Chapter 10

Parametric and Polar Curves

10.2

Polar Coordinates

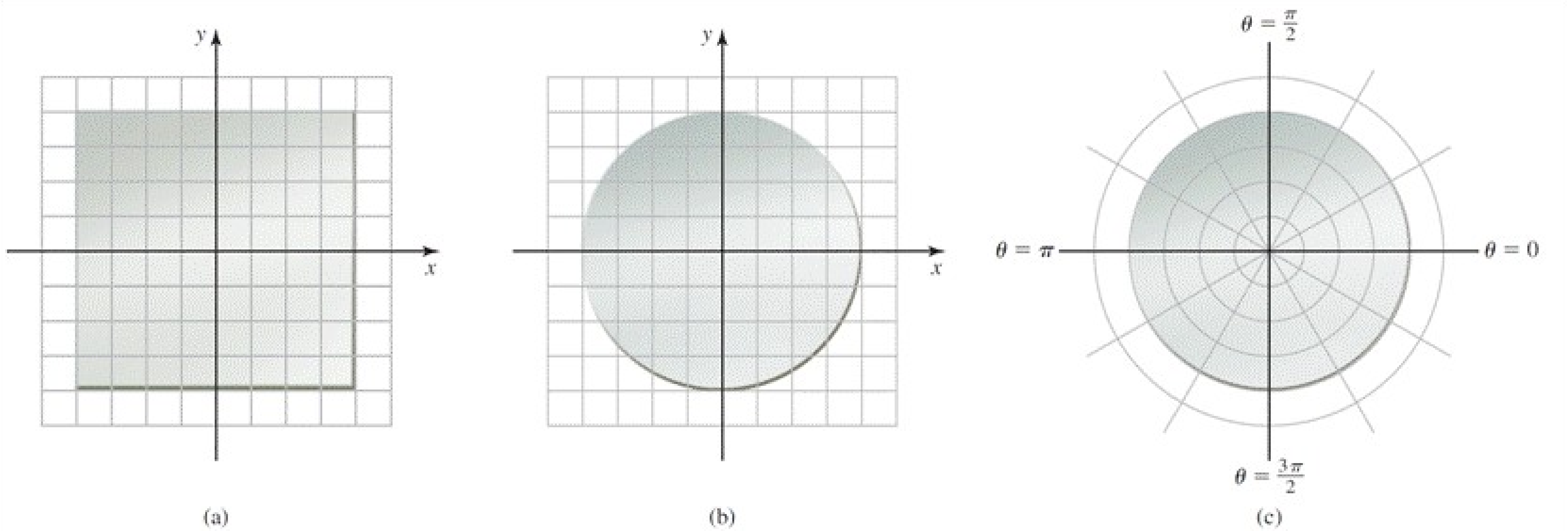


FIGURE 10.17

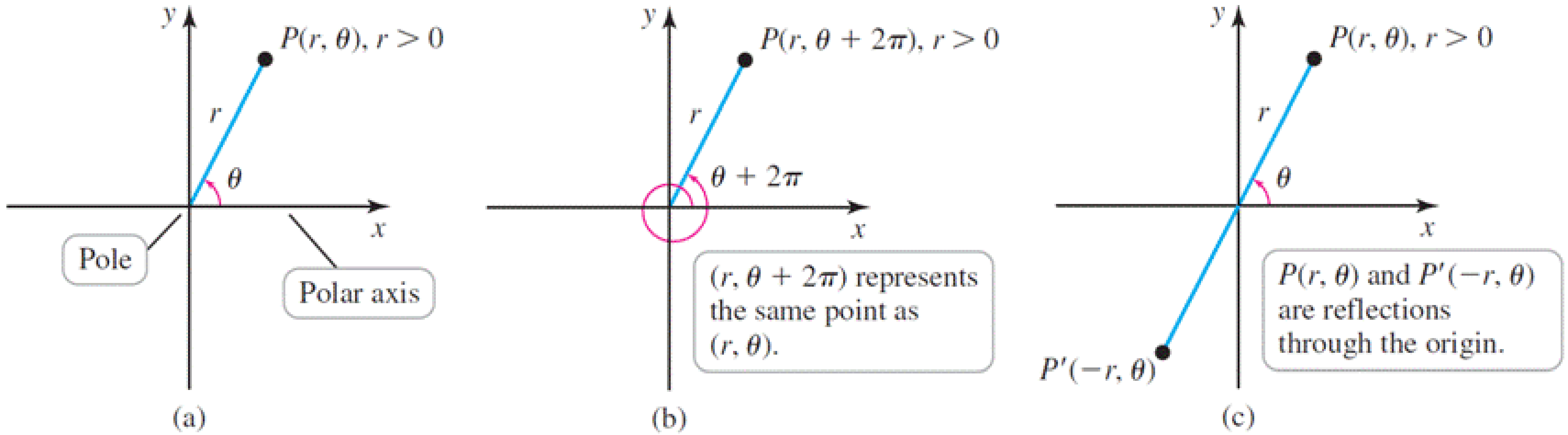
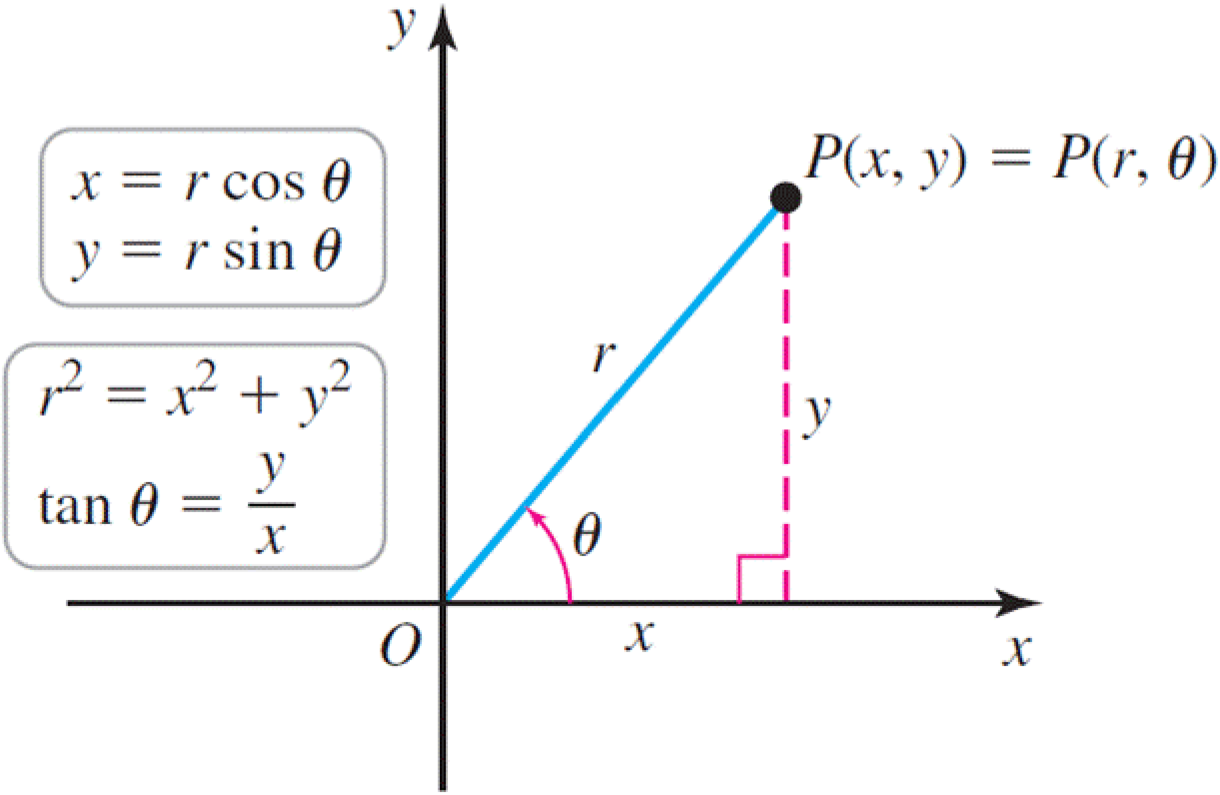
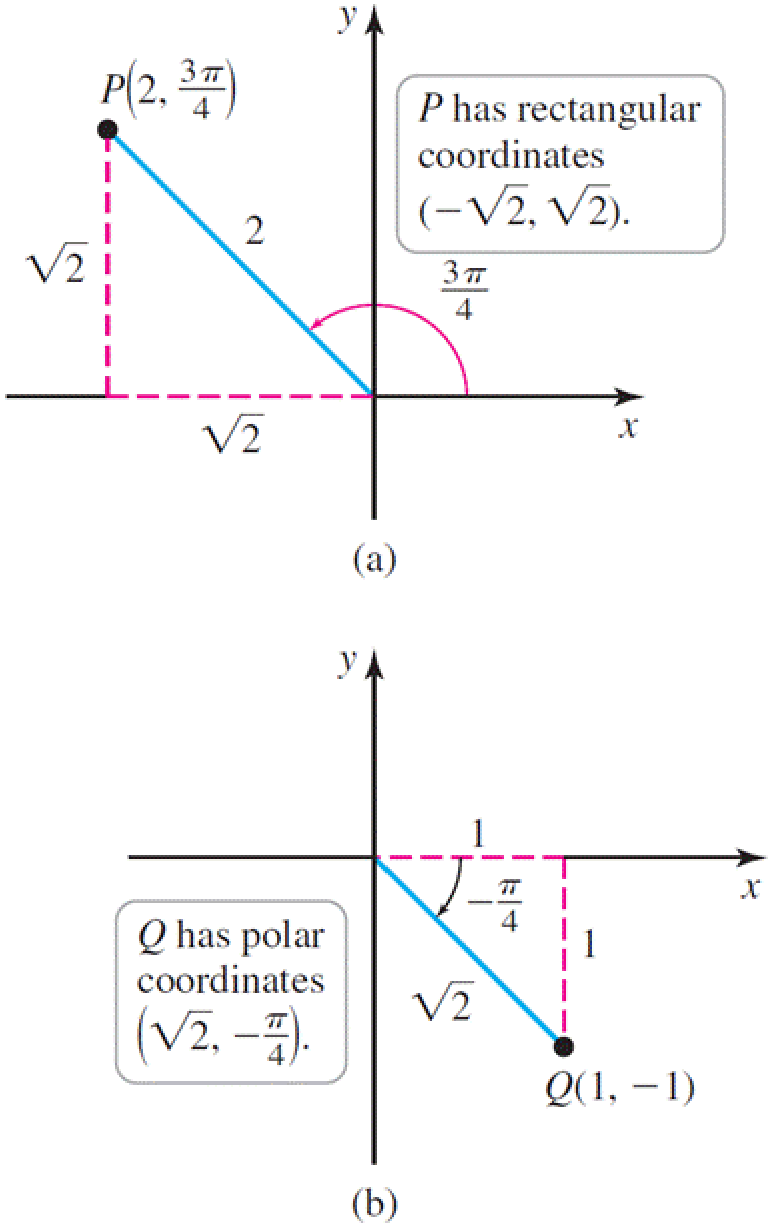

y
$P(r, \theta), r > 0$
r
θ
x
Pole
Polar axis
(a)
y
$P(r, \theta + 2\pi), r > 0$
r
$\theta + 2\pi$
x
$(r, \theta + 2\pi)$ represents the same point as (r, θ).
(b)
y
$P(r, \theta), r > 0$
r
θ
x
$P'(-r, \theta)$
$P(r, \theta)$ and $P'(-r, \theta)$ are reflections through the origin.
(c)

FIGURE 10.18



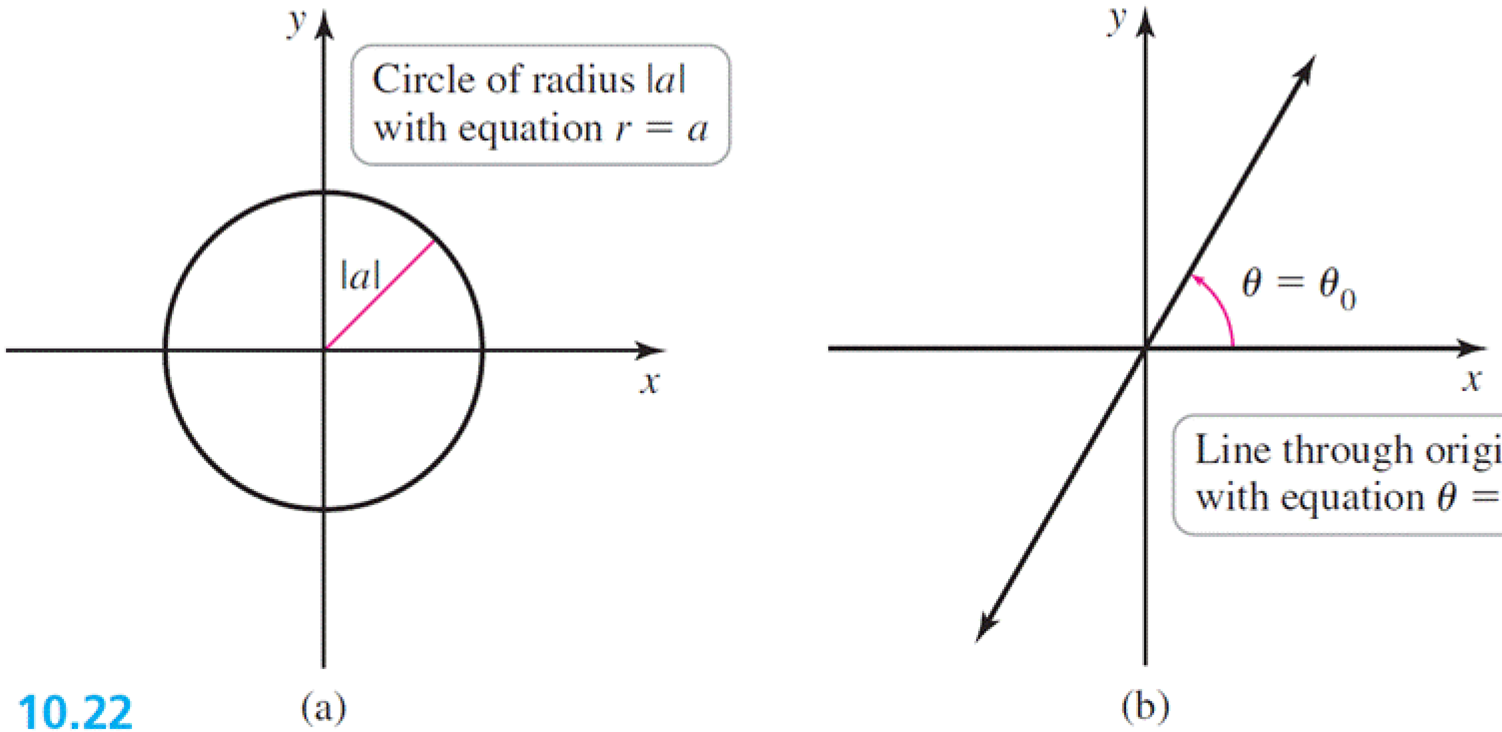
$x = r\cos\theta$
$y = r\sin\theta$
$r^2 = x^2 + y^2$
$\tan\theta = \frac{y}{x}$
y
$P(x, y) = P(r, \theta)$
r
y
θ
O
x
x

FIGURE 10.20



y
$P\left(2, \frac{3\pi}{4}\right)$
P has rectangular coordinates $(-\sqrt{2}, \sqrt{2})$.
2
$\sqrt{2}$
$\frac{3\pi}{4}$
$\sqrt{2}$
x
(a)
y
1
x
$-\frac{\pi}{4}$
Q has polar coordinates $\left(\sqrt{2}, -\frac{\pi}{4}\right)$.
1
$\sqrt{2}$
$Q(1, -1)$
(b)

FIGURE 10.21



y
Circle of radius |a|
with equation r = a
|a|
x
(a)
y
θ = θ0
x
Line through origin
with equation θ = θ0.
(b)

FIGURE 10.22

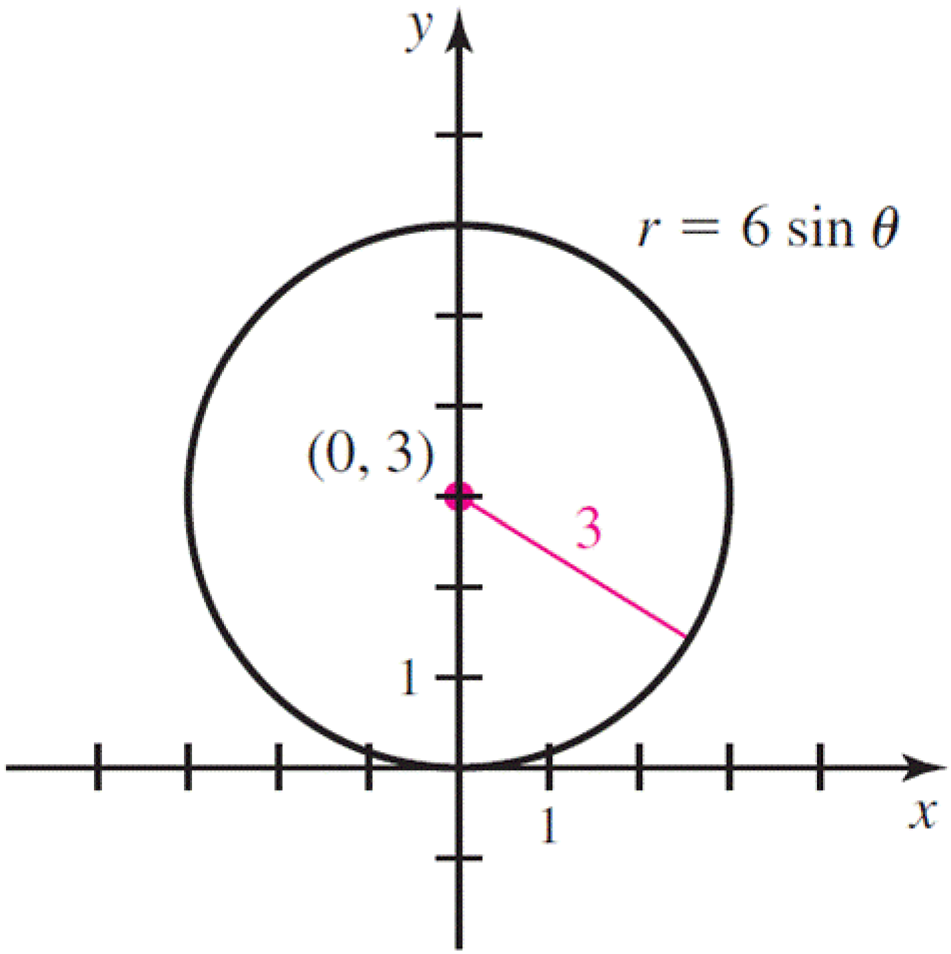
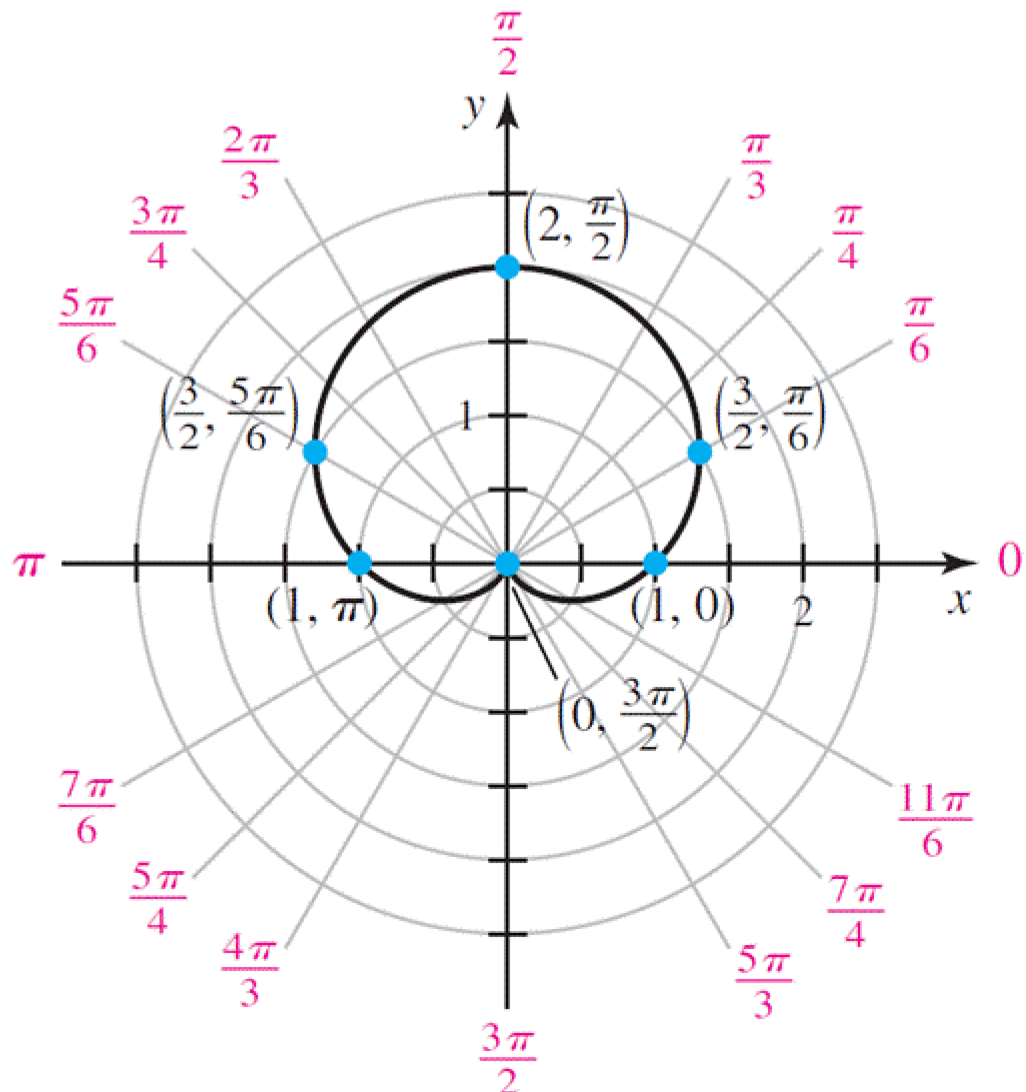


FIGURE 10.24



Cardioid $r = 1 + \sin\theta$

FIGURE 10.25

Table 10.3

θ	$r = 1 + \sin\theta$
0	1
$\pi/6$	$3/2$
$\pi/2$	2
$5\pi/6$	$3/2$
π	1
$7\pi/6$	$1/2$
$3\pi/2$	0
$11\pi/6$	$1/2$
2π	1

13.5

Triple Integrals in Cylindrical and Spherical Coordinates

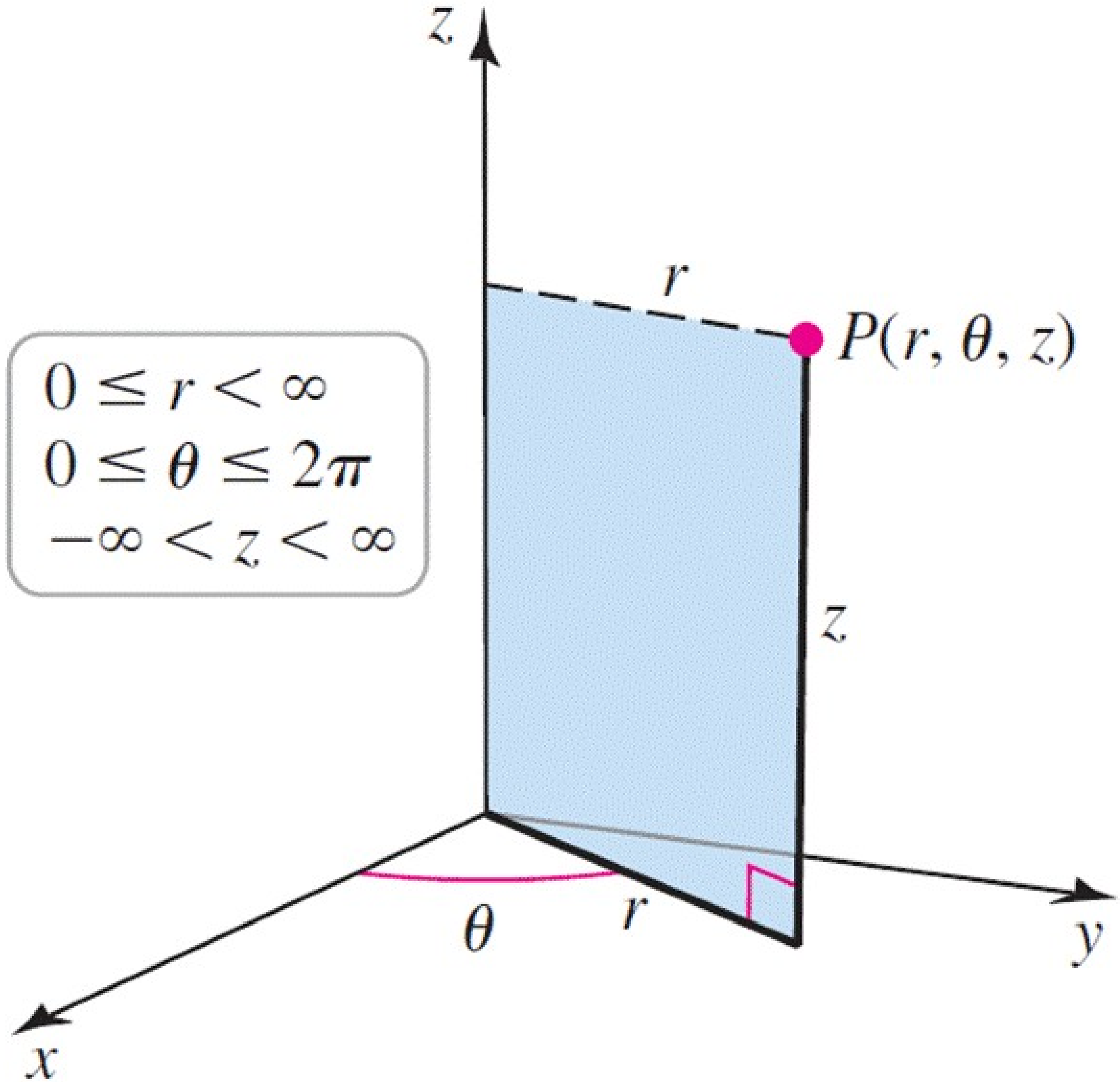


FIGURE 13.46

Table 13.3

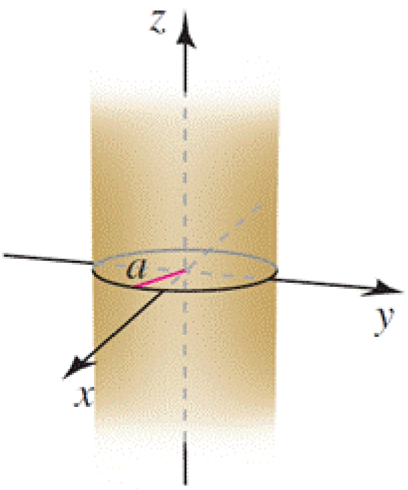
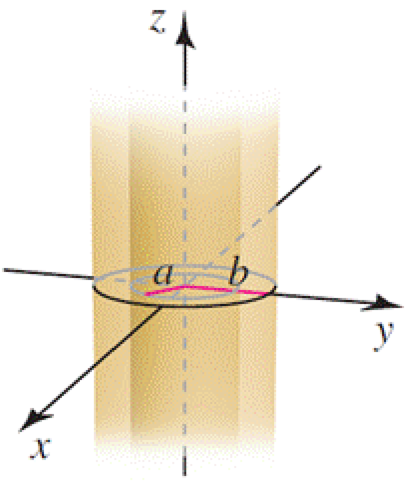
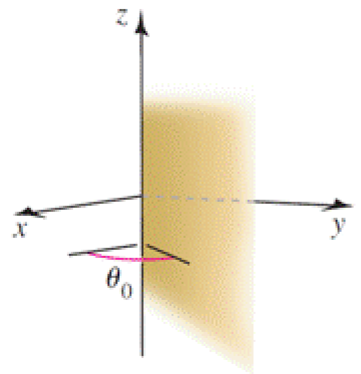
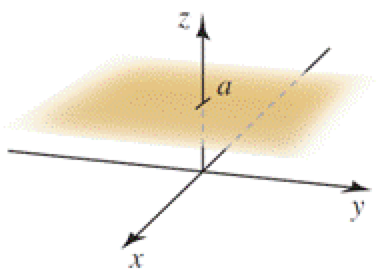
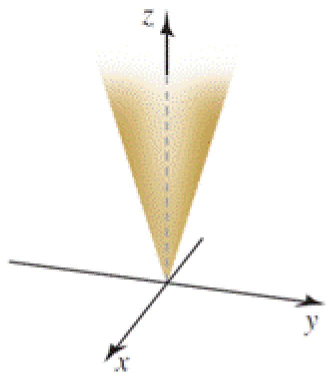
Name	Description	Example
Cylinder	$\{(r, \theta, z): r = a\}, a > 0$	
Cylindrical shell	$\{(r, \theta, z): 0 < a \le r \le b\}$	

