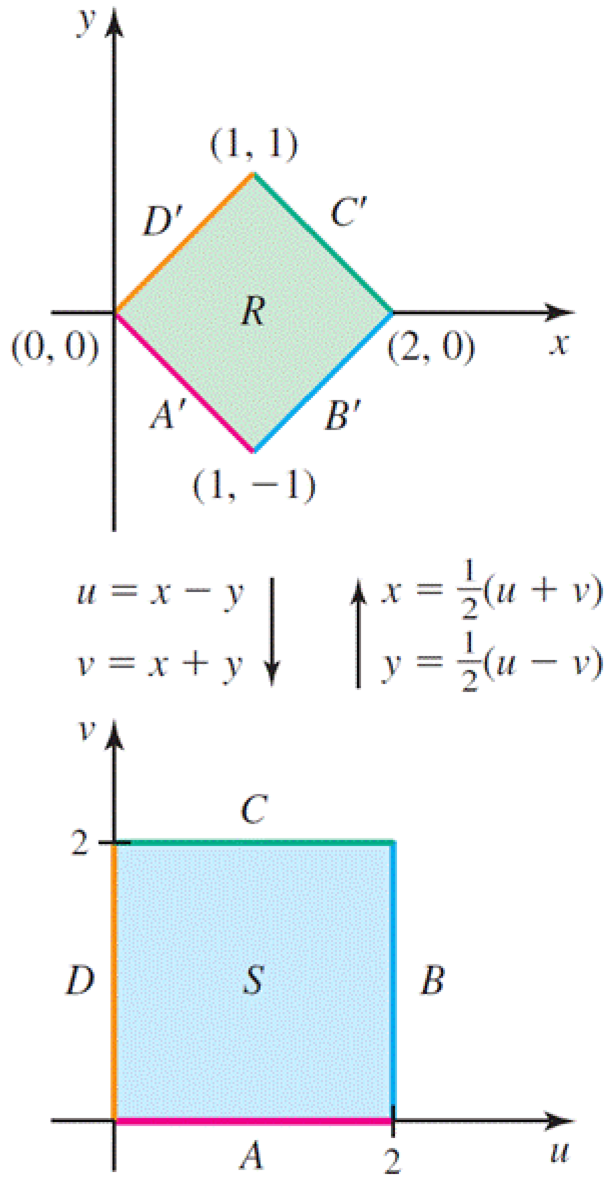

FIGURE 13.80

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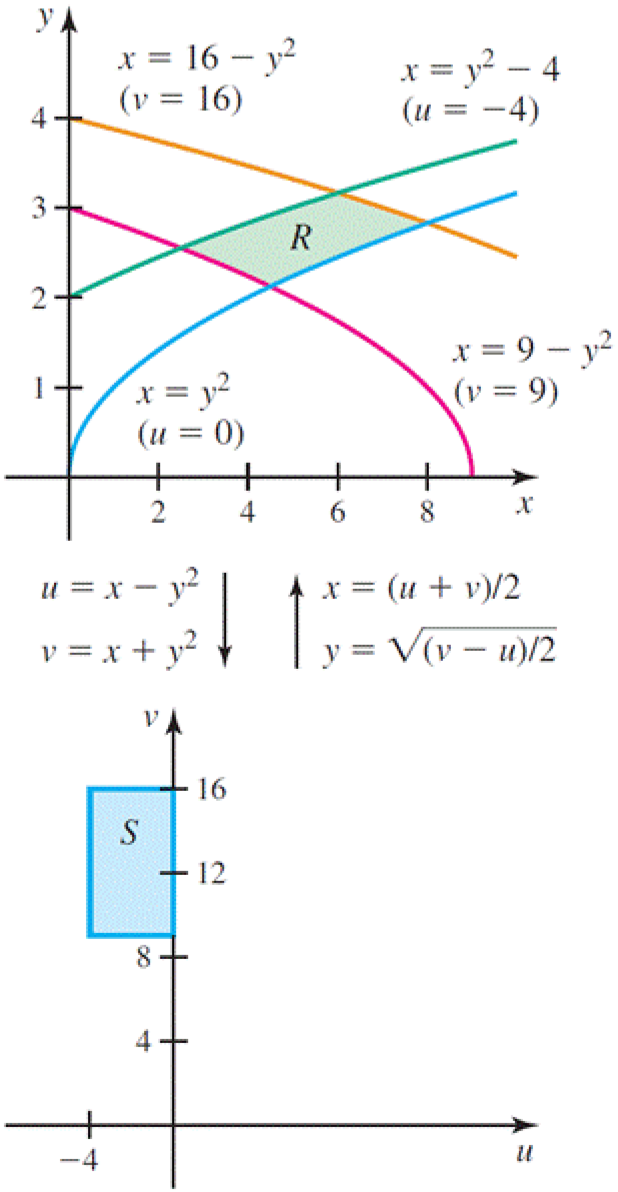