Table 13.3 (Continued)

Name	Description	Example
Vertical half plane	$\{(r, \theta, z): \theta = \theta_0\}$	
Horizontal plane	$\{(r, \theta, z): z = a\}$	
Cone	$\{(r, \theta, z): z = ar\}, a \neq 0$	

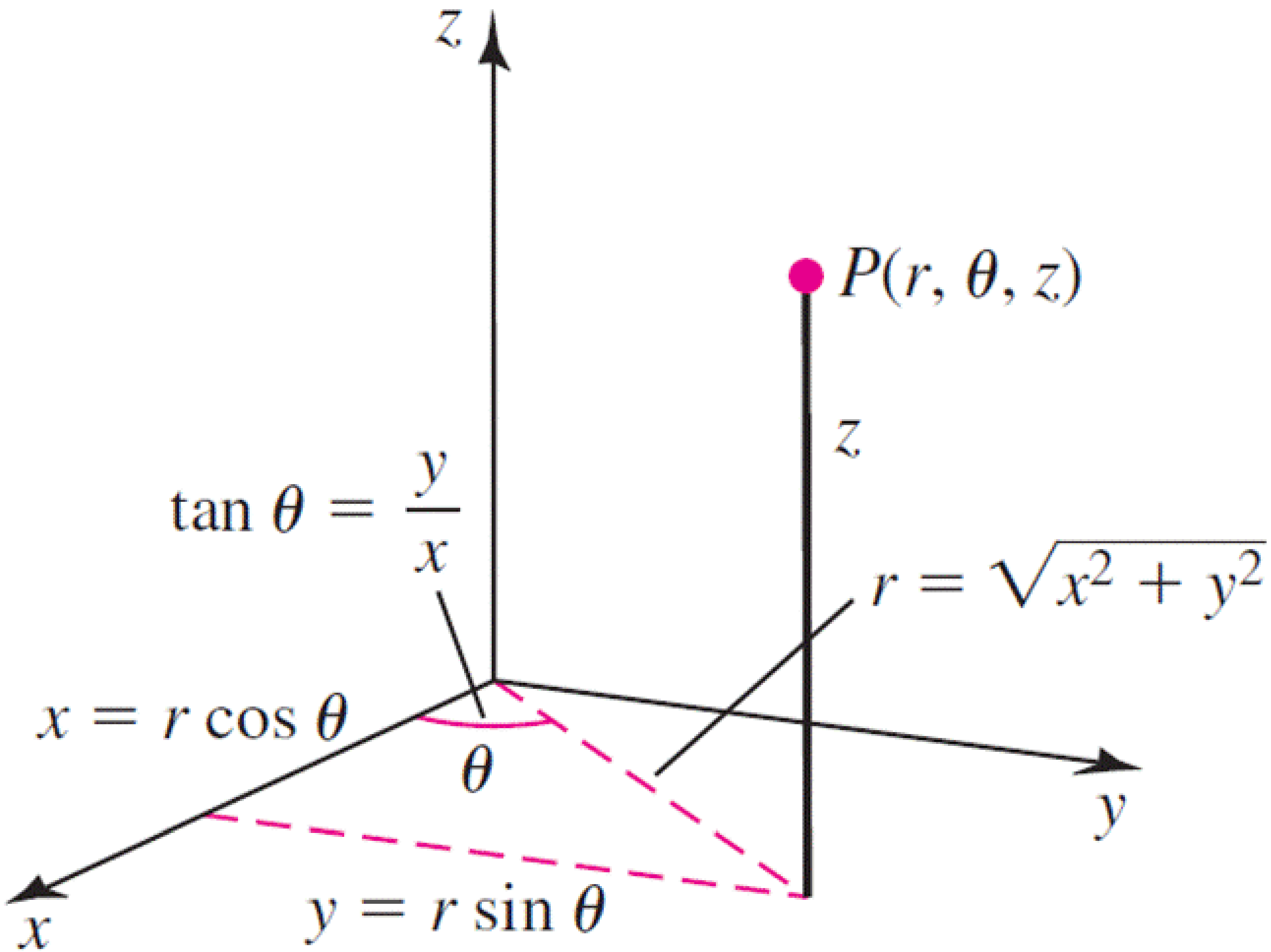
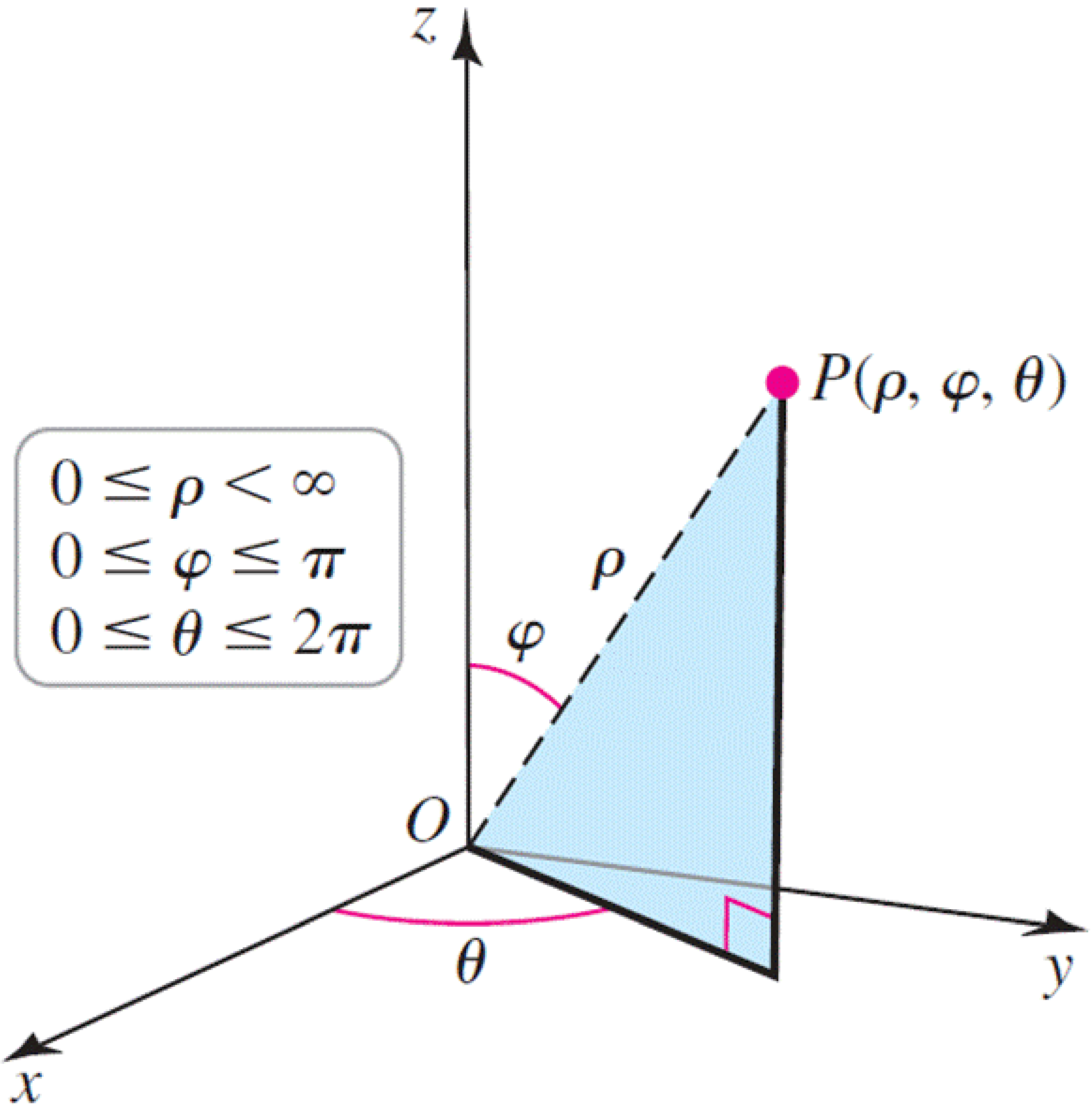


FIGURE 13.48



z
$P(\rho, \varphi, \theta)$
$0 \le \rho < \infty$
$0 \le \varphi \le \pi$
$0 \le \theta \le 2\pi$
ρ
φ
O
θ
y
x

FIGURE 13.54

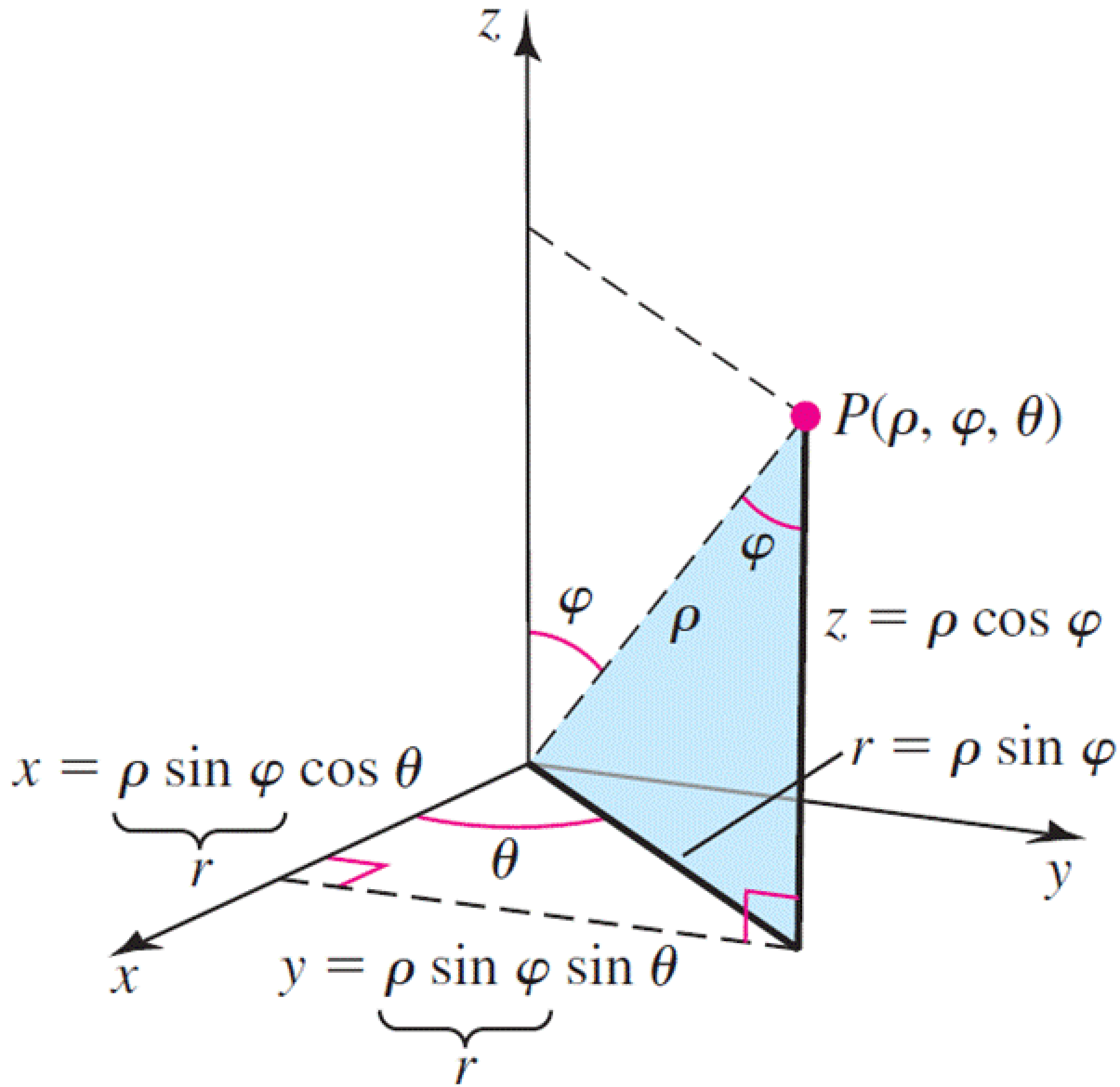

z
$P(\rho, \varphi, \theta)$
φ
φ
ρ
$z = \rho \cos \varphi$
$r = \rho \sin \varphi$
$x = \rho \sin \varphi \cos \theta$
r
θ
y
x
$y = \rho \sin \varphi \sin \theta$
r

FIGURE 13.55

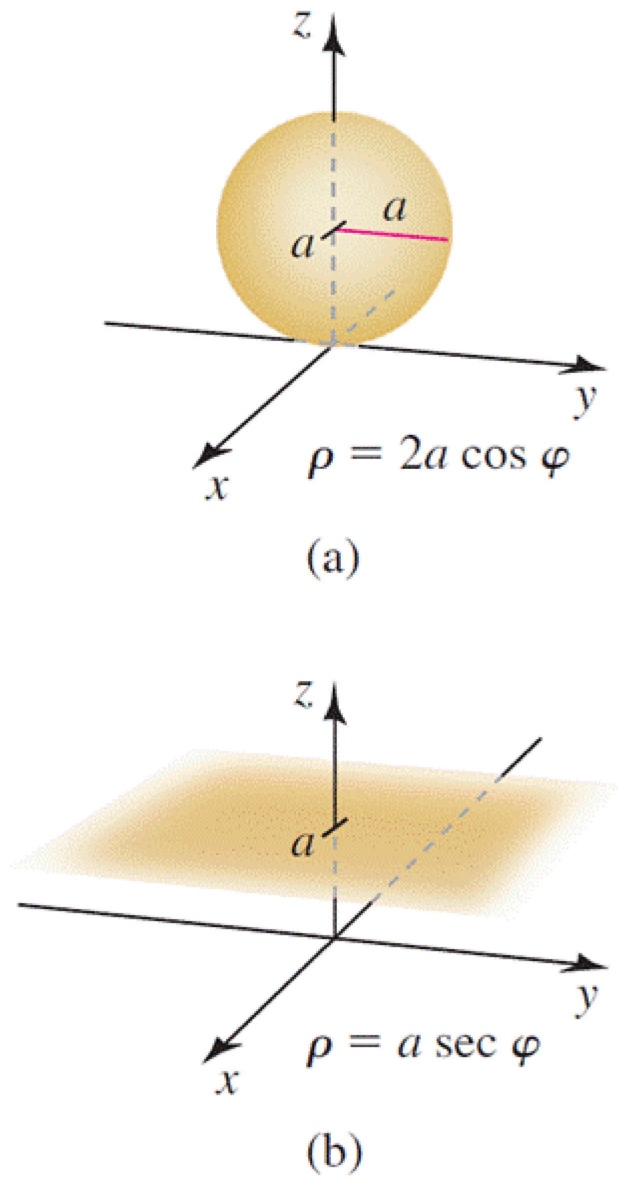
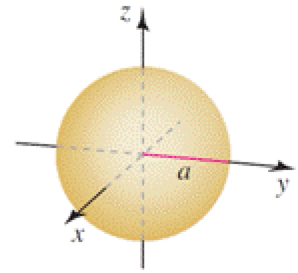
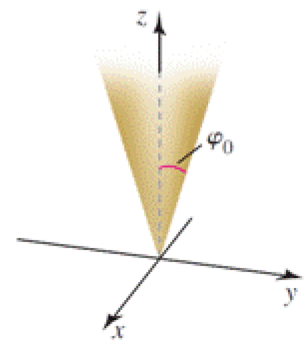
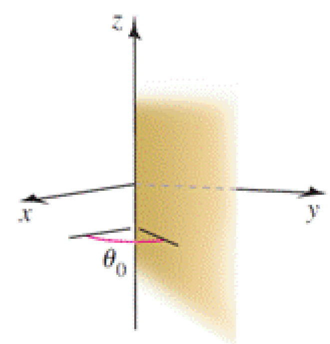


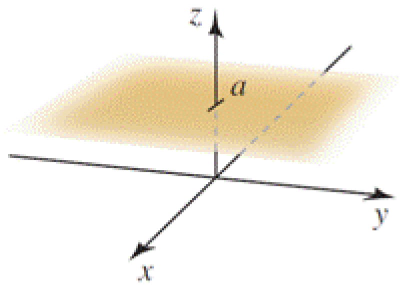
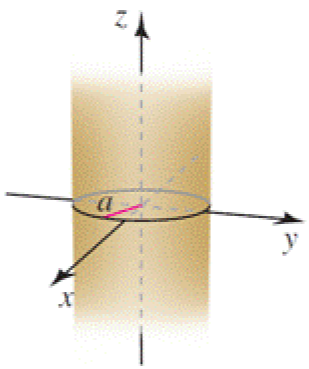
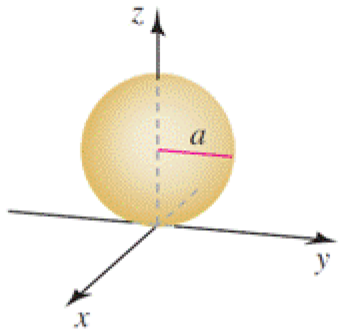
FIGURE 13.56

Table 13.4

Name	Description	Example
Sphere, radius a, center $(0, 0, 0)$	$\{(\rho, \varphi, \theta): \rho = a\}, a > 0$	
Cone	$\{(\rho, \varphi, \theta): \varphi = \varphi_0\}, \varphi_0 \neq 0, \pi/2, \pi$	
Vertical half plane	$\{(\rho, \varphi, \theta): \theta = \theta_0\}$	

(Continued)

Table 13.4 (Continued)

Name	Description	Example
Horizontal plane, $z = a$	$\{(\rho, \varphi, \theta): \rho = a \sec \varphi, 0 \le \varphi < \pi/2\}$	
Cylinder, radius $a > 0$	$\{(\rho, \varphi, \theta): \rho = a \csc \varphi, 0 < \varphi < \pi\}$	
Sphere, radius $a > 0$, center $(0, 0, a)$	$\{(\rho, \varphi, \theta): \rho = 2a \cos \varphi, 0 \le \varphi \le \pi/2\}$	

Chapter 11

Vectors and Vector-Valued Functions

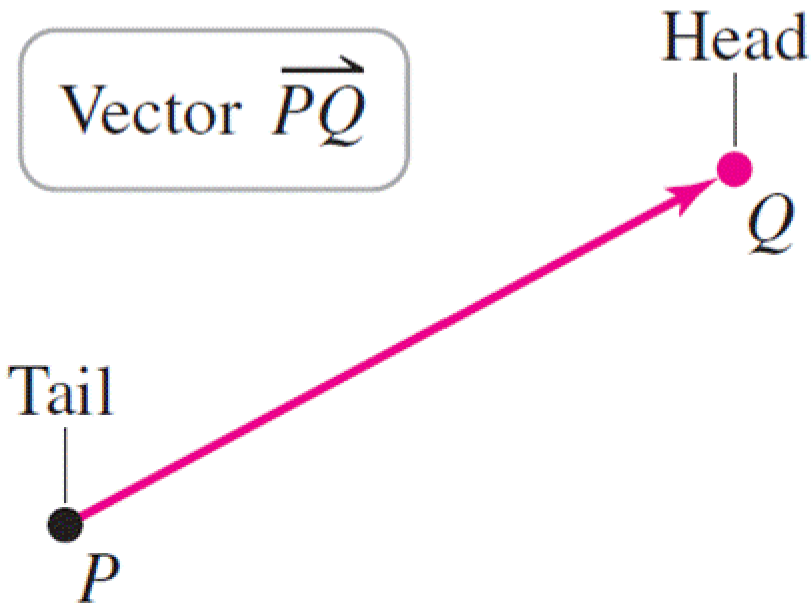


FIGURE 11.3

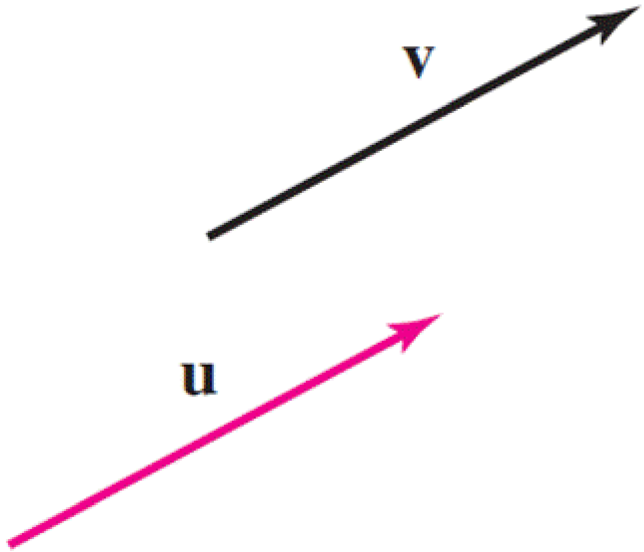
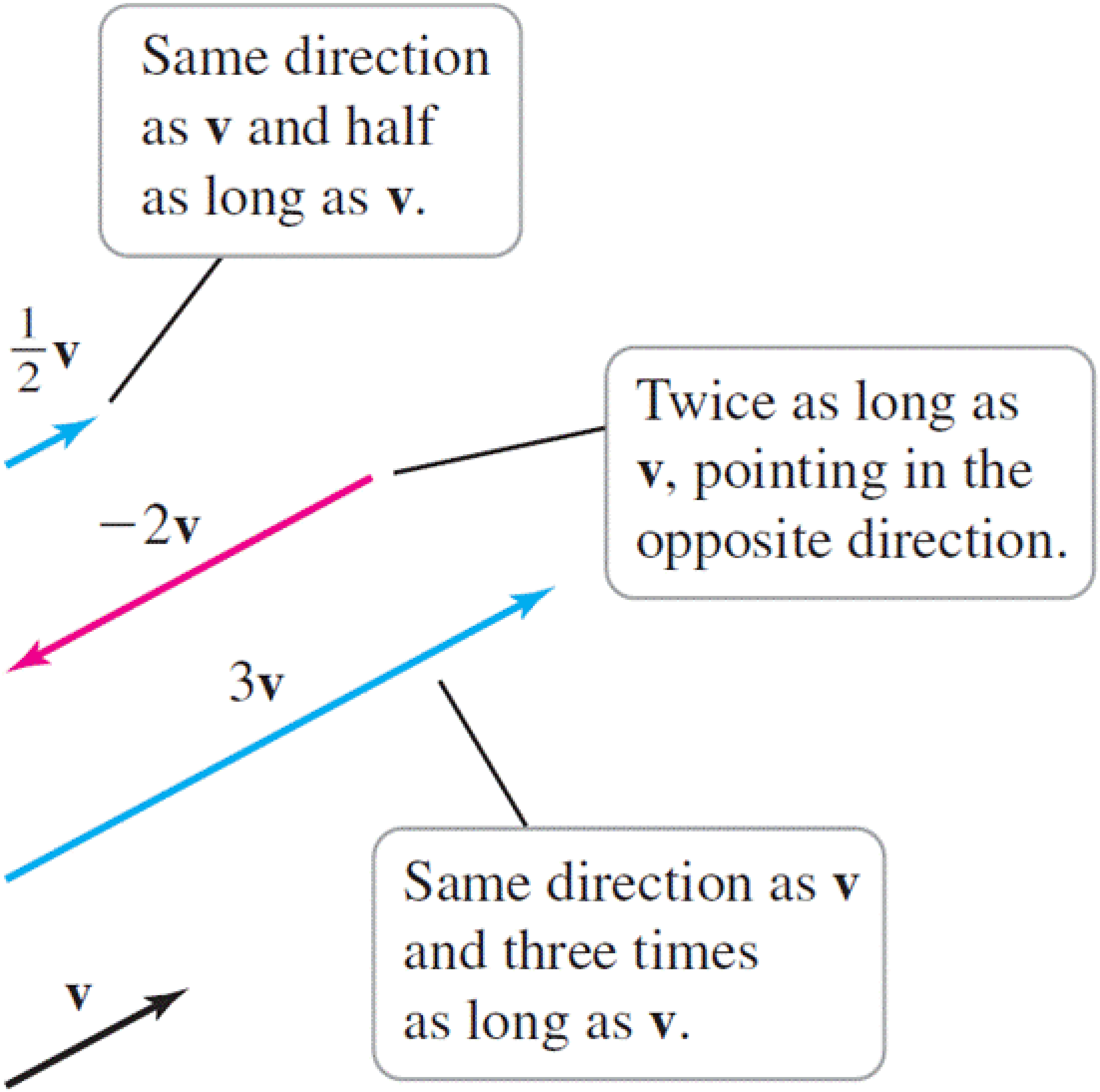


FIGURE 11.4

Vectors **u** and **v** are *equal* if they have the same length and direction.



Same direction as **v** and half as long as **v**.
$\frac{1}{2}\mathbf{v}$
Twice as long as **v**, pointing in the opposite direction.
$-2\mathbf{v}$
$3\mathbf{v}$
Same direction as **v** and three times as long as **v**.
v

FIGURE 11.5

DEFINITION **Scalar Multiples and Parallel Vectors**

Given a scalar c and a vector $\mathbf{v}$, the **scalar multiple** $c\mathbf{v}$ is a vector whose magnitude is $|c|$ multiplied by the magnitude of $\mathbf{v}$. If $c > 0$, then $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If $c < 0$, then $c\mathbf{v}$ and $\mathbf{v}$ point in opposite directions. Two vectors are **parallel** if they are scalar multiples of each other.

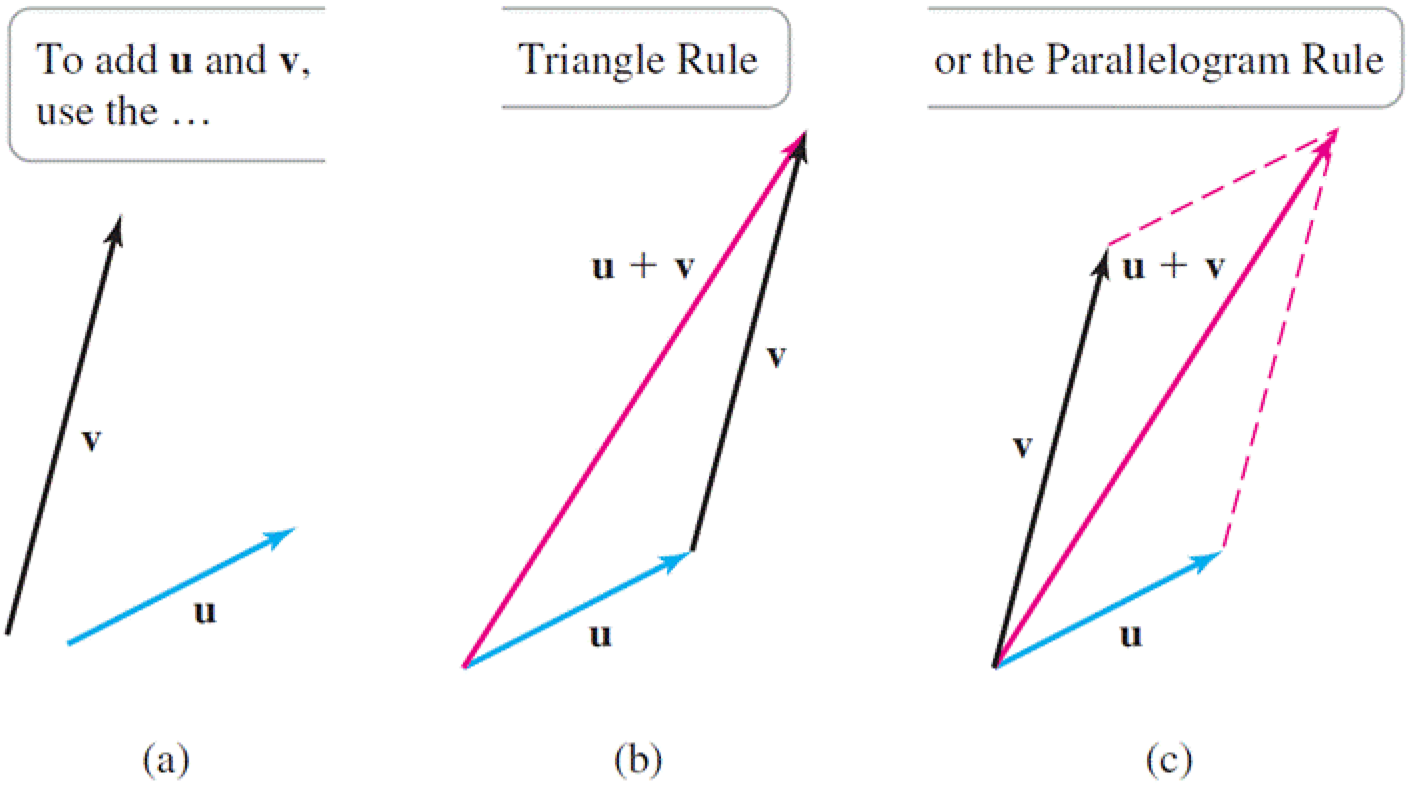

To add **u** and **v**, use the …
Triangle Rule
or the Parallelogram Rule
v
u
u + v
v
u
v
u + v
u
(a)
(b)
(c)

FIGURE 11.8

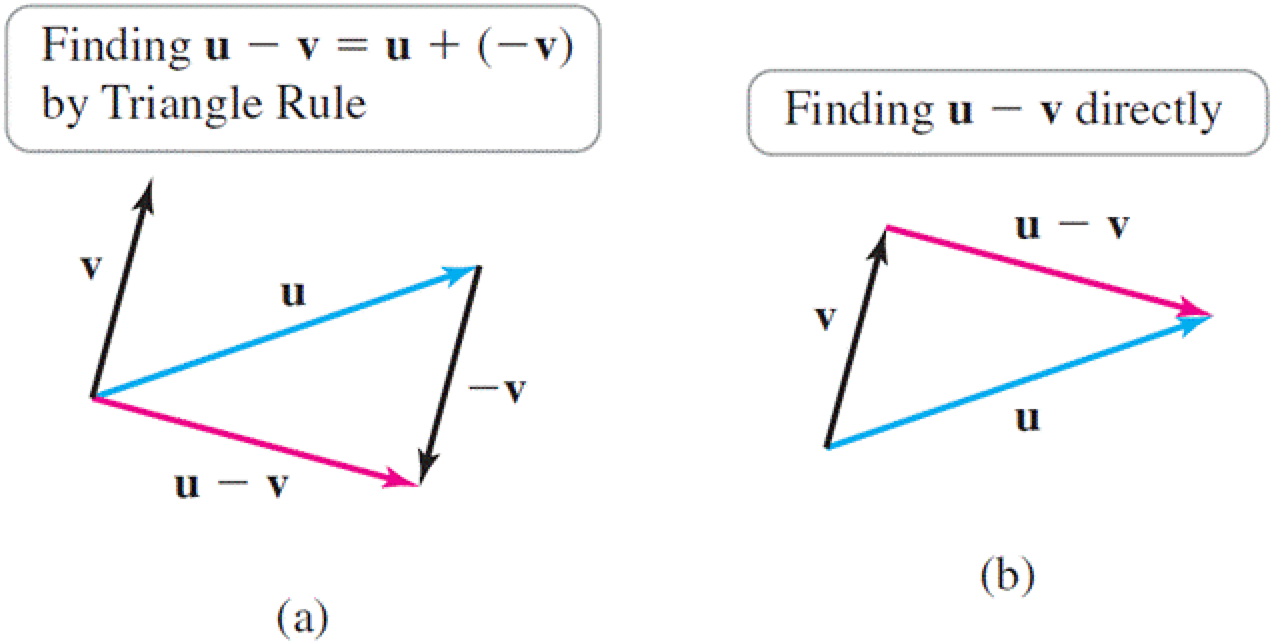
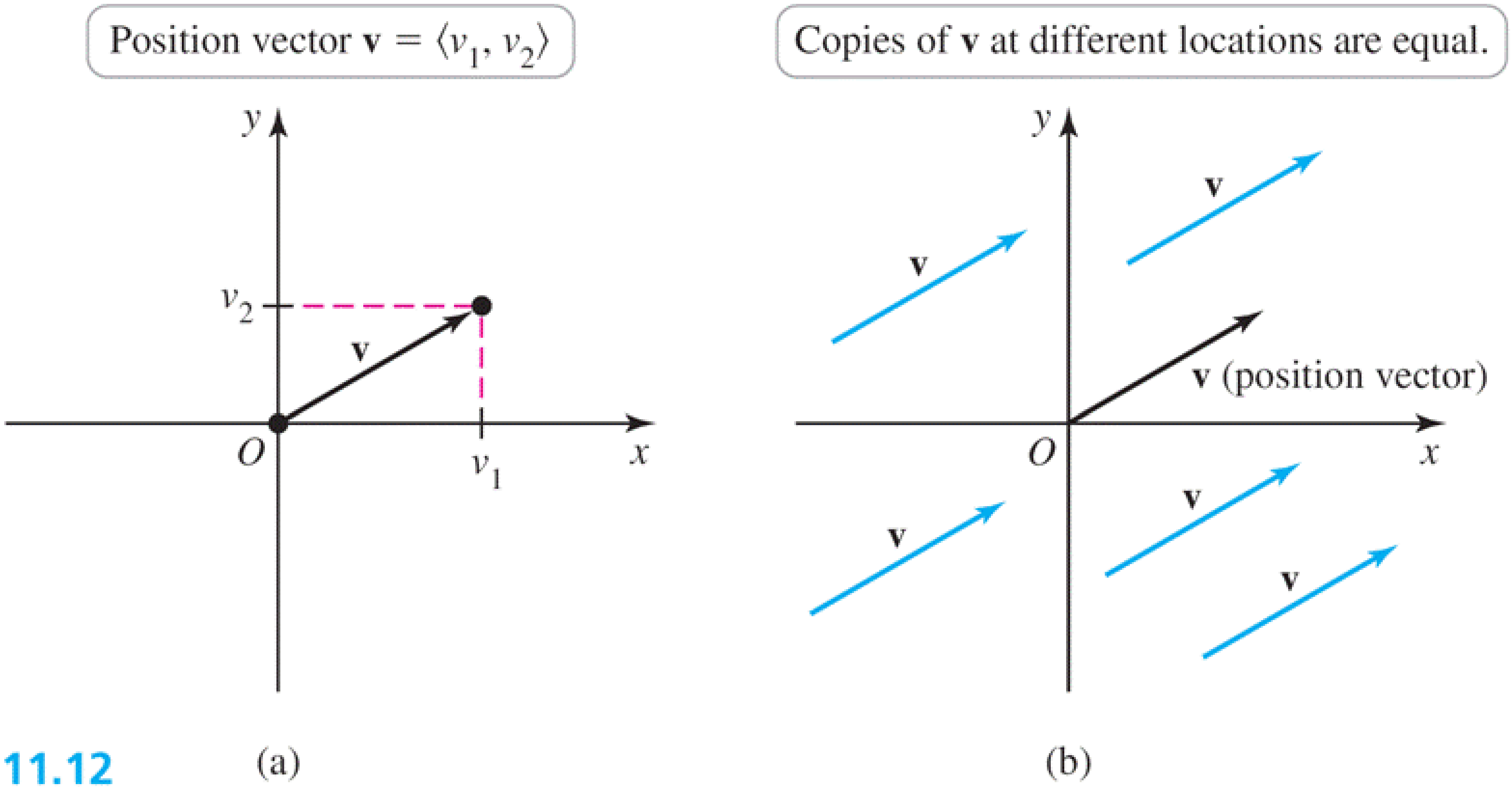

Finding **u** − **v** = **u** + (−**v**)
by Triangle Rule
v
u
−**v**
u − **v**
(a)
Finding **u** − **v** directly
u − **v**
v
u
(b)

FIGURE 11.9

DEFINITION Position Vectors and Vector Components

A vector **v** with its tail at the origin and head at (v_1, v_2) is called a **position vector** (or is said to be in **standard position**) and is written $\langle v_1, v_2 \rangle$. The real numbers v_1 and v_2 are the x- and y-**components** of **v**, respectively. The position vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.



Position vector $\mathbf{v} = \langle v_1, v_2 \rangle$
y
v_2
v
O
v_1
x
(a)
Copies of **v** at different locations are equal.
y
v
v
v (position vector)
O
x
v
v
v
(b)

FIGURE 11.12

F

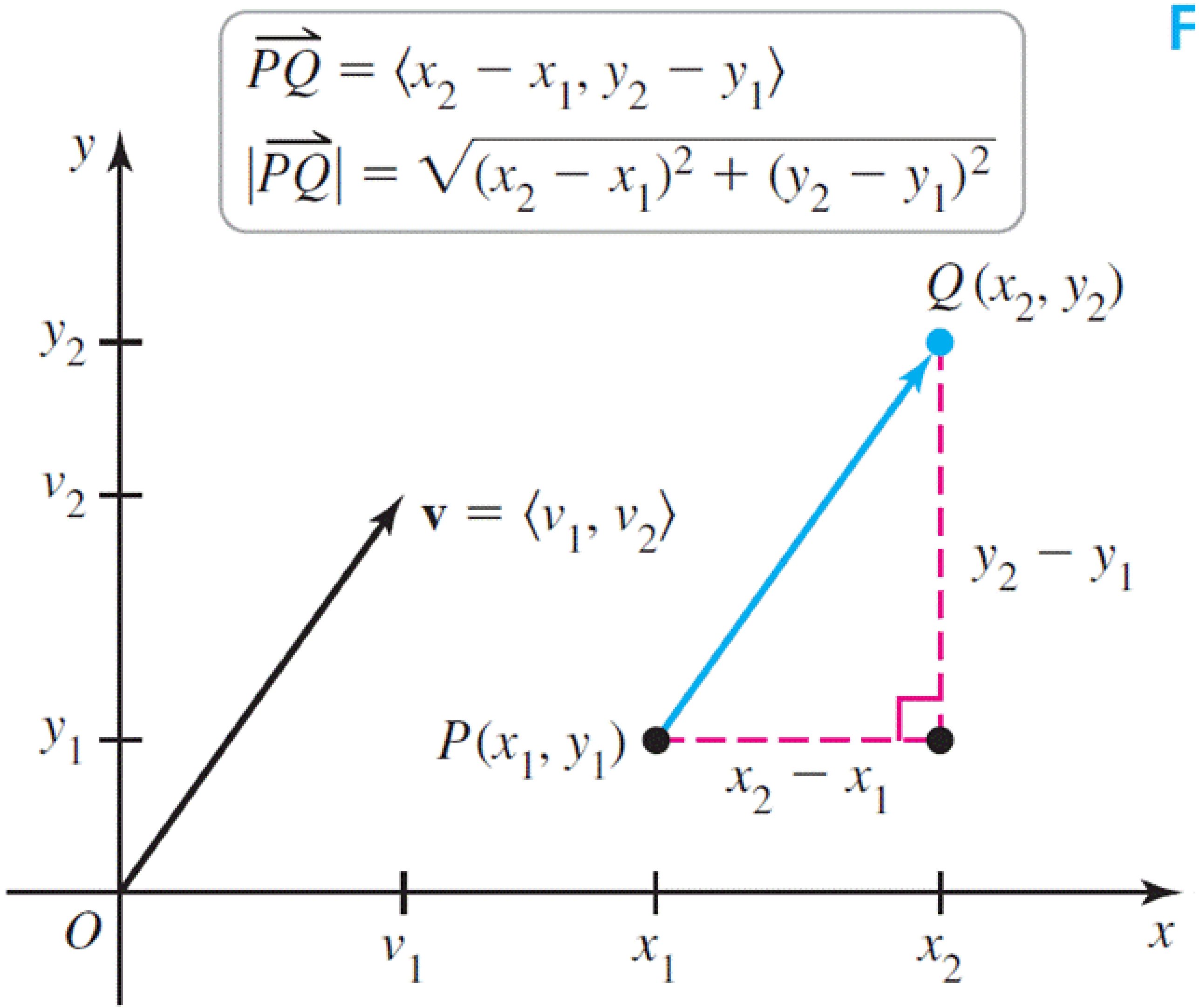


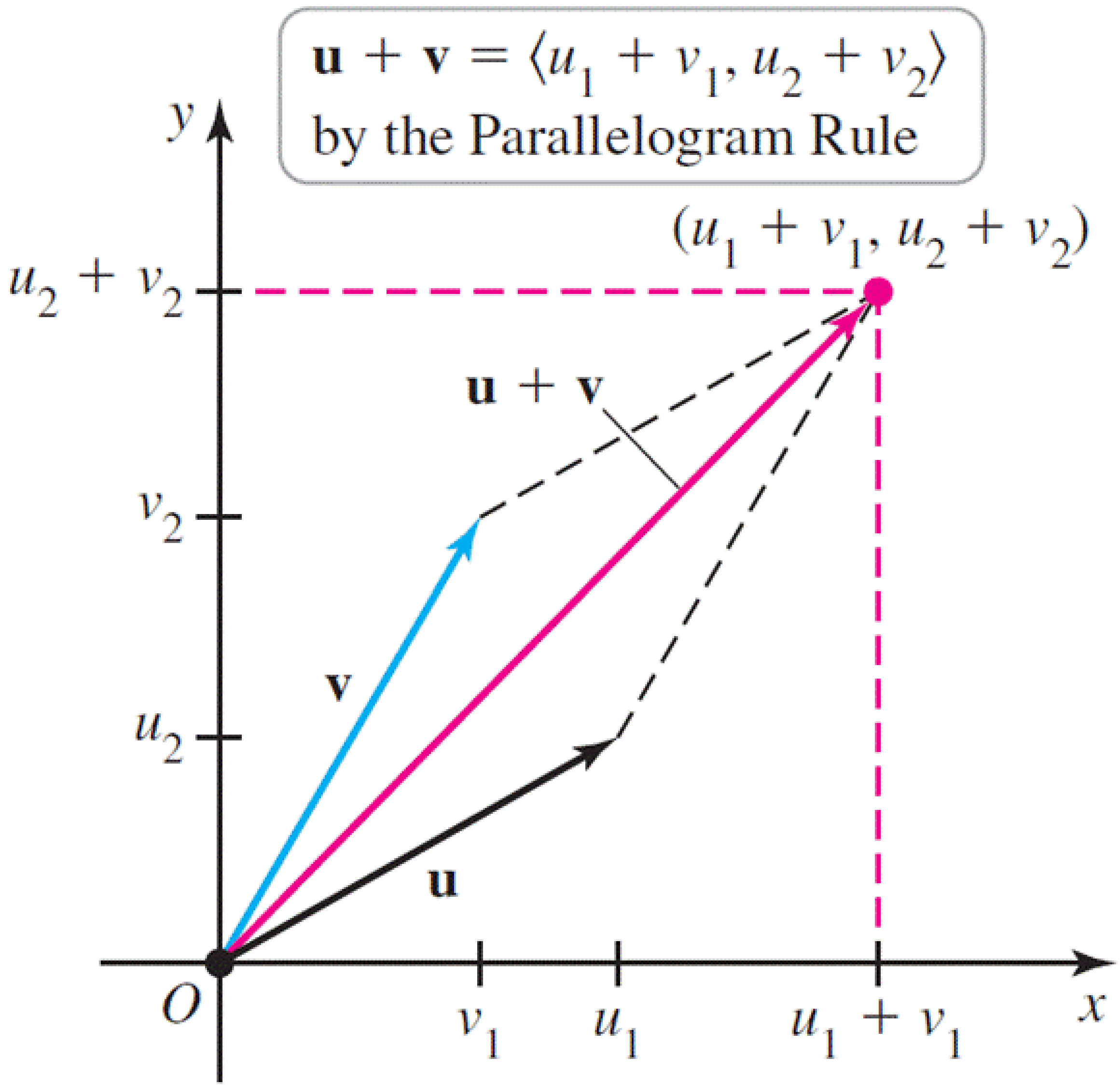
FIGURE 11.13

DEFINITION **Magnitude of a Vector**

Given the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $|\overrightarrow{PQ}|$, is the distance between P and Q:

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of the position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$.



u + **v** = ⟨$u_1 + v_1, u_2 + v_2$⟩
by the Parallelogram Rule
$(u_1 + v_1, u_2 + v_2)$
y
$u_2 + v_2$
u + **v**
v_2
v
u_2
u
O
v_1
u_1
$u_1 + v_1$
x

FIGURE 11.14

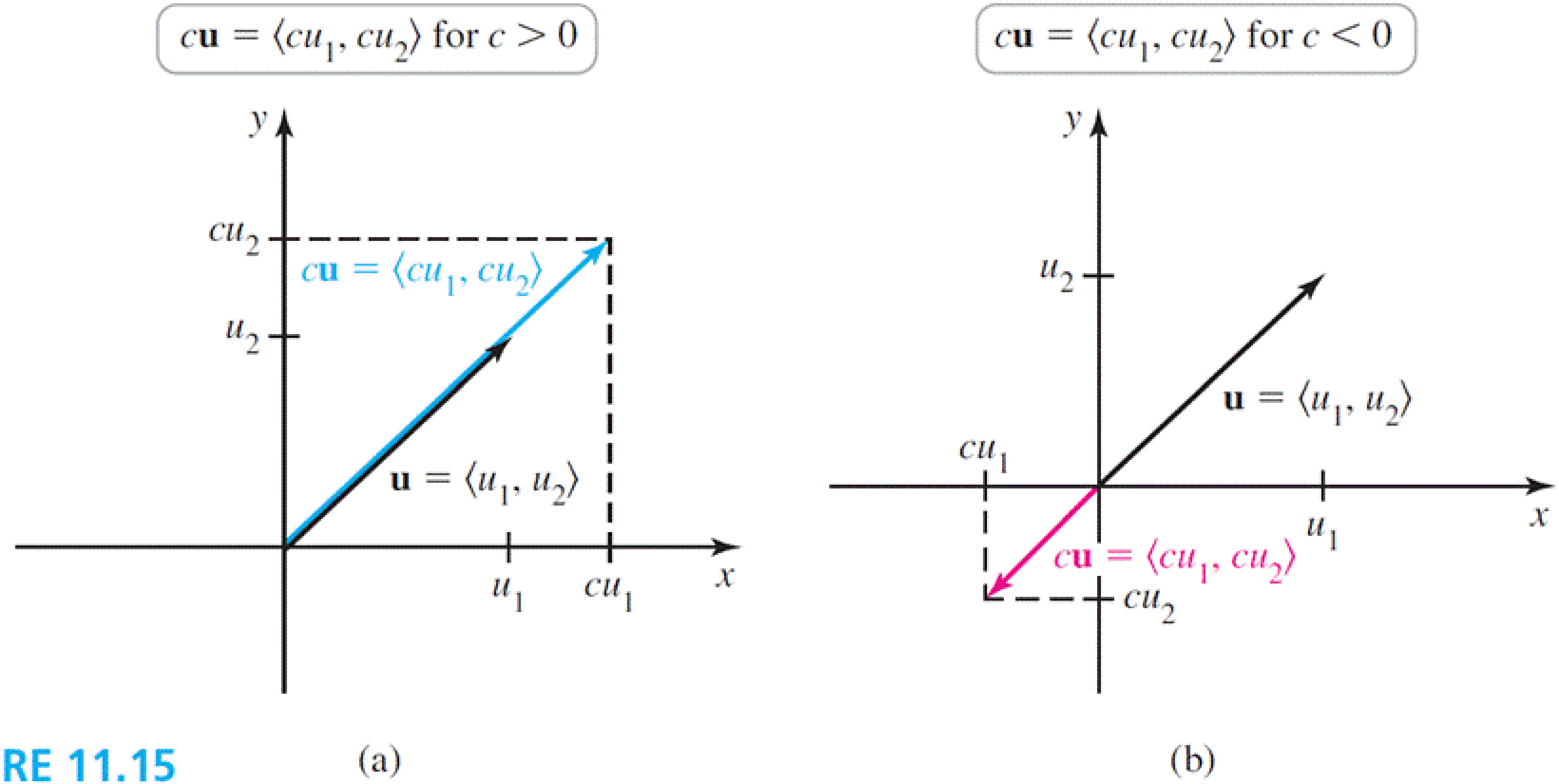

$c\mathbf{u} = \langle cu_1, cu_2 \rangle$ for $c > 0$
y
cu_2
$c\mathbf{u} = \langle cu_1, cu_2 \rangle$
u_2
$\mathbf{u} = \langle u_1, u_2 \rangle$
u_1
cu_1
x
(a)
$c\mathbf{u} = \langle cu_1, cu_2 \rangle$ for $c < 0$
y
u_2
$\mathbf{u} = \langle u_1, u_2 \rangle$
cu_1
x
u_1
$c\mathbf{u} = \langle cu_1, cu_2 \rangle$
cu_2
(b)