FIGURE 13.81

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DEFINITION Jacobian Determinant of a Transformation of Three Variables

Given a transformation T: $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = p(u, v, w)$, where g, h, and p are differentiable on a region of uvw-space, the **Jacobian determinant** (or **Jacobian**) of T is

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

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THEOREM 13.9 Change of Variables for Triple Integrals

Let $T: x = g(u, v, w)$, $y = h(u, v, w)$, and $z = p(u, v, w)$ be a transformation that maps a closed bounded region S in uvw-space to a region $D = T(S)$ in xyz-space. Assume that T is one-to-one on the interior of S and that g, h, and p have continuous first partial derivatives there. If f is continuous on D, then

$$\iiint_D f(x, y, z)\, dV$$

$$= \iiint_S f(g(u, v, w), h(u, v, w), p(u, v, w))\, |J(u, v, w)|\, dV.$$

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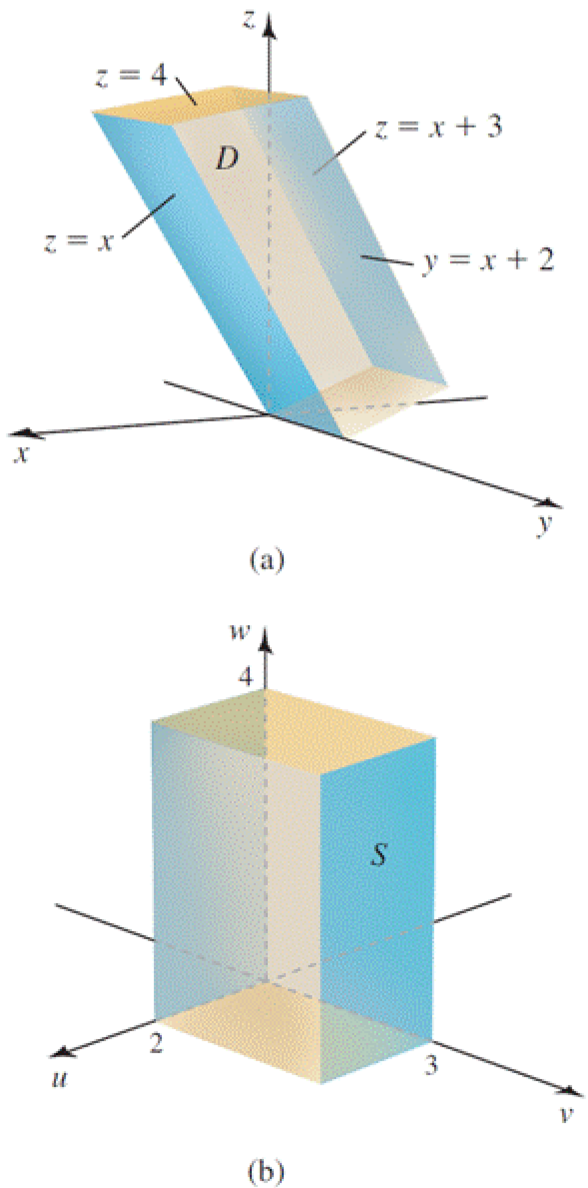