FIGURE 11.15

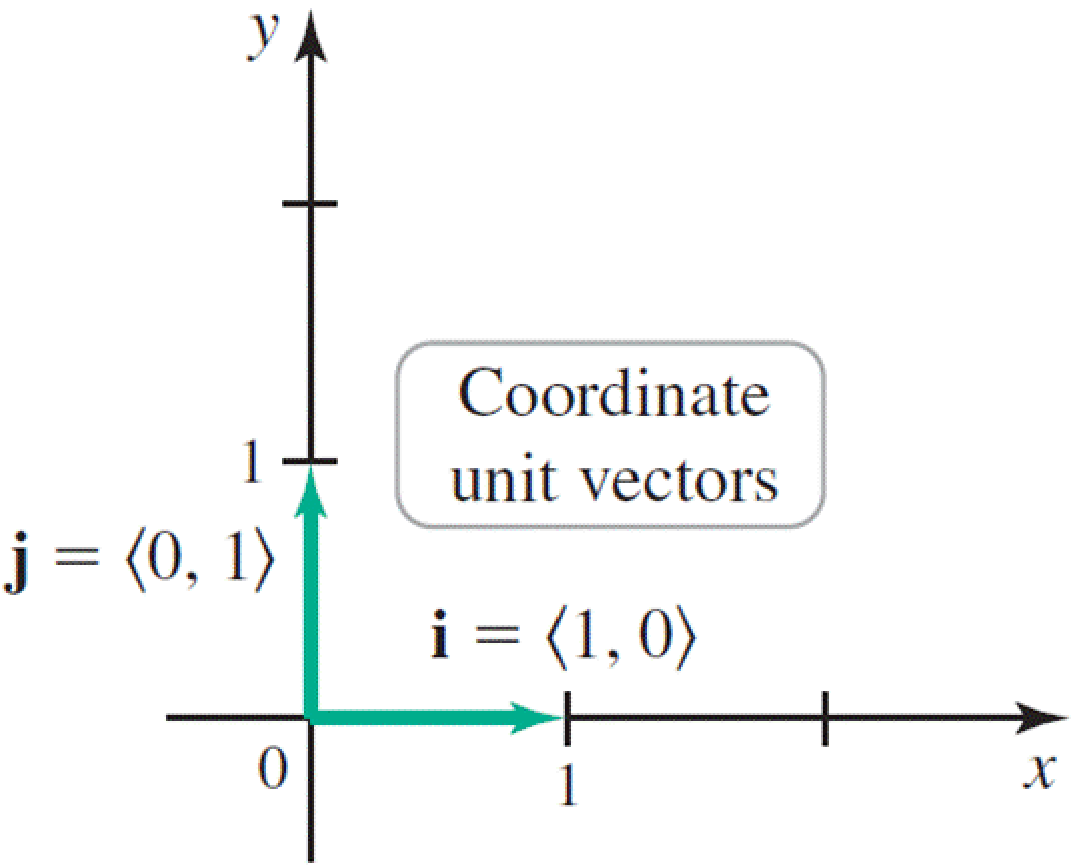
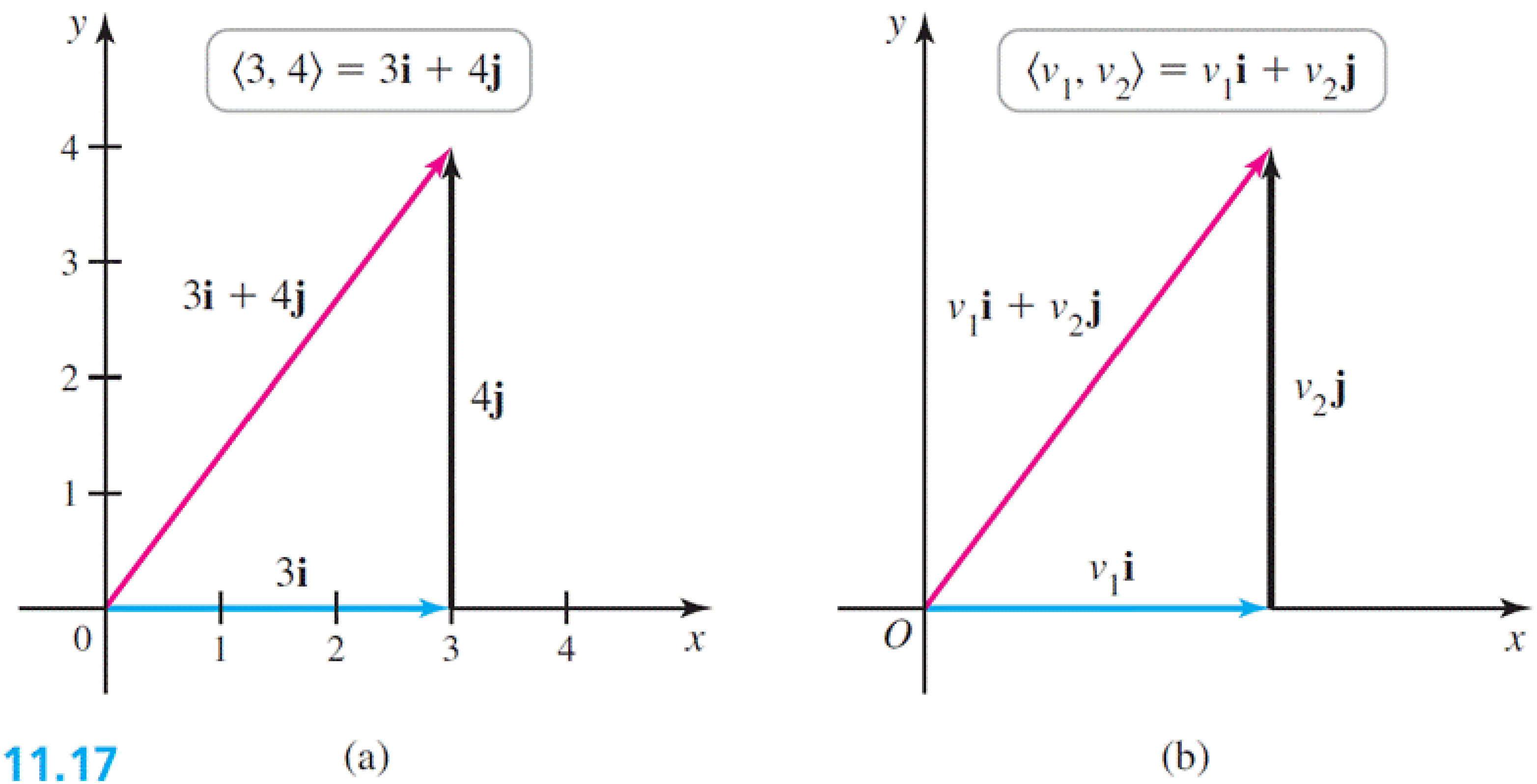


FIGURE 11.16



⟨3, 4⟩ = 3i + 4j
y
4
3
2
1
0
1
2
3
4
x
3i + 4j
4j
3i
(a)
⟨v1, v2⟩ = v1i + v2j
y
v1i + v2j
v2j
v1i
O
x
(b)

FIGURE 11.17

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \text{ and } -\mathbf{u} = -\frac{\mathbf{v}}{|\mathbf{v}|} \text{ have length 1.}$$

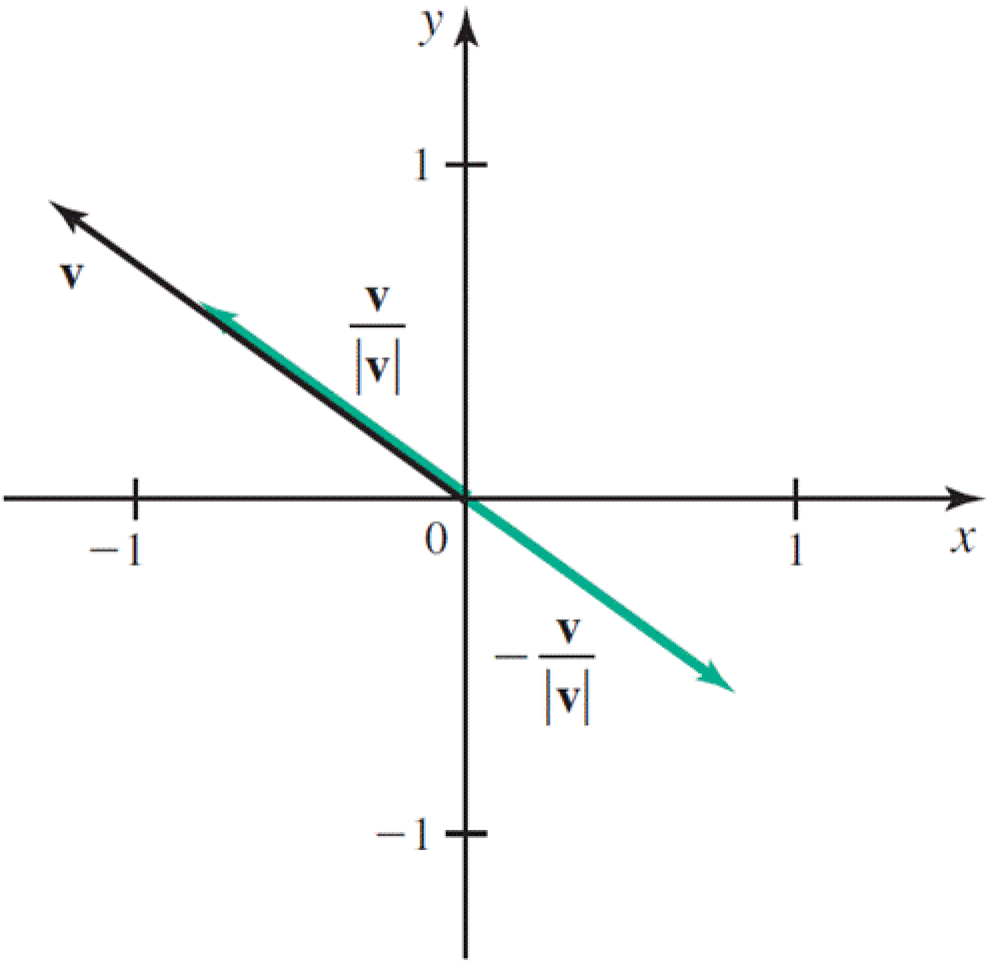


FIGURE 11.18

DEFINITION **Unit Vectors and Vectors of a Specified Length**

A **unit vector** is any vector with length 1. Given a nonzero vector $\mathbf{v}$, $\pm\frac{\mathbf{v}}{|\mathbf{v}|}$ are unit vectors parallel to $\mathbf{v}$. For a scalar $c > 0$, the vectors $\pm\frac{c\mathbf{v}}{|\mathbf{v}|}$ are vectors of length c parallel to $\mathbf{v}$.

11.2

Vectors in Three Dimensions

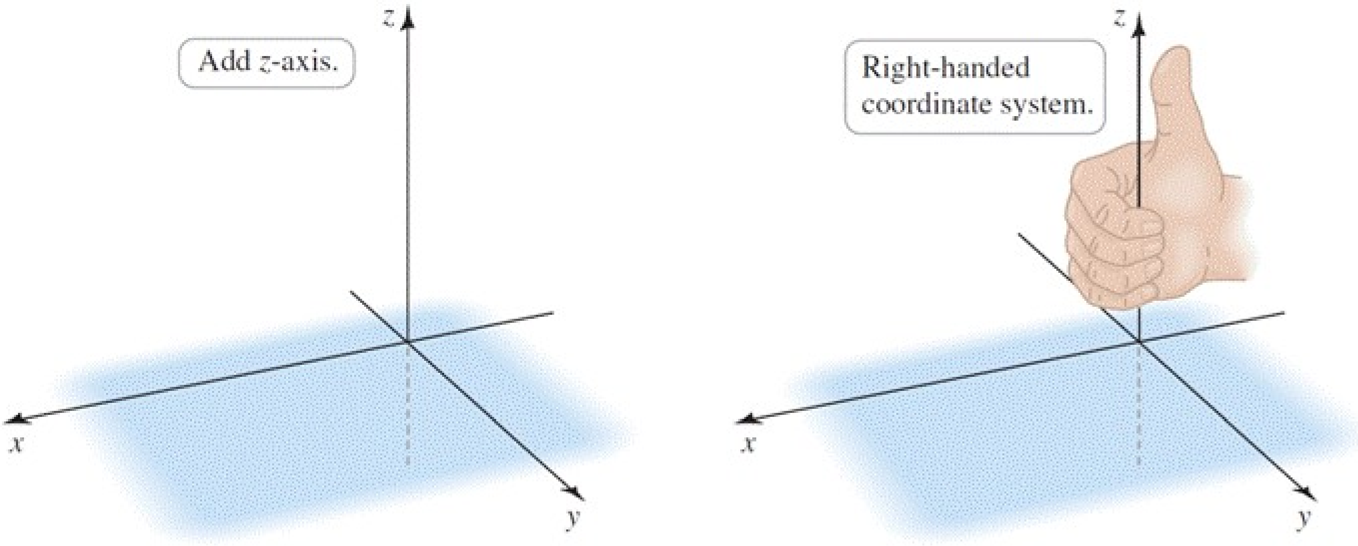


FIGURE 11.25

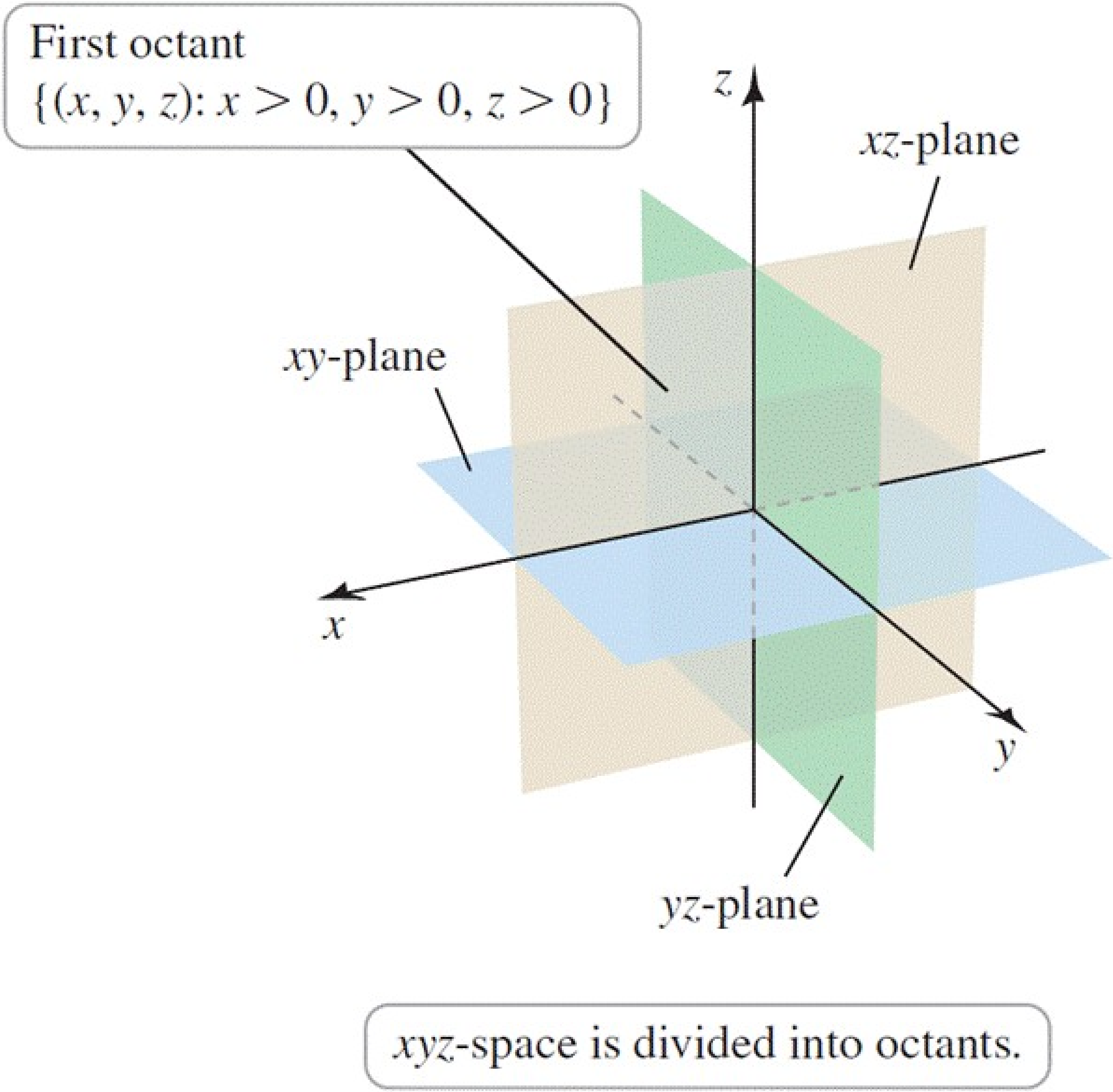

First octant
{(x, y, z): x > 0, y > 0, z > 0}
z
xz-plane
xy-plane
x
y
yz-plane
xyz-space is divided into octants.

FIGURE 11.26

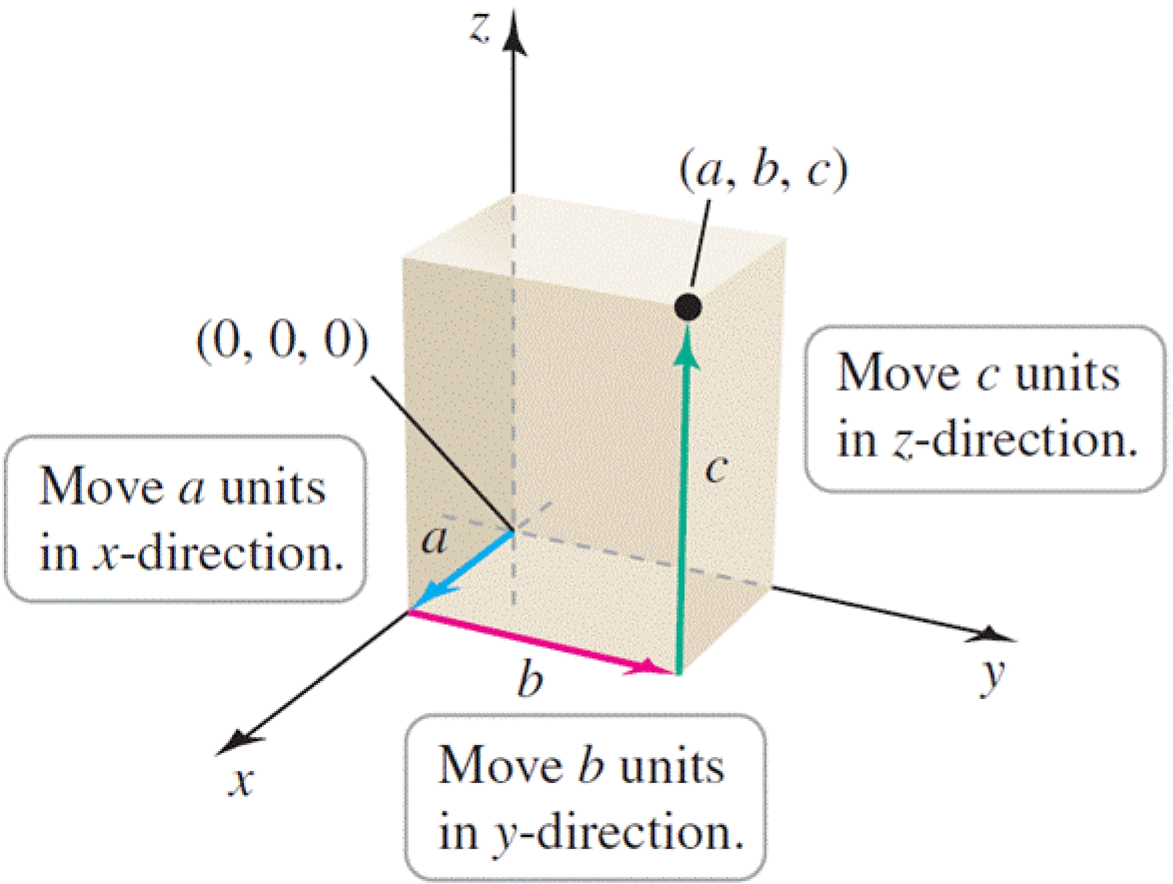

z
(a, b, c)
(0, 0, 0)
Move c units
in z-direction.
Move a units
in x-direction.
a
c
b
y
x
Move b units
in y-direction.

FIGURE 11.27

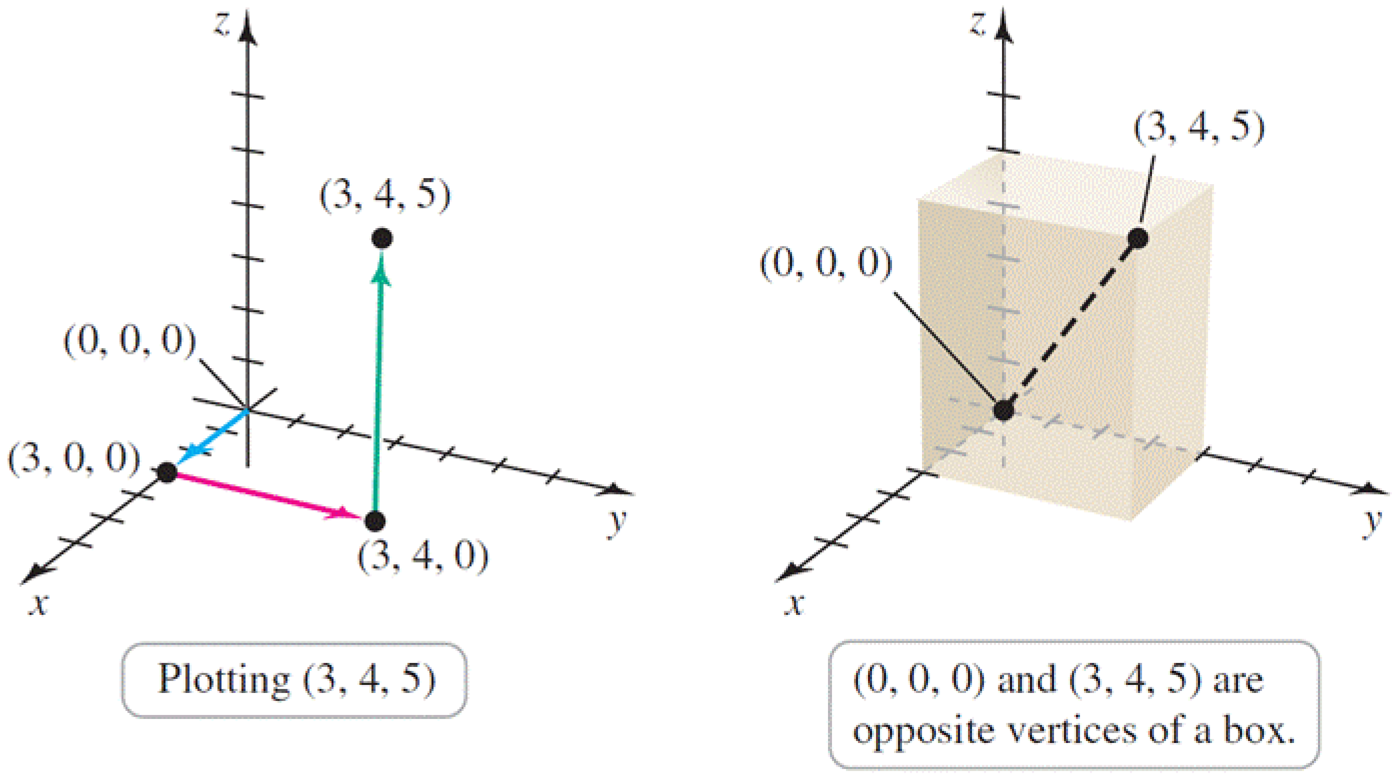

z
(3, 4, 5)
(0, 0, 0)
(3, 0, 0)
(3, 4, 0)
y
x
Plotting (3, 4, 5)
z
(3, 4, 5)
(0, 0, 0)
y
x
(0, 0, 0) and (3, 4, 5) are opposite vertices of a box.

FIGURE 11.28

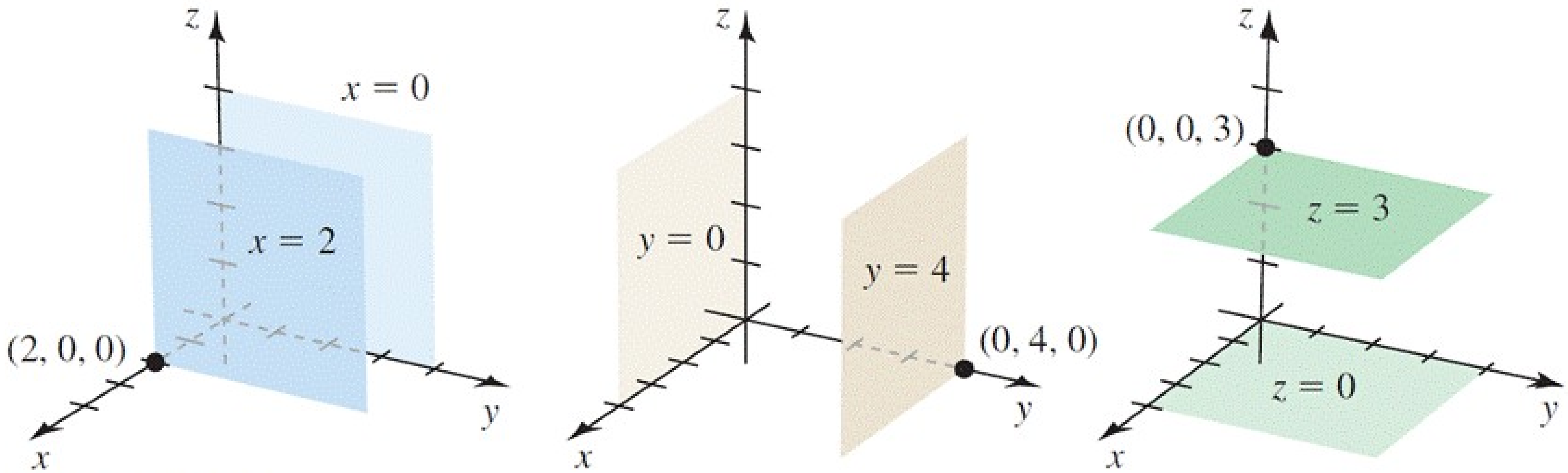
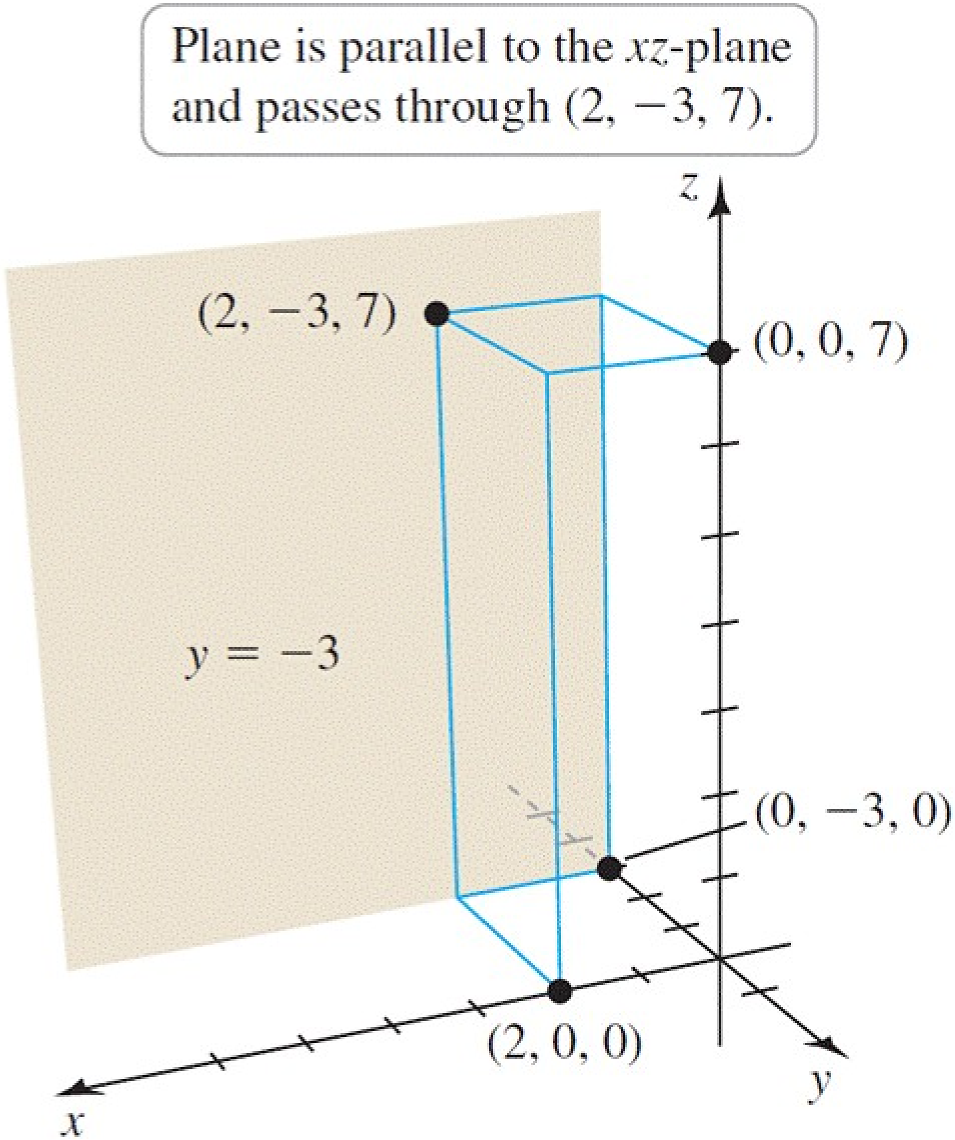
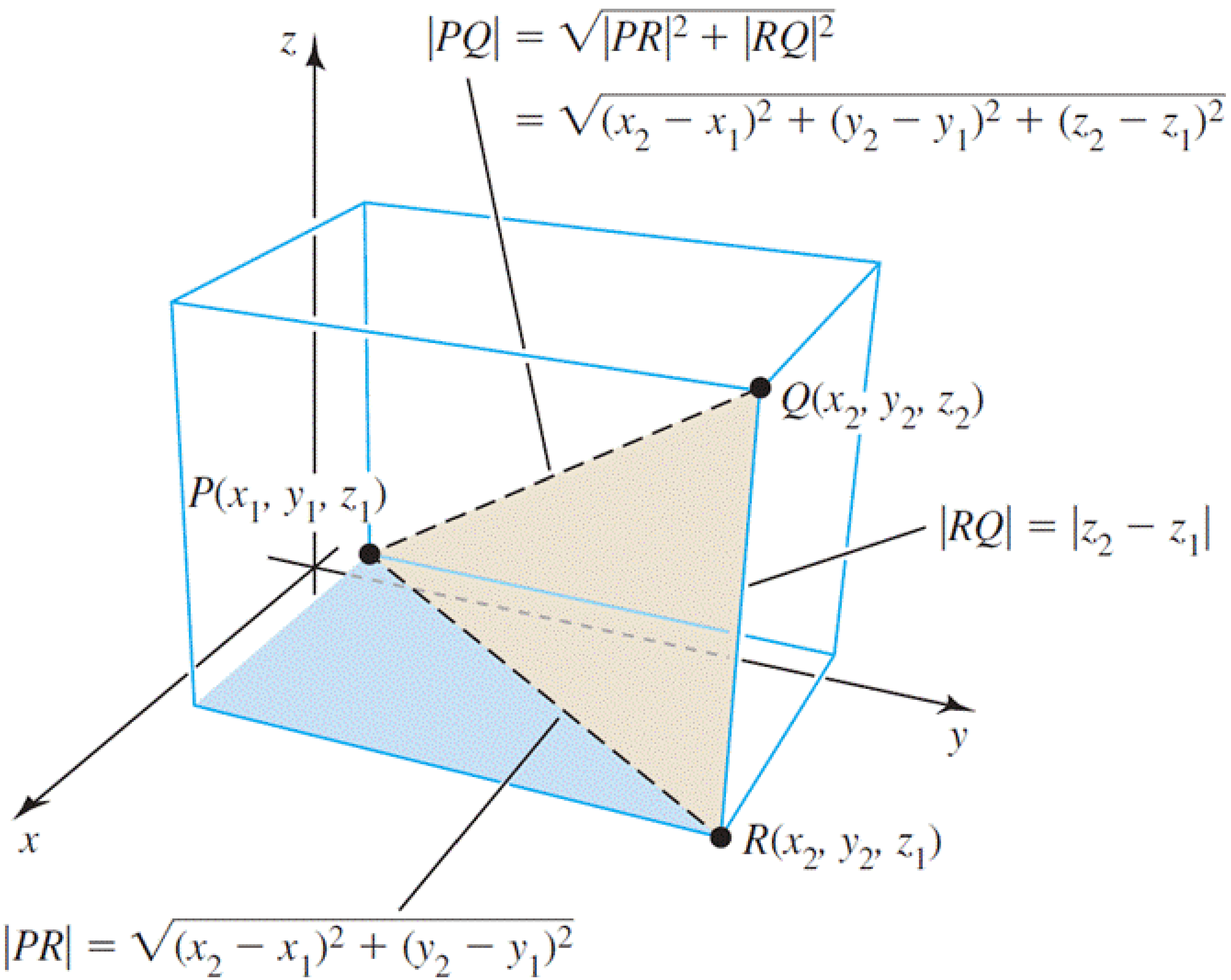


FIGURE 11.30


Plane is parallel to the xz-plane and passes through (2, −3, 7).
z
(2, −3, 7)
(0, 0, 7)
y = −3
(0, −3, 0)
(2, 0, 0)
y
x
FIGURE 11.31



$|PQ| = \sqrt{|PR|^2 + |RQ|^2}$
$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
z
$Q(x_2, y_2, z_2)$
$P(x_1, y_1, z_1)$
$|RQ| = |z_2 - z_1|$
y
x
$R(x_2, y_2, z_1)$
$|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

FIGURE 11.32

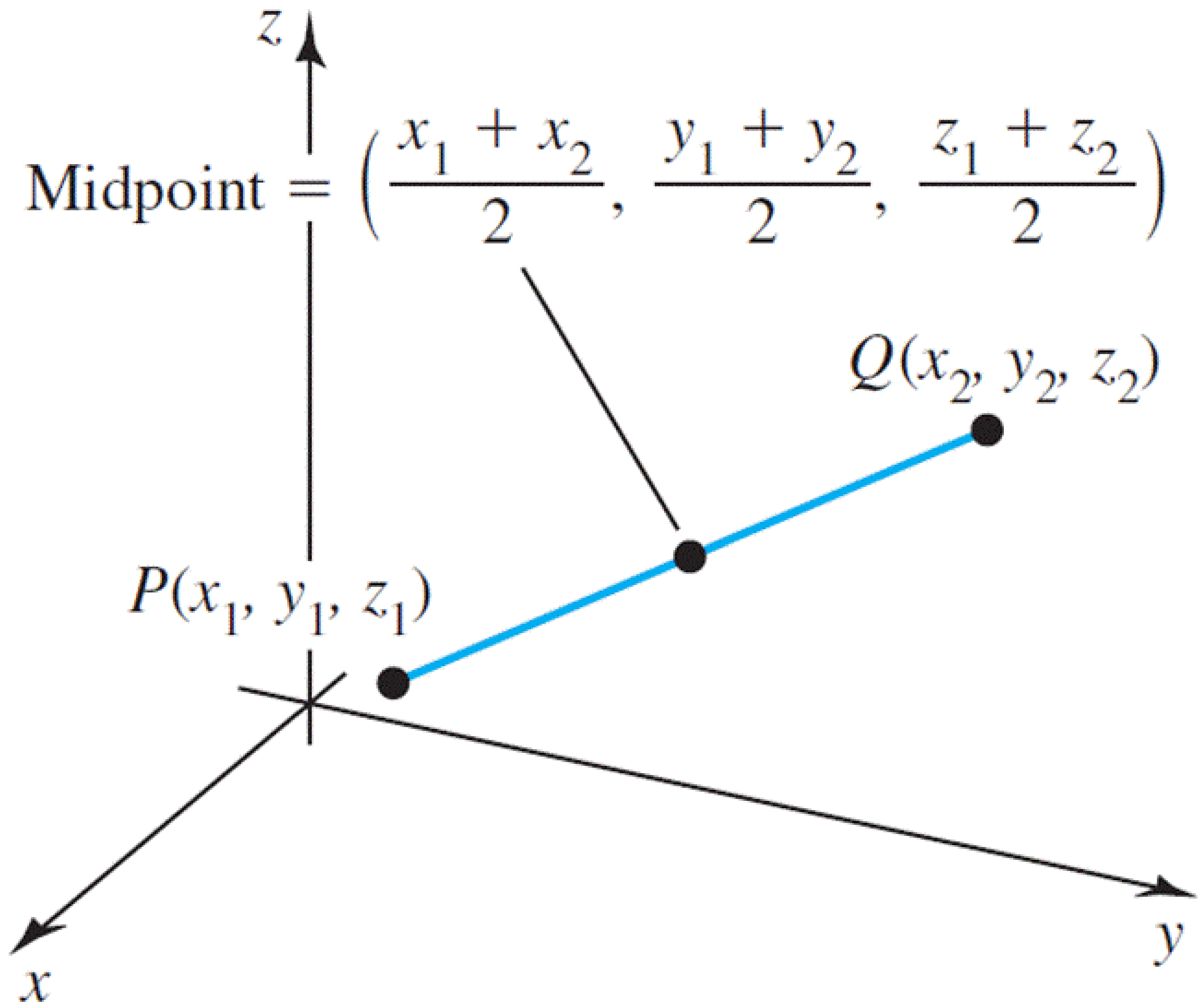
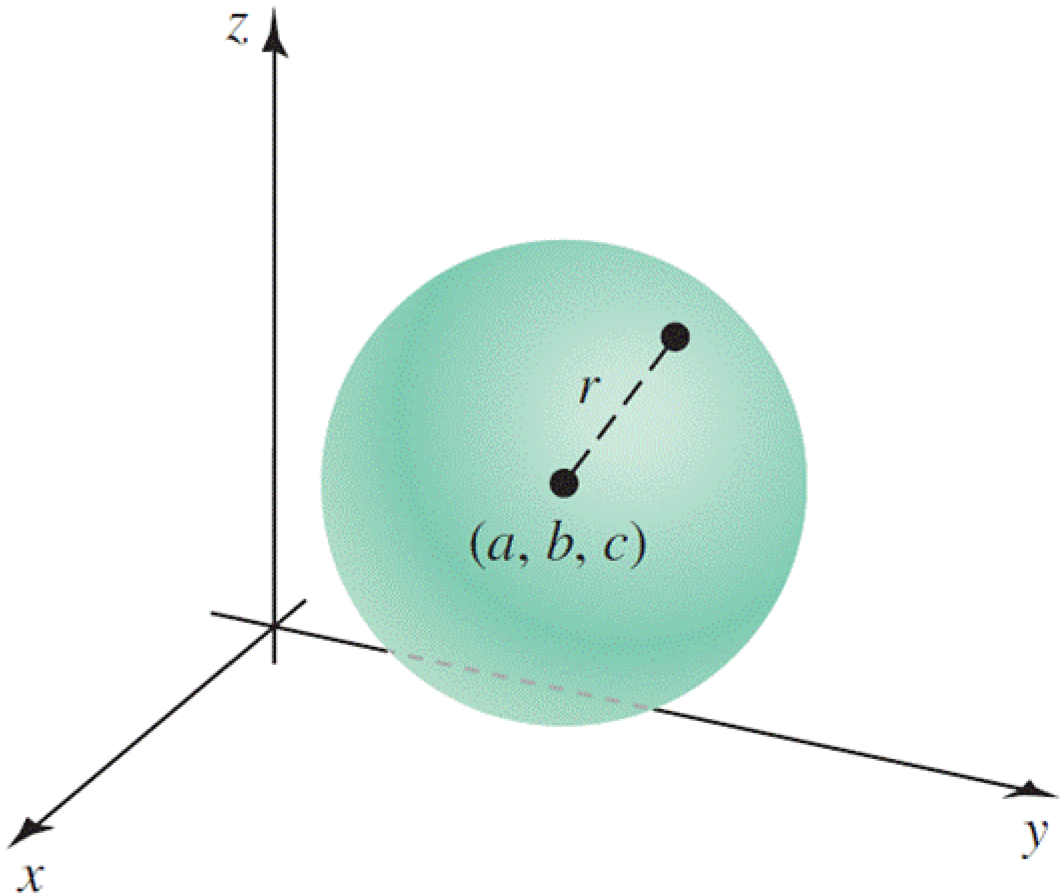

z
Midpoint = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
$Q(x_2, y_2, z_2)$
$P(x_1, y_1, z_1)$
x
y

FIGURE 11.33



$$\text{Sphere: } (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

$$\text{Ball: } (x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2$$

FIGURE 11.34

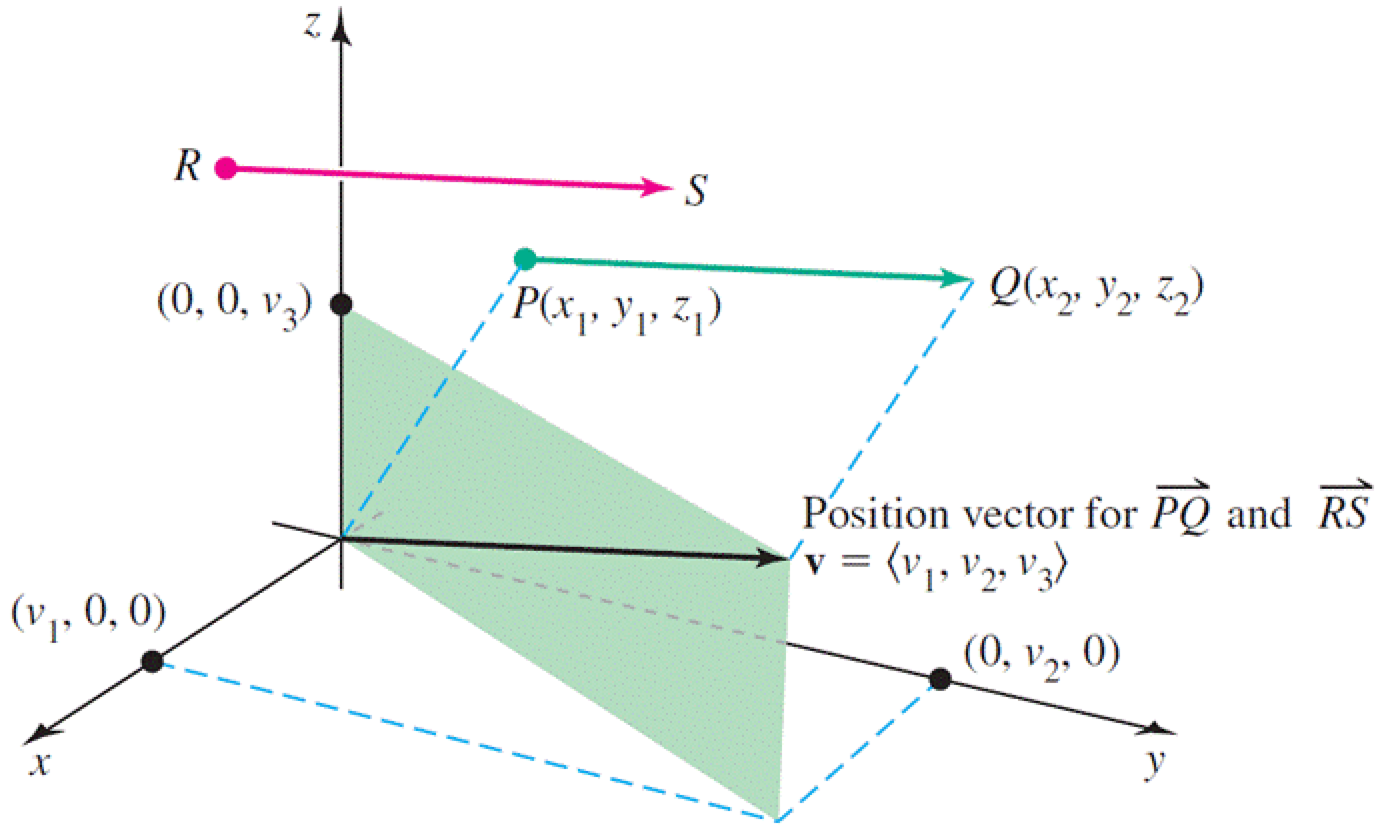

z
R
S
(0, 0, v_3)
$P(x_1, y_1, z_1)$
$Q(x_2, y_2, z_2)$
Position vector for $\overrightarrow{PQ}$ and $\overrightarrow{RS}$
$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$
(v_1, 0, 0)
(0, v_2, 0)
x
y

FIGURE 11.35

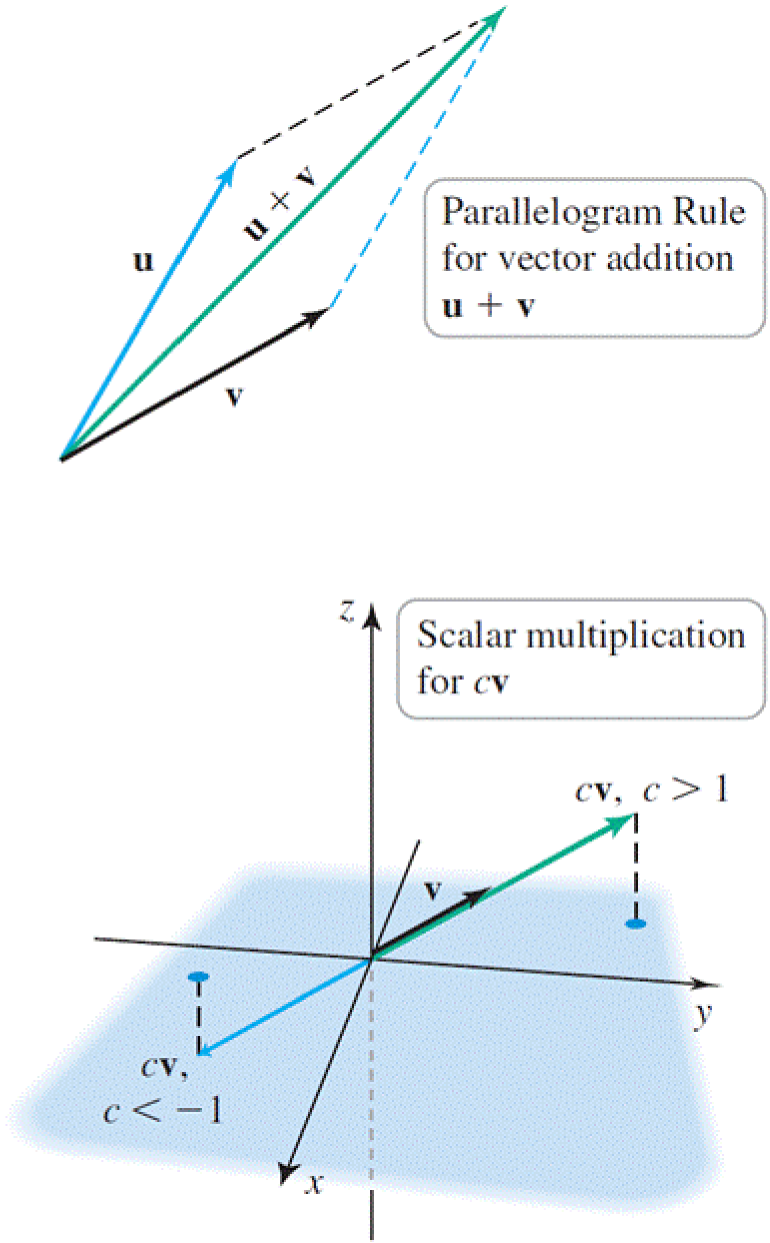

Parallelogram Rule
for vector addition
u + v
u
v
u + v
Scalar multiplication
for cv
cv, c > 1
v
cv,
c < −1
z
y
x

FIGURE 11.36

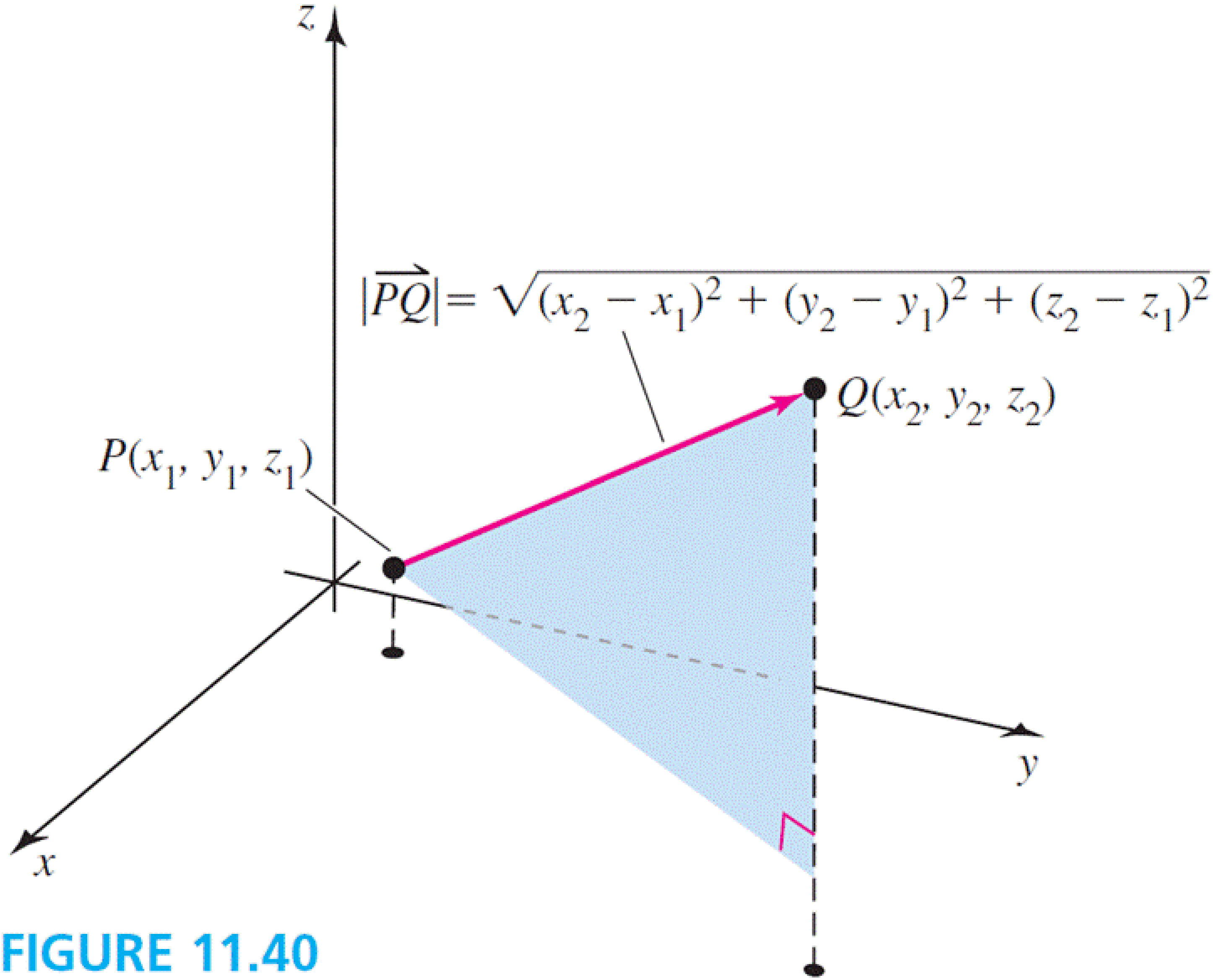

z
$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
$Q(x_2, y_2, z_2)$
$P(x_1, y_1, z_1)$
y
x

FIGURE 11.40

DEFINTION **Magnitude of a Vector**

The **magnitude** (or **length**) of the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

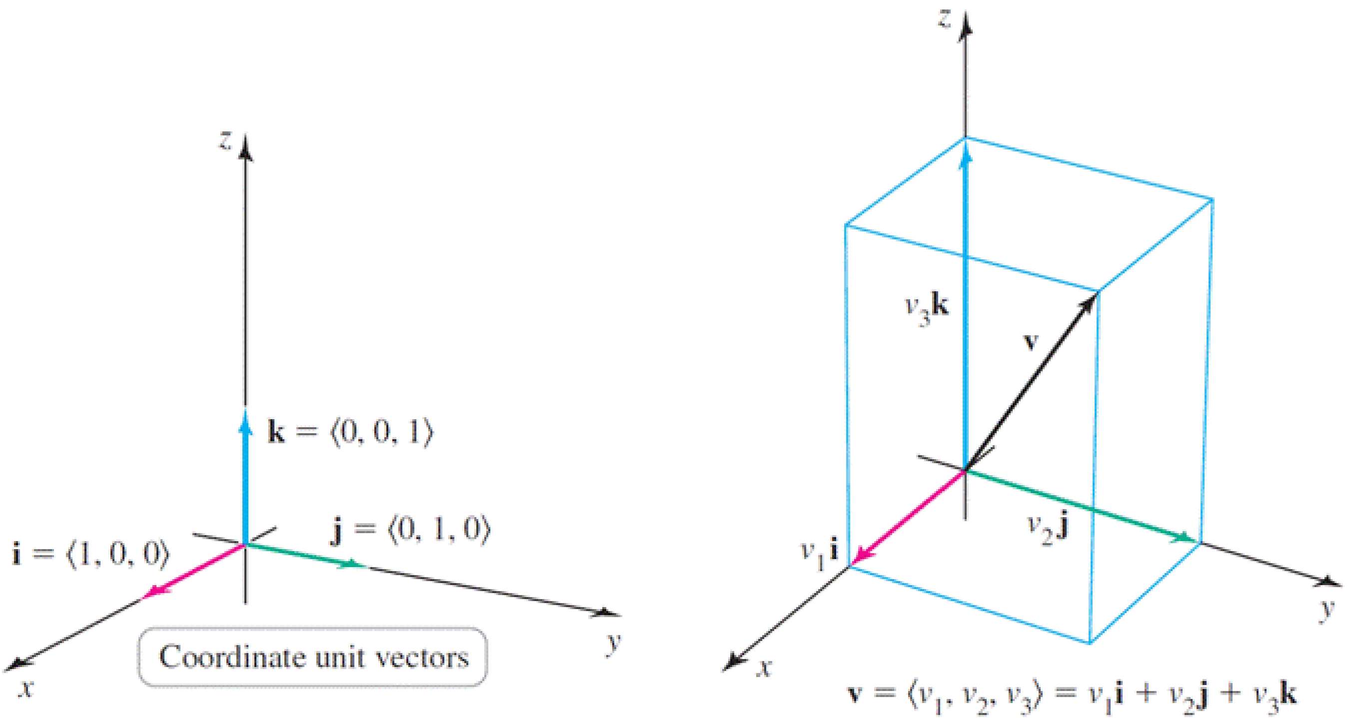

z
k = ⟨0, 0, 1⟩
j = ⟨0, 1, 0⟩
i = ⟨1, 0, 0⟩
y
x
Coordinate unit vectors
z
$v_3\mathbf{k}$
v
$v_2\mathbf{j}$
$v_1\mathbf{i}$
y
x
$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

FIGURE 11.41

11.3

Dot Products

DEFINITION **Dot Product**

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ with $0 \leq \theta \leq \pi$ (Figure 11.44). If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.