FIGURE 13.82

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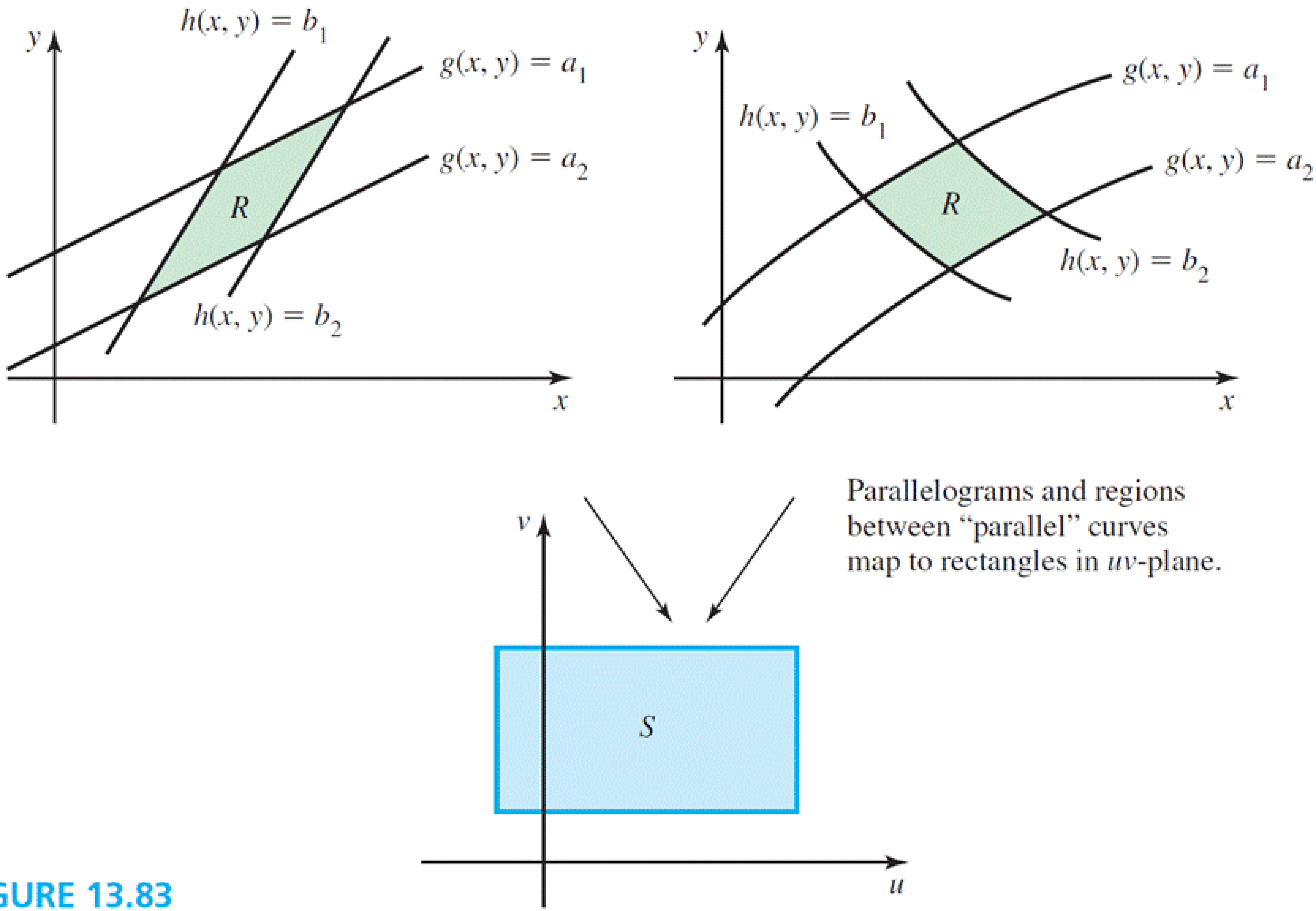

FIGURE 13.83

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