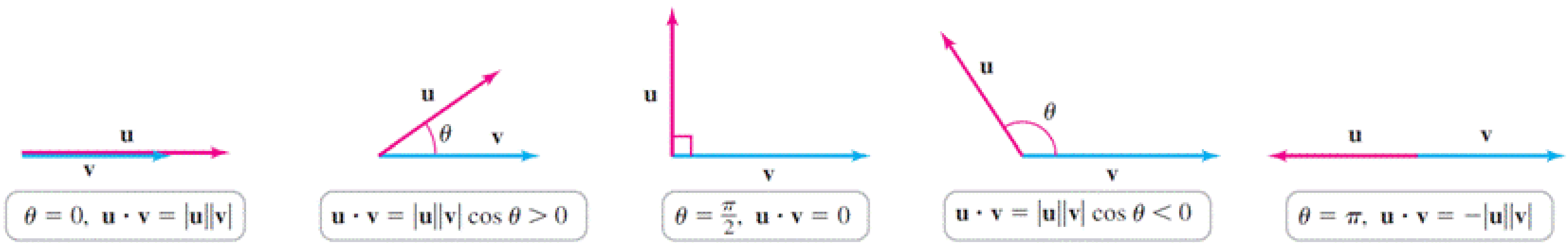

u
v
$\theta = 0,\ \mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|$
u
θ
v
$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta > 0$
u
v
$\theta = \frac{\pi}{2},\ \mathbf{u}\cdot\mathbf{v} = 0$
u
θ
v
$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta < 0$
u
v
$\theta = \pi,\ \mathbf{u}\cdot\mathbf{v} = -|\mathbf{u}||\mathbf{v}|$

FIGURE 11.44

DEFINITION Orthogonal Vectors

Two vectors **u** and **v** are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimesions, two nonzero orthogonal vectors are perpendicular to each other.

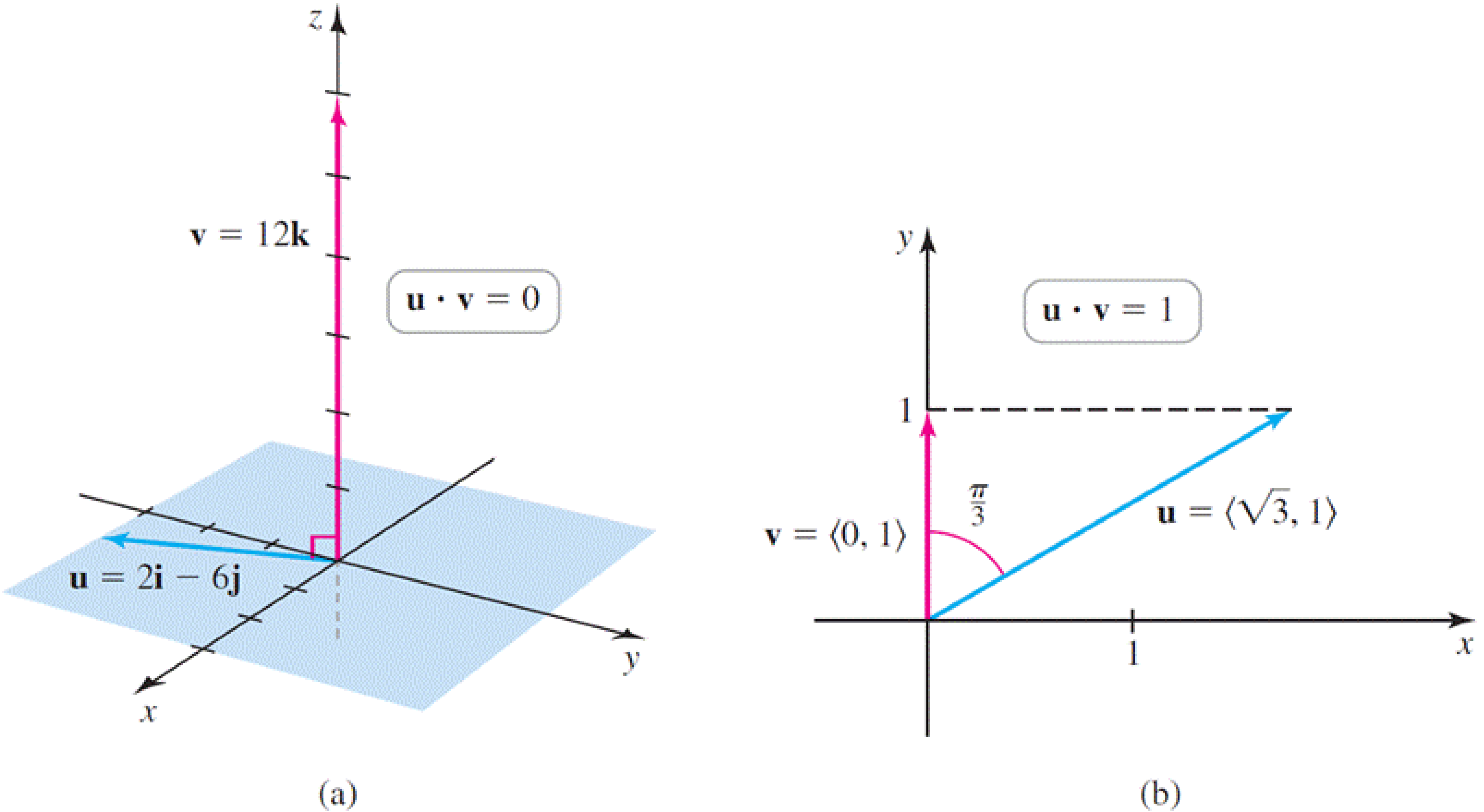


FIGURE 11.45

Related Exercises 9–12

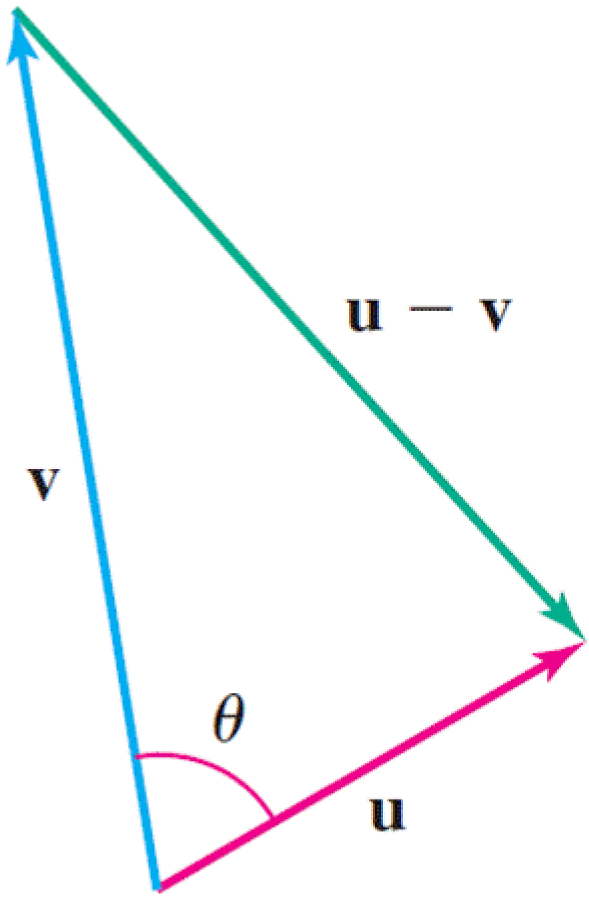


FIGURE 11.46

THEOREM 11.1 Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

THEOREM 11.2 Properties of the Dot Product

Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — Commutative property
2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ — Associative property
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ — Distributive property

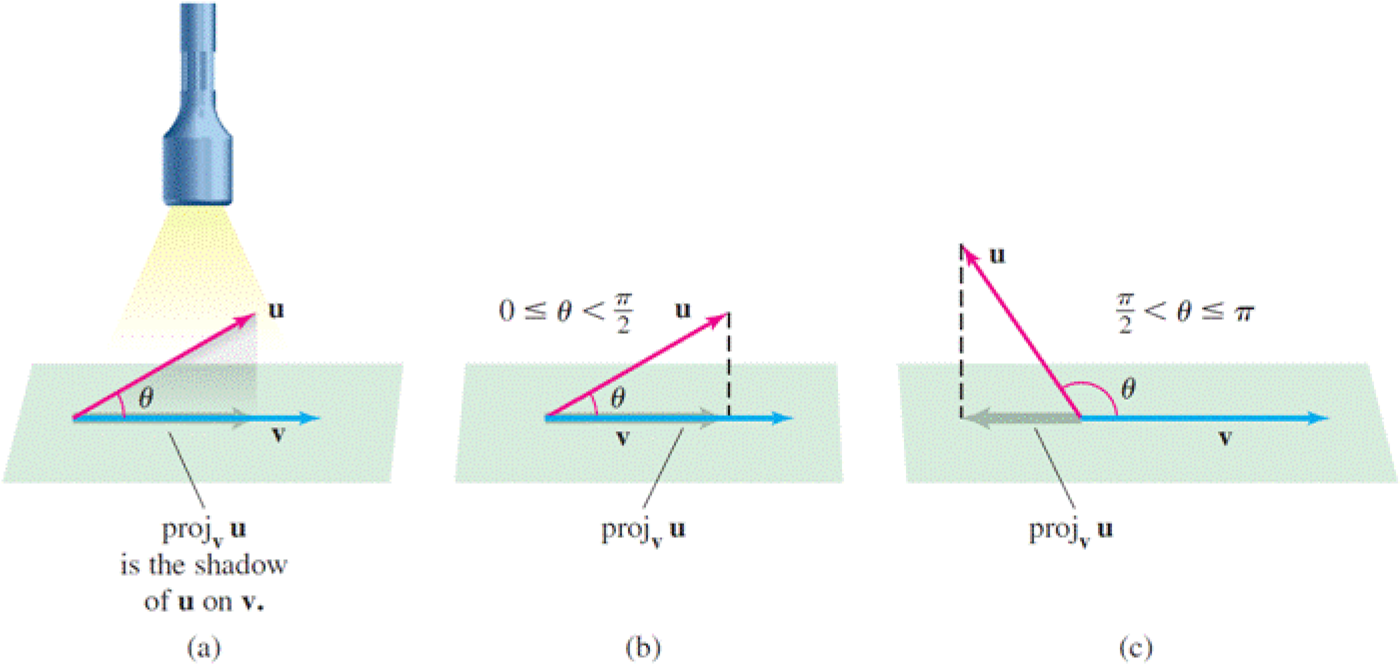

u
θ
v
proj$_{\mathbf{v}}$ u
is the shadow
of u on v.
(a)
$0 \le \theta < \frac{\pi}{2}$
u
θ
v
proj$_{\mathbf{v}}$ u
(b)
u
$\frac{\pi}{2} < \theta \le \pi$
θ
v
proj$_{\mathbf{v}}$ u
(c)

FIGURE 11.47

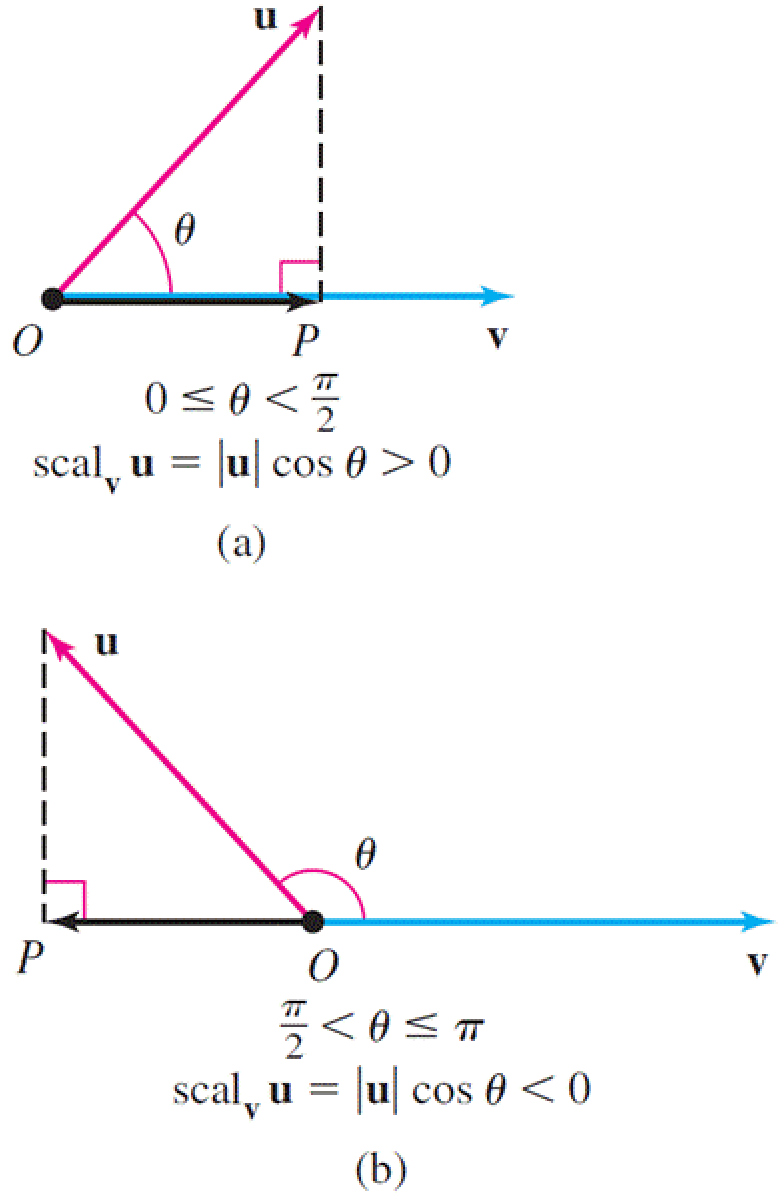

u
θ
O
P
v
$0 \le \theta < \frac{\pi}{2}$
$\text{scal}_{\mathbf{v}}\,\mathbf{u} = |\mathbf{u}| \cos\theta > 0$
(a)
u
θ
P
O
v
$\frac{\pi}{2} < \theta \le \pi$
$\text{scal}_{\mathbf{v}}\,\mathbf{u} = |\mathbf{u}| \cos\theta < 0$
(b)

FIGURE 11.48

DEFINITION (Orthogonal) Projection of u onto v

The **orthogonal projection of u onto v**, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right).$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v},$$

where the **scalar component of u in the direction of v** is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

11.4

Cross Products

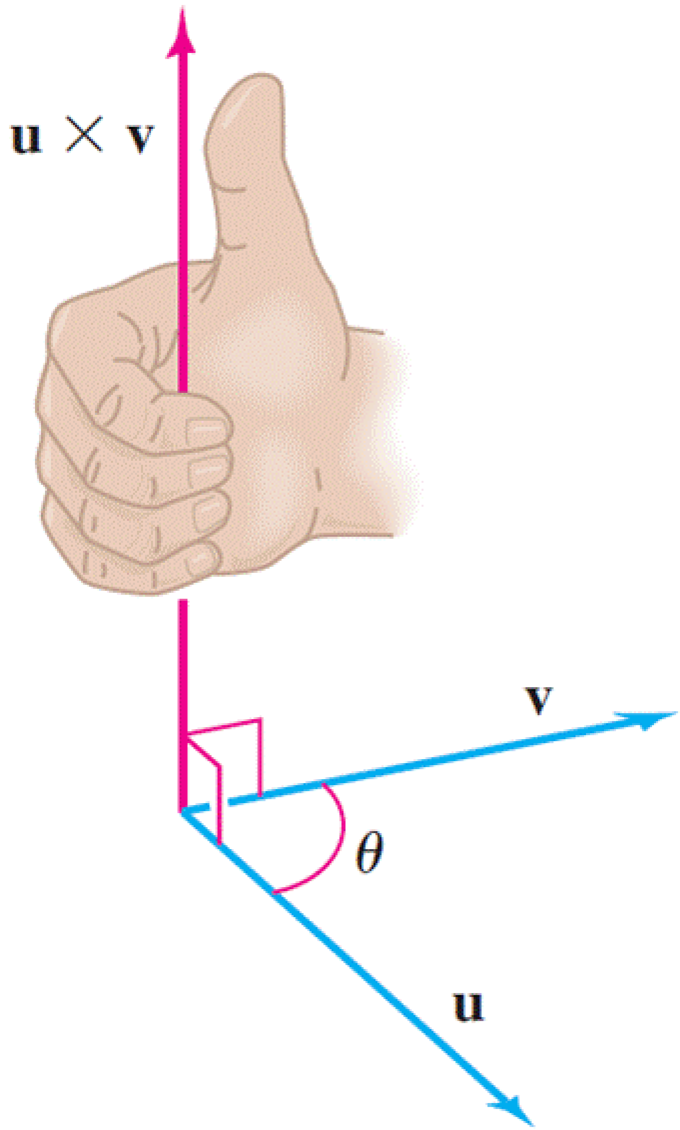


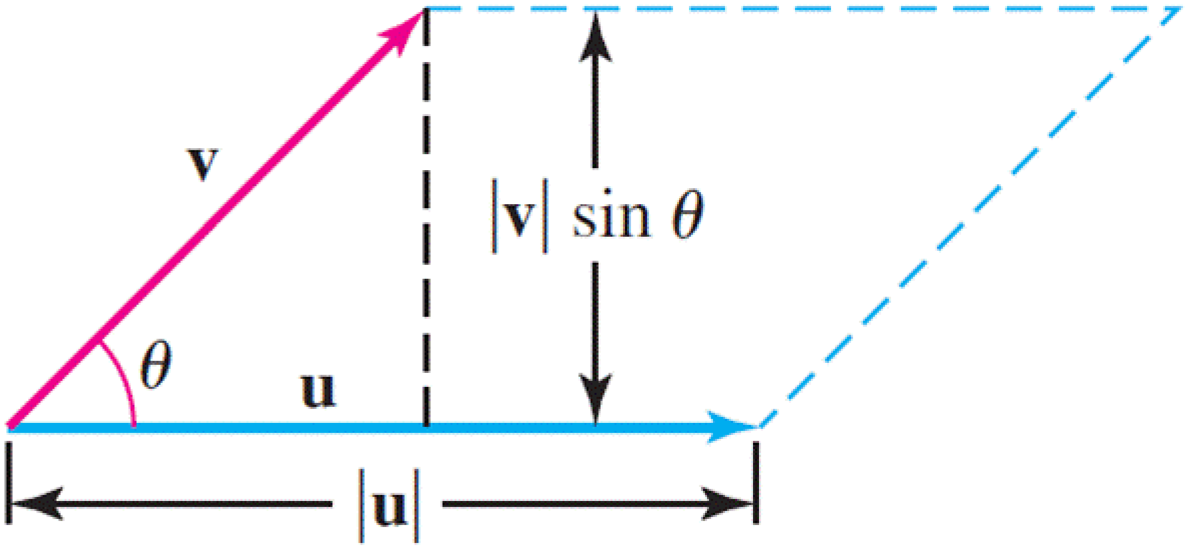
FIGURE 11.56

DEFINITION **Cross Product**

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta,$$

where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**: When you put the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$, the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ (Figure 11.56). When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.



$$\begin{aligned}\text{Area} &= \text{base} \times \text{height}\\ &= |\mathbf{u}||\mathbf{v}|\sin\theta\\ &= |\mathbf{u}\times\mathbf{v}|\end{aligned}$$

FIGURE 11.57

THEOREM 11.3 Geometry of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbf{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram (Figure 11.57), then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta.$$

THEOREM 11.4 Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbf{R}^3$, and let a and b be scalars.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ — Anticommutative property
2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ — Associative property
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ — Distributive property
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ — Distributive property

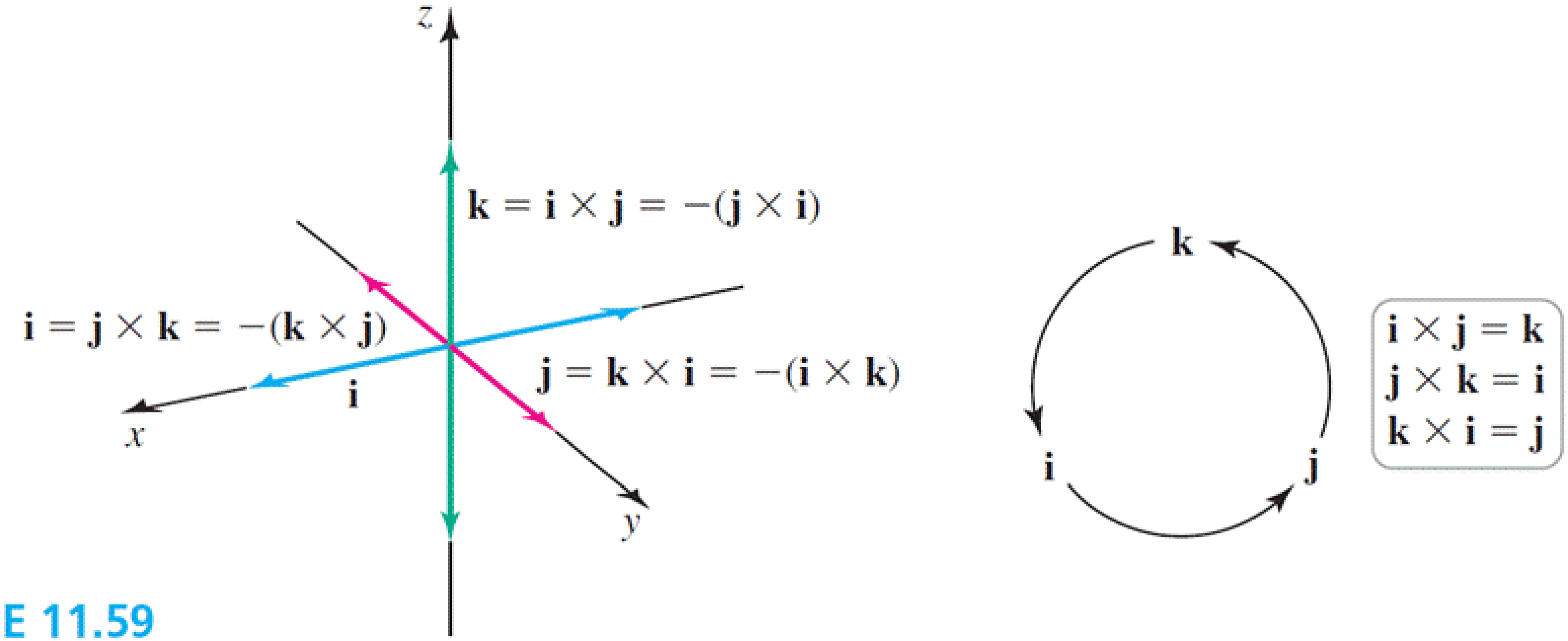


FIGURE 11.59

THEOREM 11.5 Cross Products of Coordinate Unit Vectors

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j} \qquad \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

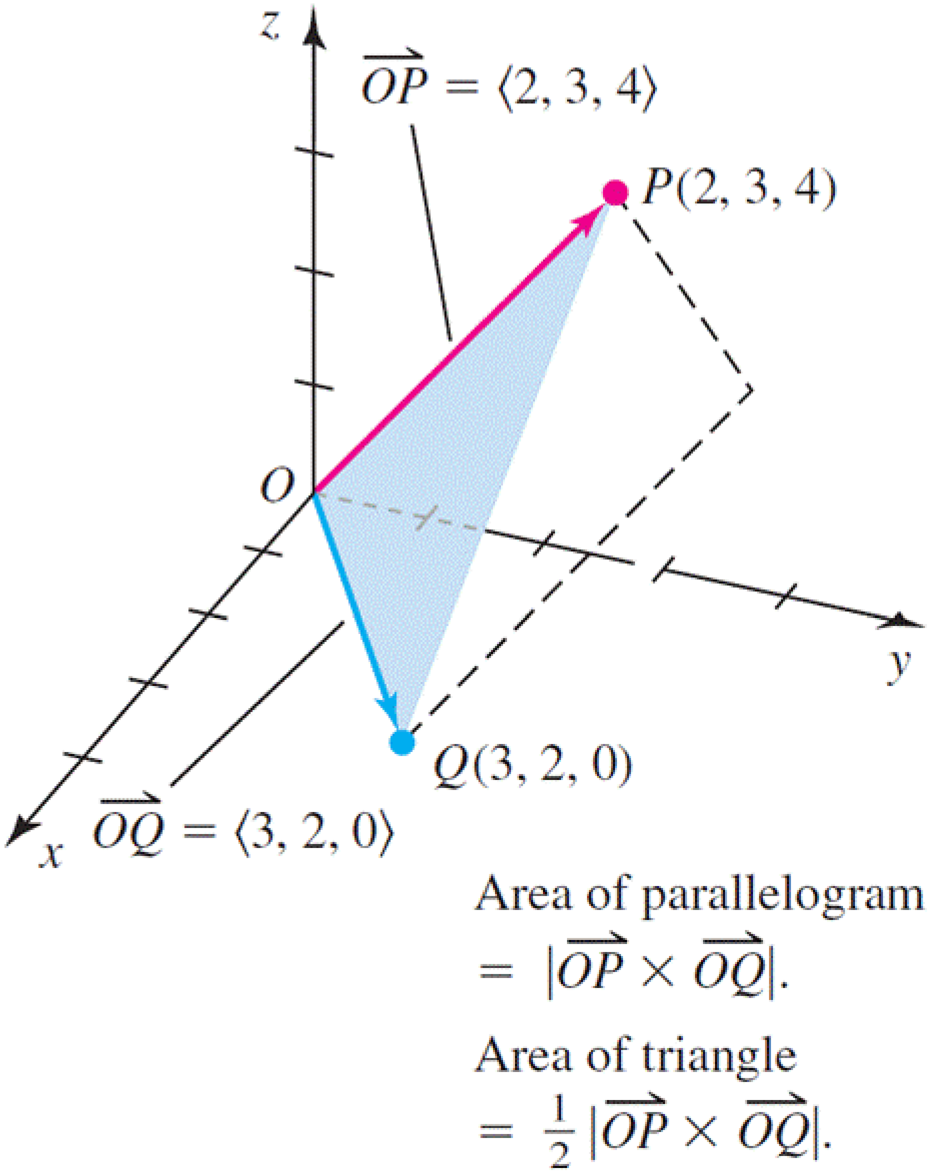

z
$\overrightarrow{OP} = \langle 2, 3, 4 \rangle$
$P(2, 3, 4)$
O
y
$Q(3, 2, 0)$
x
$\overrightarrow{OQ} = \langle 3, 2, 0 \rangle$
Area of parallelogram
$= |\overrightarrow{OP} \times \overrightarrow{OQ}|$.
Area of triangle
$= \frac{1}{2}|\overrightarrow{OP} \times \overrightarrow{OQ}|$.

FIGURE 11.60

THEOREM 11.6 Evaluating the Cross Product

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{k}.$$

11.5

Lines and Curves in Space

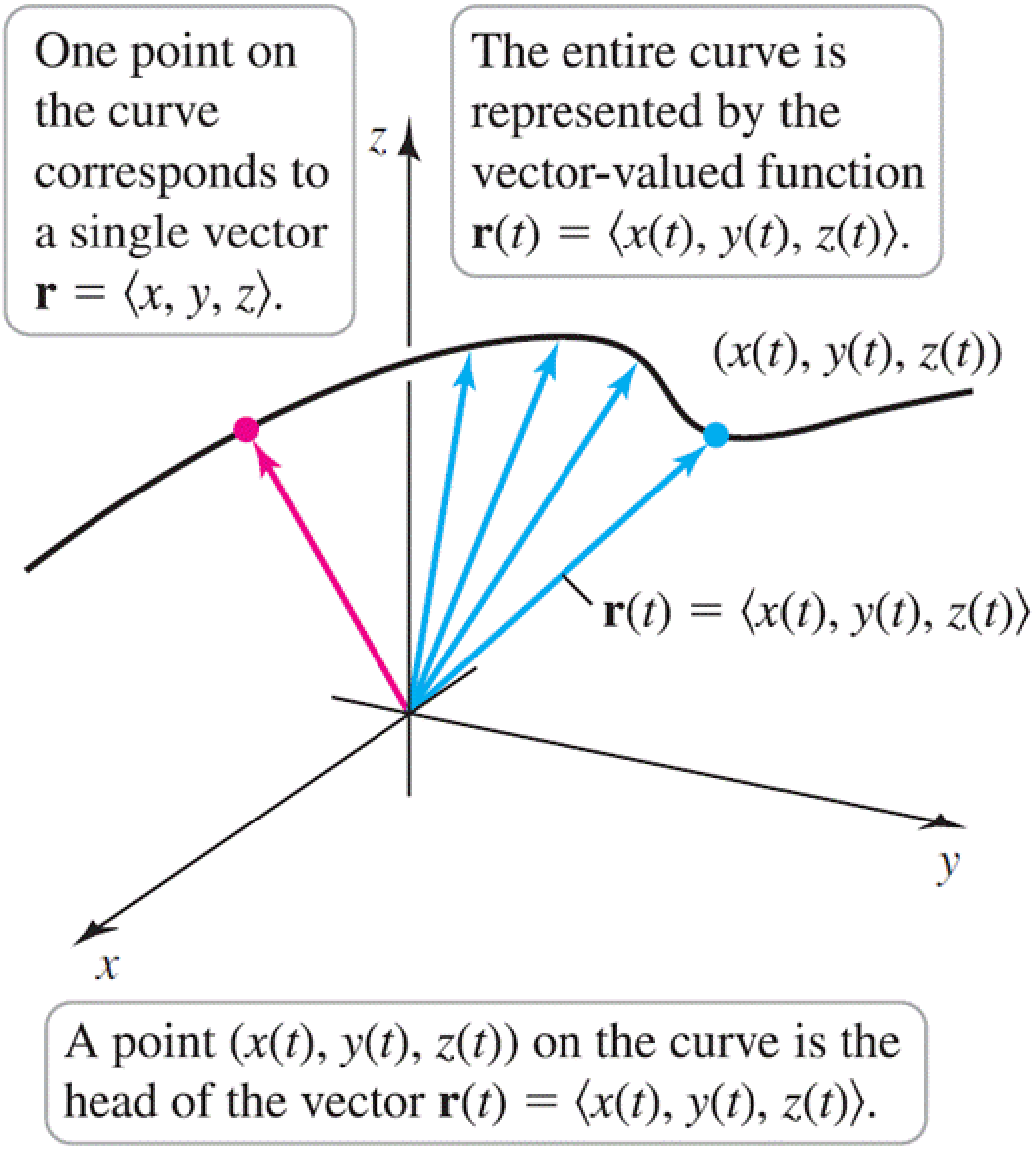

One point on the curve corresponds to a single vector $\mathbf{r} = \langle x, y, z \rangle$.
The entire curve is represented by the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.
z
$(x(t), y(t), z(t))$
$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$
y
x
A point $(x(t), y(t), z(t))$ on the curve is the head of the vector $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

FIGURE 11.66

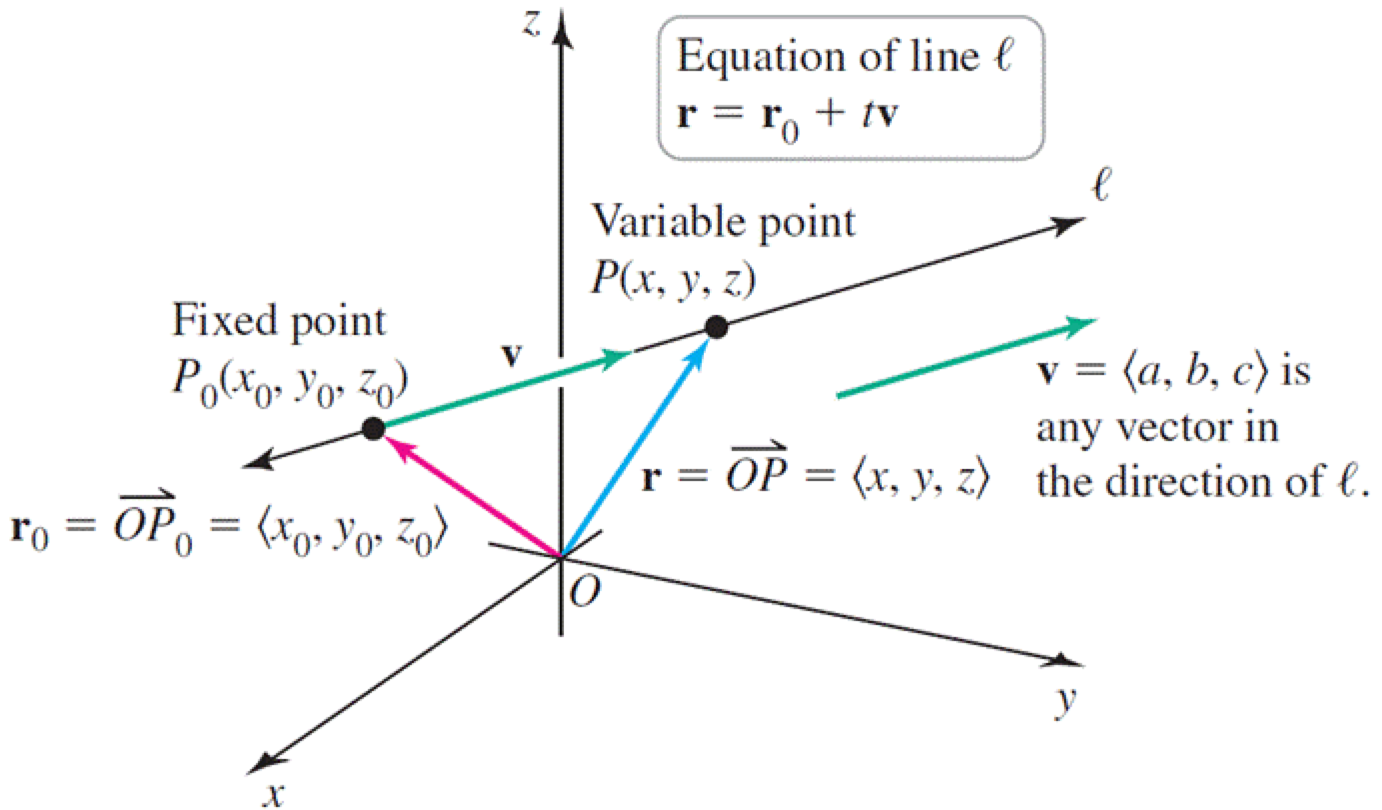

z
Equation of line ℓ
$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$
ℓ
Variable point
$P(x, y, z)$
Fixed point
$P_0(x_0, y_0, z_0)$
v
$\mathbf{v} = \langle a, b, c \rangle$ is
any vector in
the direction of ℓ.
$\mathbf{r} = \overrightarrow{OP} = \langle x, y, z \rangle$
$\mathbf{r}_0 = \overrightarrow{OP_0} = \langle x_0, y_0, z_0 \rangle$
O
y
x

FIGURE 11.67

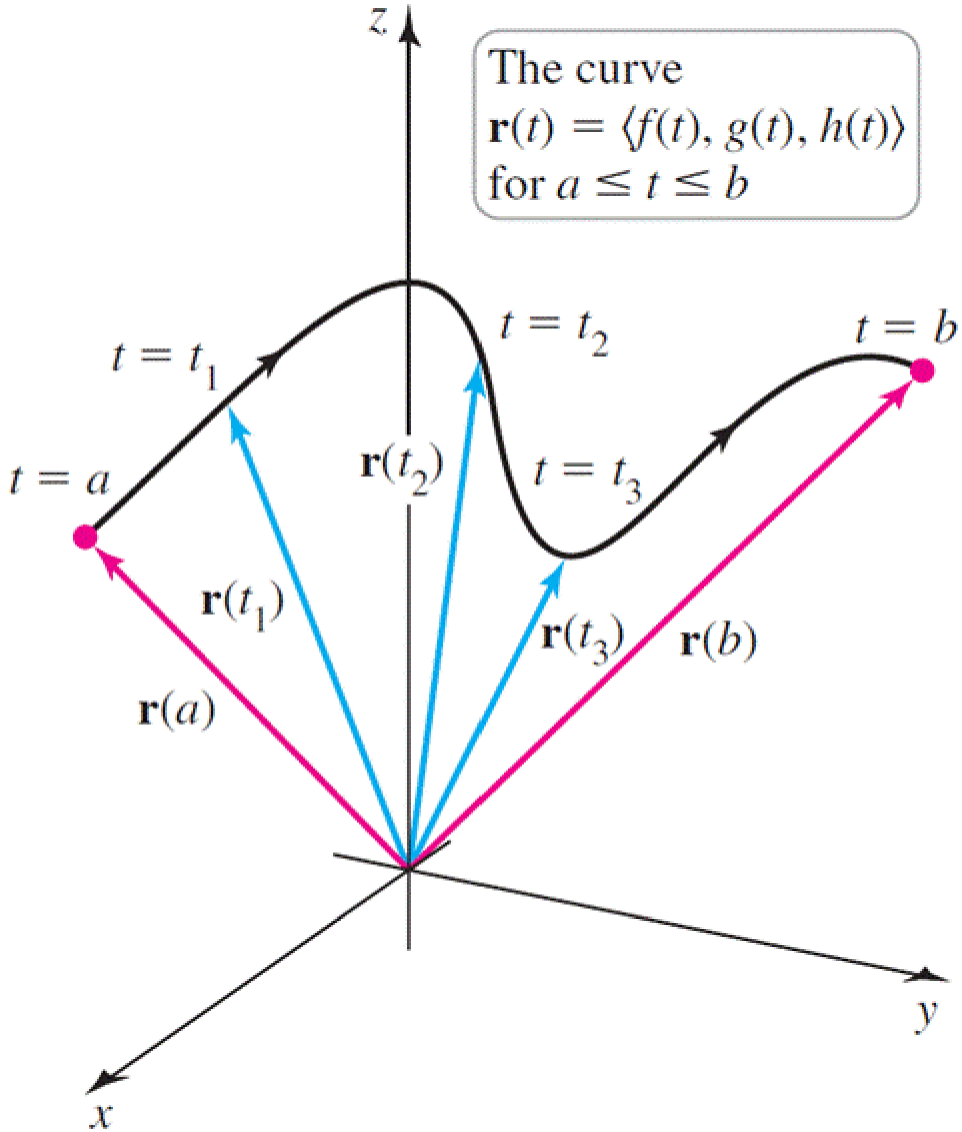

z
The curve
$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$
for $a \le t \le b$
$t = t_1$
$t = t_2$
$t = b$
$t = a$
$\mathbf{r}(t_2)$
$t = t_3$
$\mathbf{r}(t_1)$
$\mathbf{r}(t_3)$
$\mathbf{r}(b)$
$\mathbf{r}(a)$
y
x

FIGURE 11.71

DEFINITION **Limit of a Vector-Valued Function**

A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$.

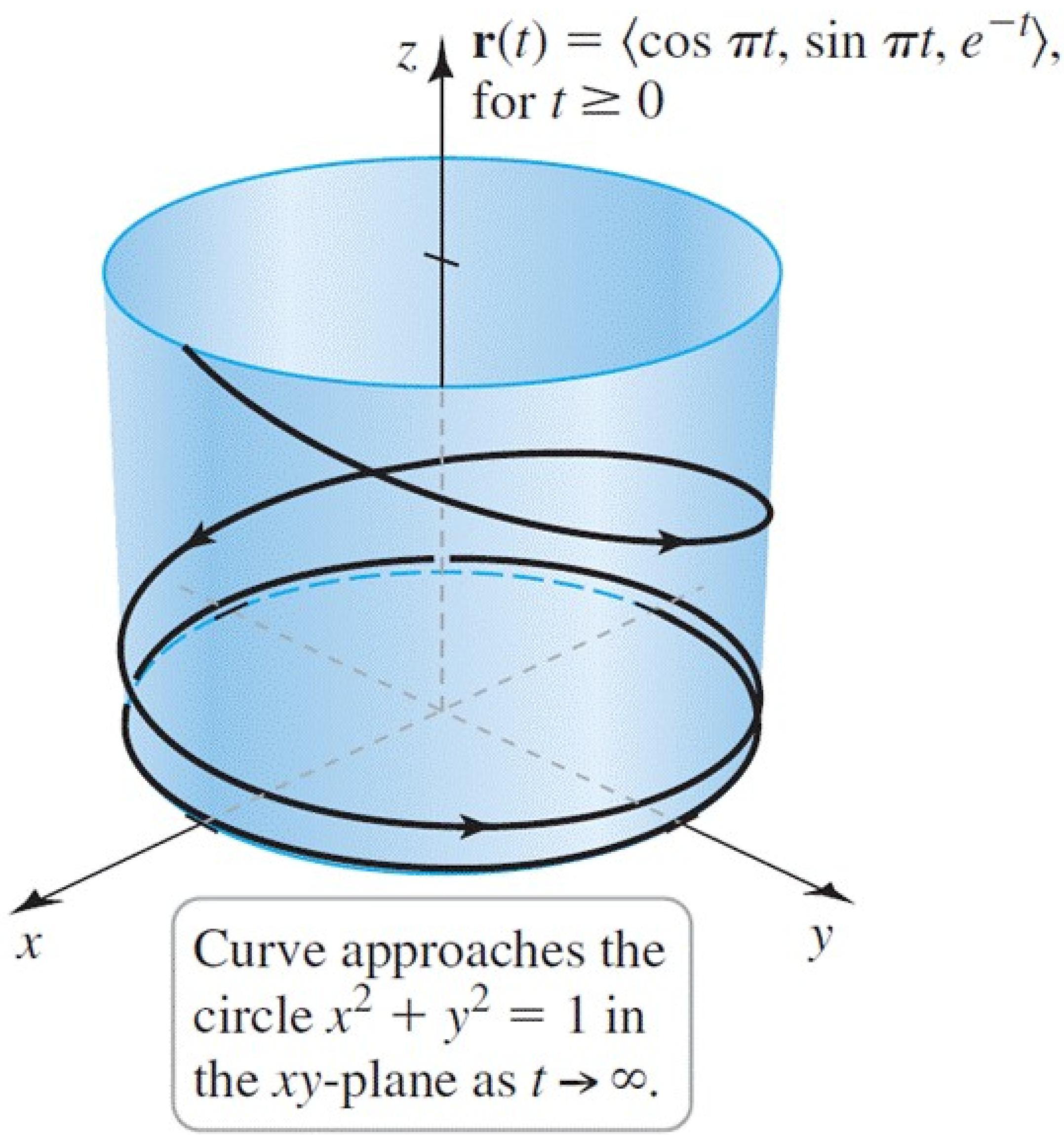

$\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, e^{-t} \rangle$,
for $t \geq 0$
z
x
y
Curve approaches the circle $x^2 + y^2 = 1$ in the xy-plane as $t \to \infty$.

FIGURE 11.76

11.6

Calculus of Vector-Valued Functions

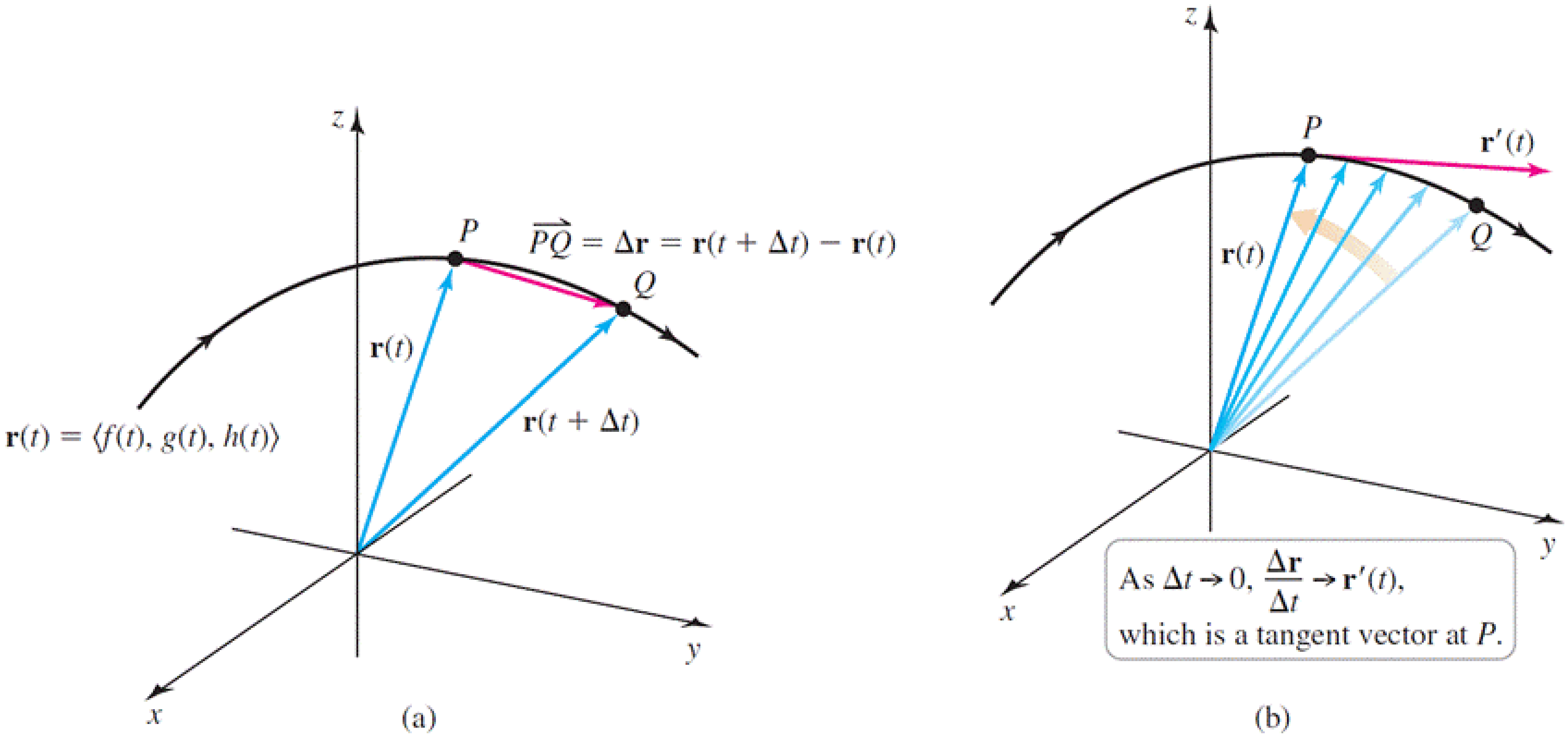

z
P
PQ = Δr = r(t + Δt) − r(t)
Q
r(t)
r(t + Δt)
r(t) = ⟨f(t), g(t), h(t)⟩
y
x
(a)
z
P
r′(t)
r(t)
Q
y
x
As Δt → 0, Δr/Δt → r′(t),
which is a tangent vector at P.
(b)

FIGURE 11.77

DEFINITION **Derivative and Tangent Vector**

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions on (a, b). Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on (a, b) and

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** (or velocity vector) at the point corresponding to $\mathbf{r}(t)$.

DEFINITION **Unit Tangent Vector**

Let $\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a smooth parameterized curve for $a \le t \le b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

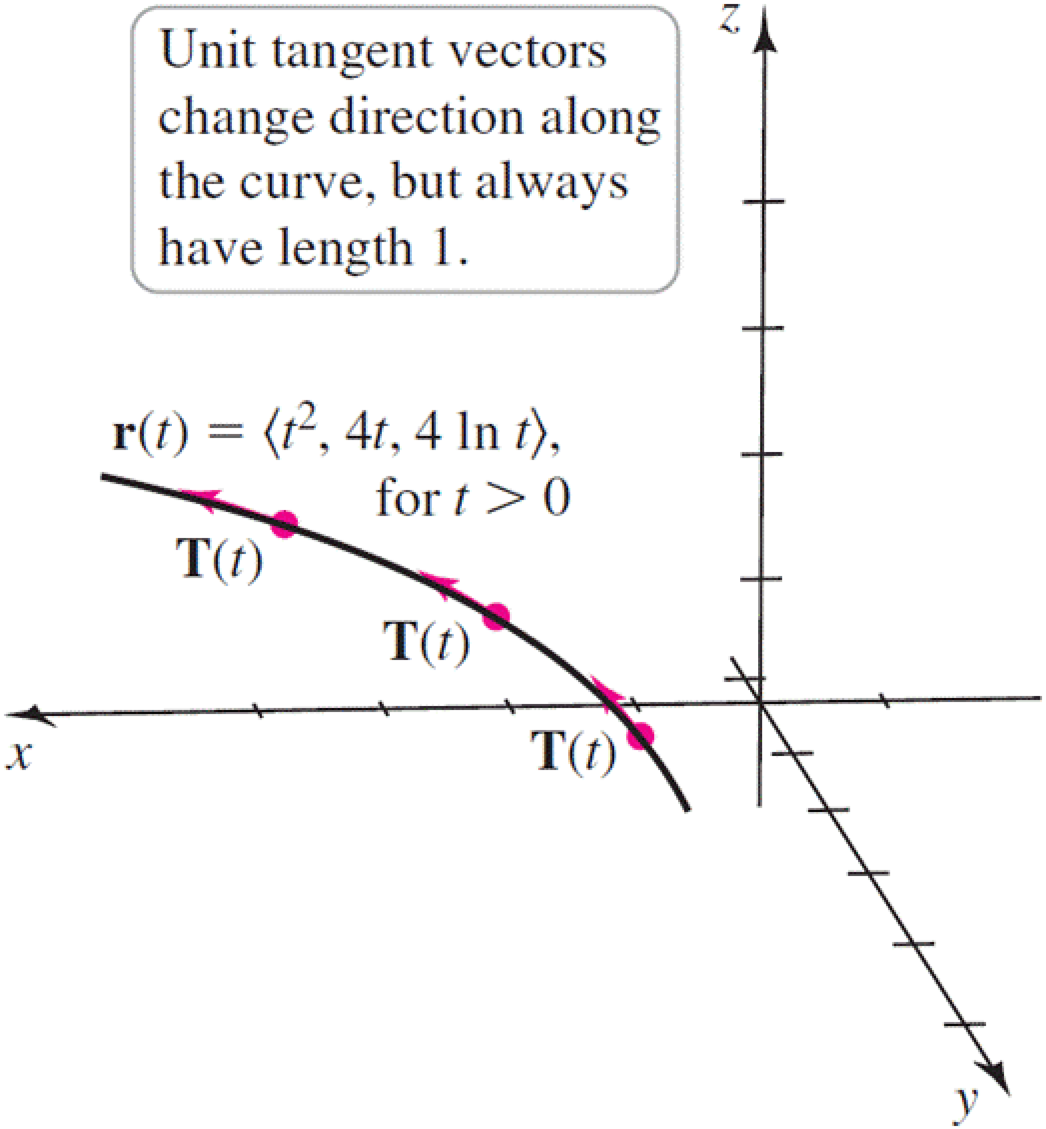

Unit tangent vectors change direction along the curve, but always have length 1.
r(t) = ⟨t², 4t, 4 ln t⟩, for t > 0
T(t)
T(t)
T(t)
x
y
z

FIGURE 11.80

THEOREM 11.7 Derivative Rules

Let **u** and **v** be differentiable vector-valued functions and let f be a differentiable scalar-valued function, all at a point t. Let **c** be a constant vector. The following rules apply.

1. $\frac{d}{dt}(\mathbf{c}) = \mathbf{0}$ Constant Rule
2. $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ Sum Rule
3. $\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ Product Rule
4. $\frac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$ Chain Rule
5. $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ Dot Product Rule
6. $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ Cross Product Rule

DEFINITION Indefinite Integral of a Vector-Valued Function

Let $\mathbf{r} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$ be a vector function and let $\mathbf{R} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The indefinite integral of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\, dt = \mathbf{R}(t) + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector.

DEFINITION Definite Integral of a Vector-Valued Function

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are integrable on the interval $[a, b]$.

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}$$

11.7

Motion in Space

PEARSON

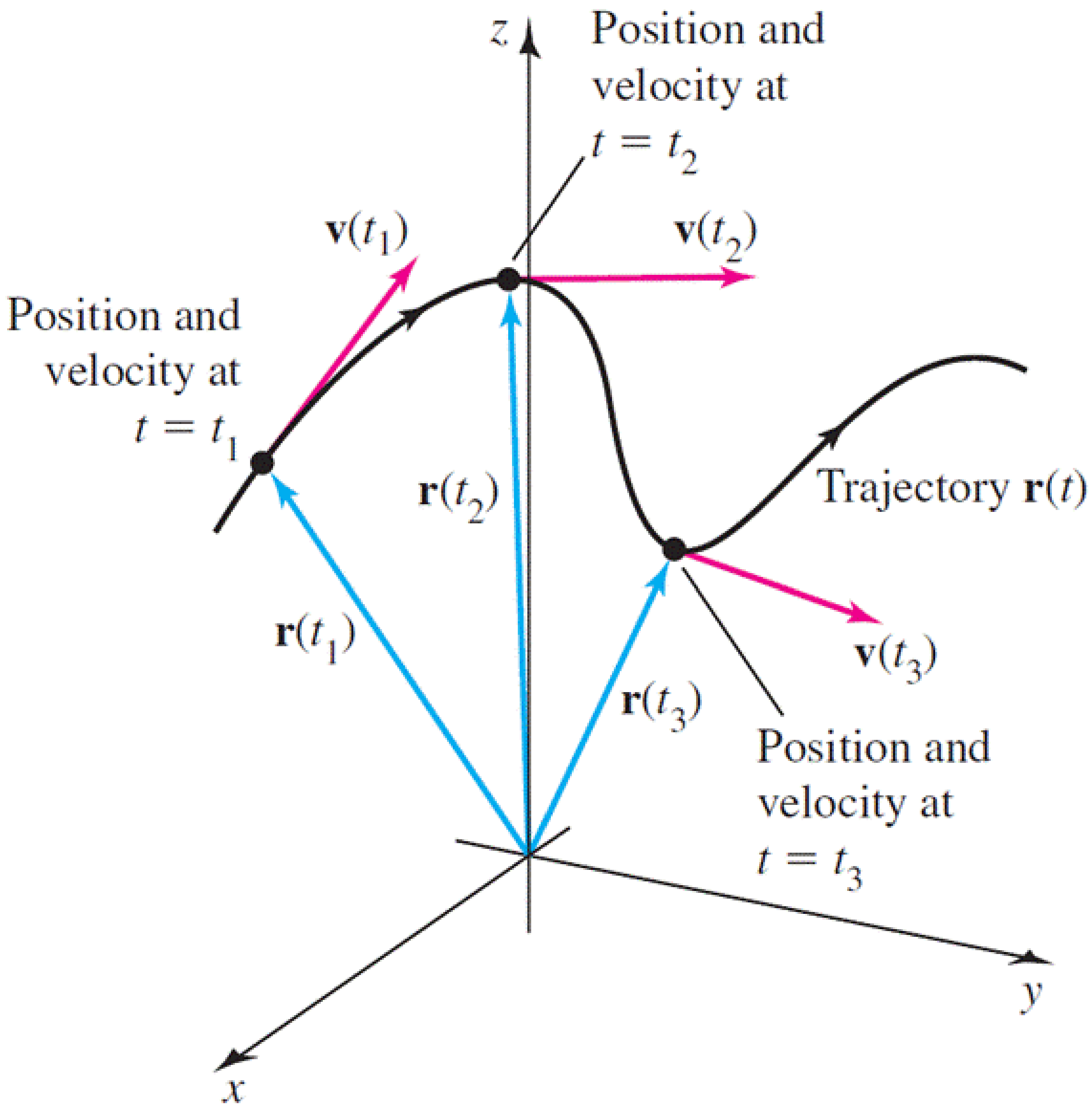

z
Position and velocity at $t = t_2$
$\mathbf{v}(t_1)$
$\mathbf{v}(t_2)$
Position and velocity at $t = t_1$
$\mathbf{r}(t_2)$
Trajectory $\mathbf{r}(t)$
$\mathbf{r}(t_1)$
$\mathbf{v}(t_3)$
$\mathbf{r}(t_3)$
Position and velocity at $t = t_3$
y
x

FIGURE 11.81

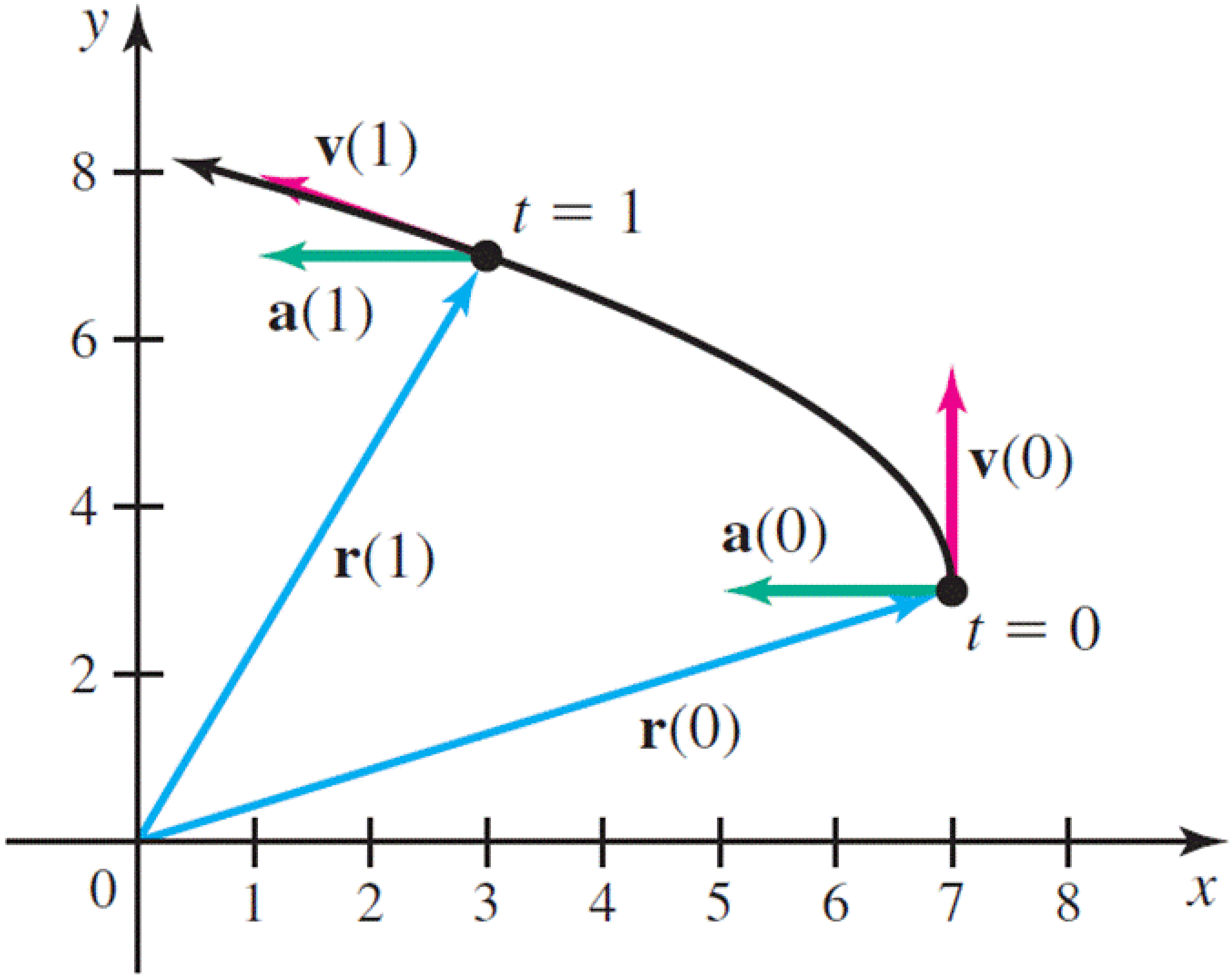


FIGURE 11.82

DEFINITION **Position, Velocity, Speed, Acceleration**

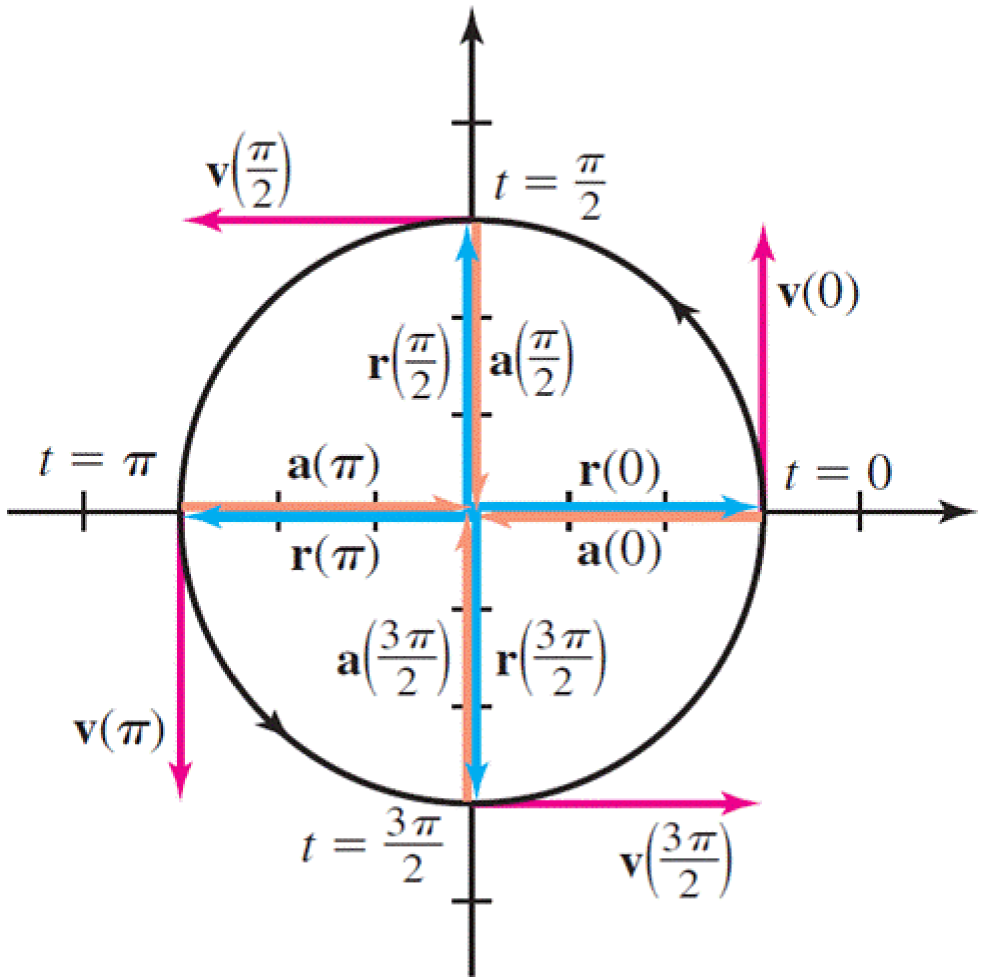
Let the **position** of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $t \geq 0$. The **velocity** of the object is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.



Circular motion: At all times $\mathbf{a}(t) = -\mathbf{r}(t)$ and $\mathbf{v}(t)$ is orthogonal to $\mathbf{r}(t)$ and $\mathbf{a}(t)$.

FIGURE 11.83

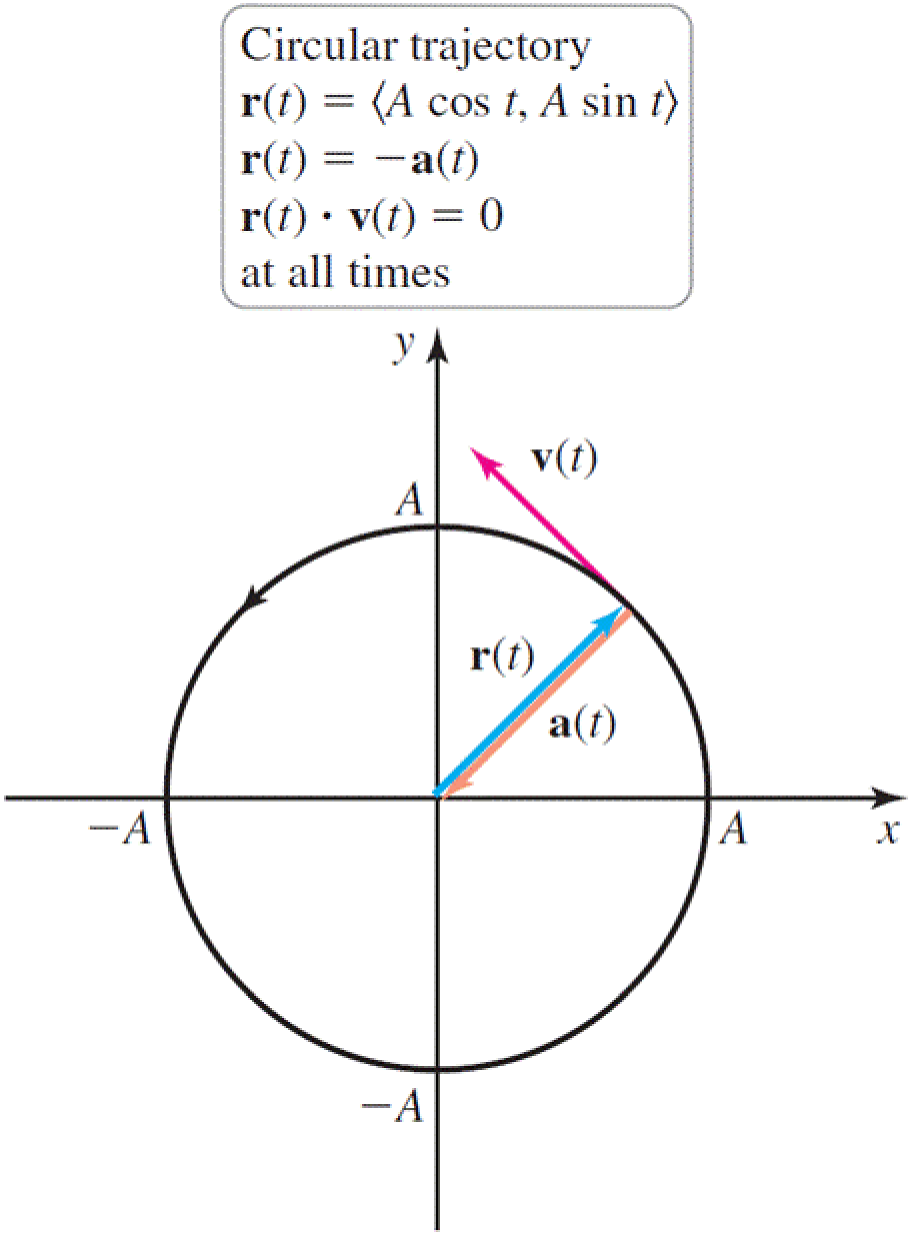
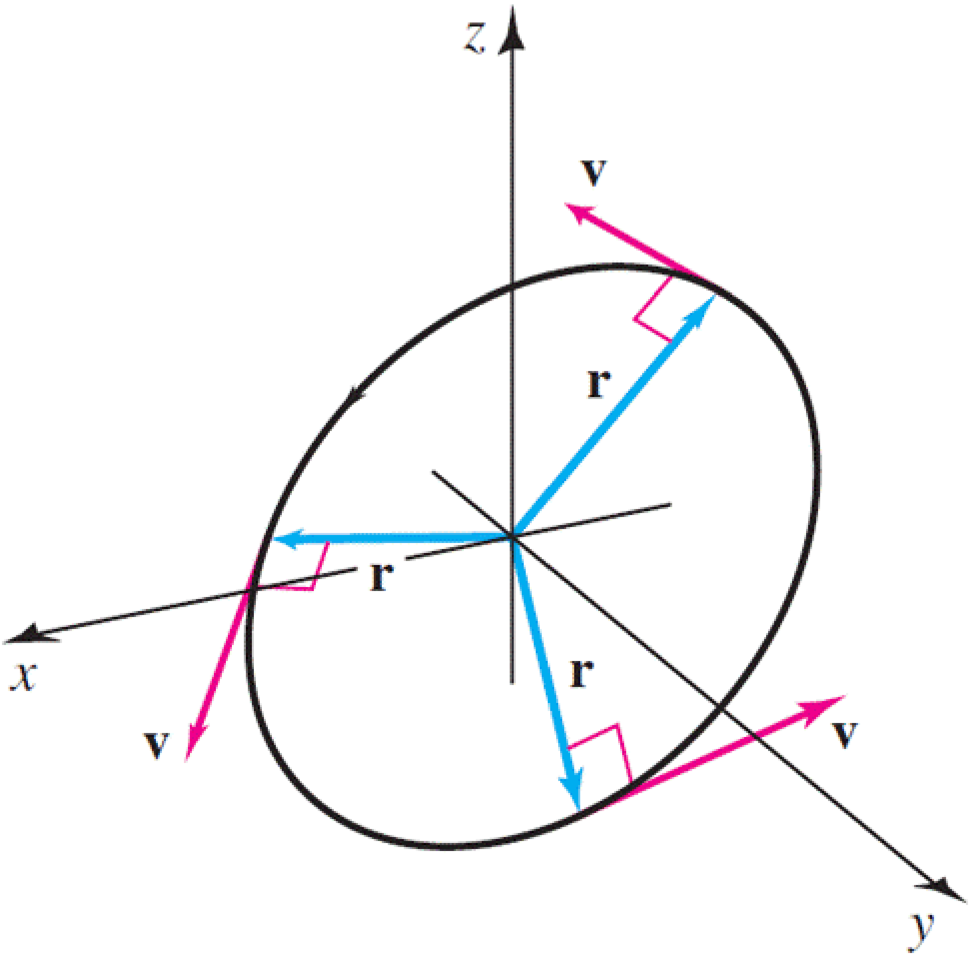

Circular trajectory
r(t) = ⟨A cos t, A sin t⟩
r(t) = −a(t)
r(t) · v(t) = 0
at all times
y
v(t)
A
r(t)
a(t)
−A
A
x
−A

FIGURE 11.86



On a trajectory on which $|\mathbf{r}(t)|$ is constant, $\mathbf{v}$ is orthogonal to $\mathbf{r}$ at all points.

FIGURE 11.87

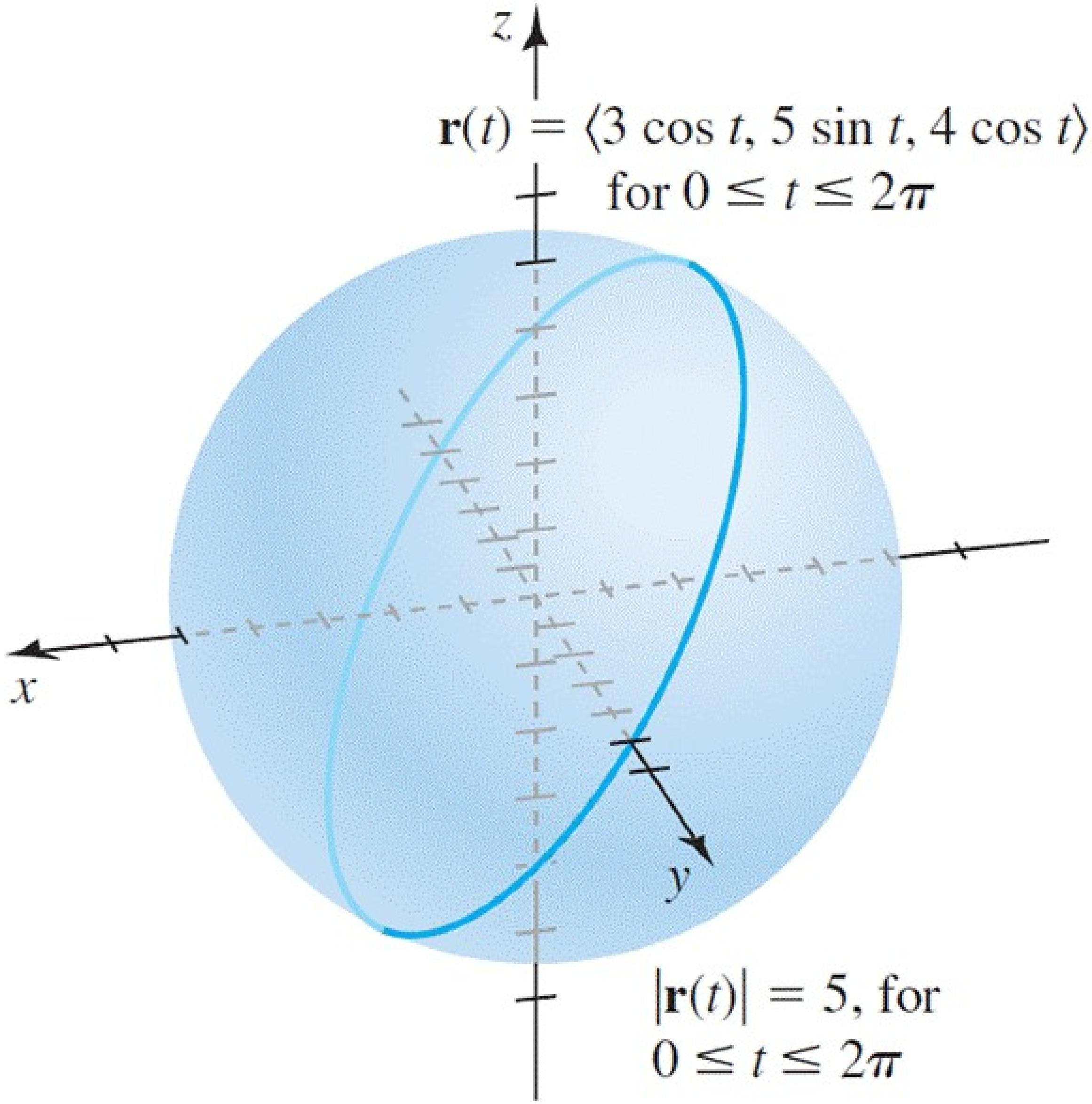
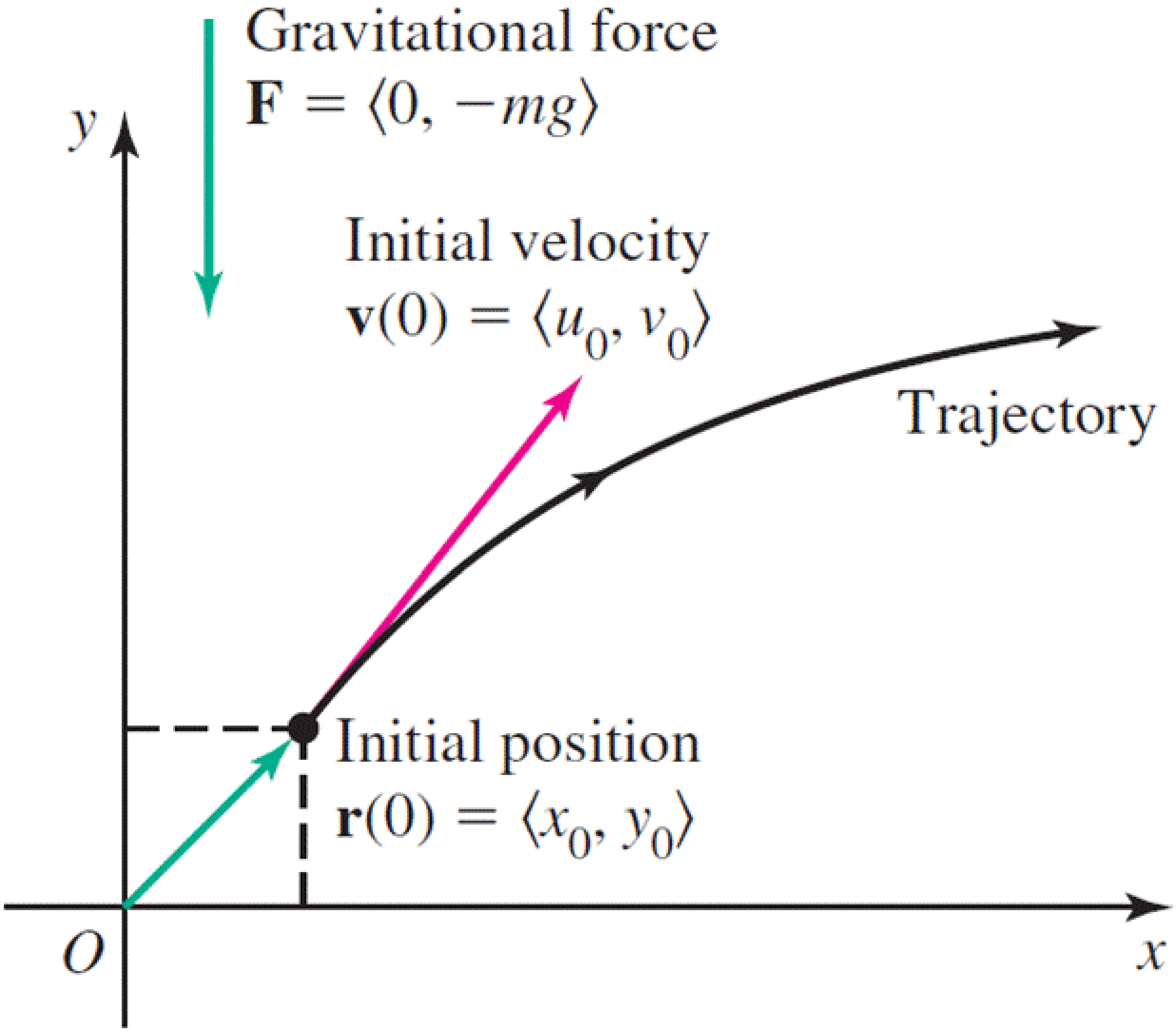


FIGURE 11.88



Gravitational force
F = ⟨0, −mg⟩
y
Initial velocity
v(0) = ⟨u0, v0⟩
Trajectory
Initial position
r(0) = ⟨x0, y0⟩
O
x

FIGURE 11.89

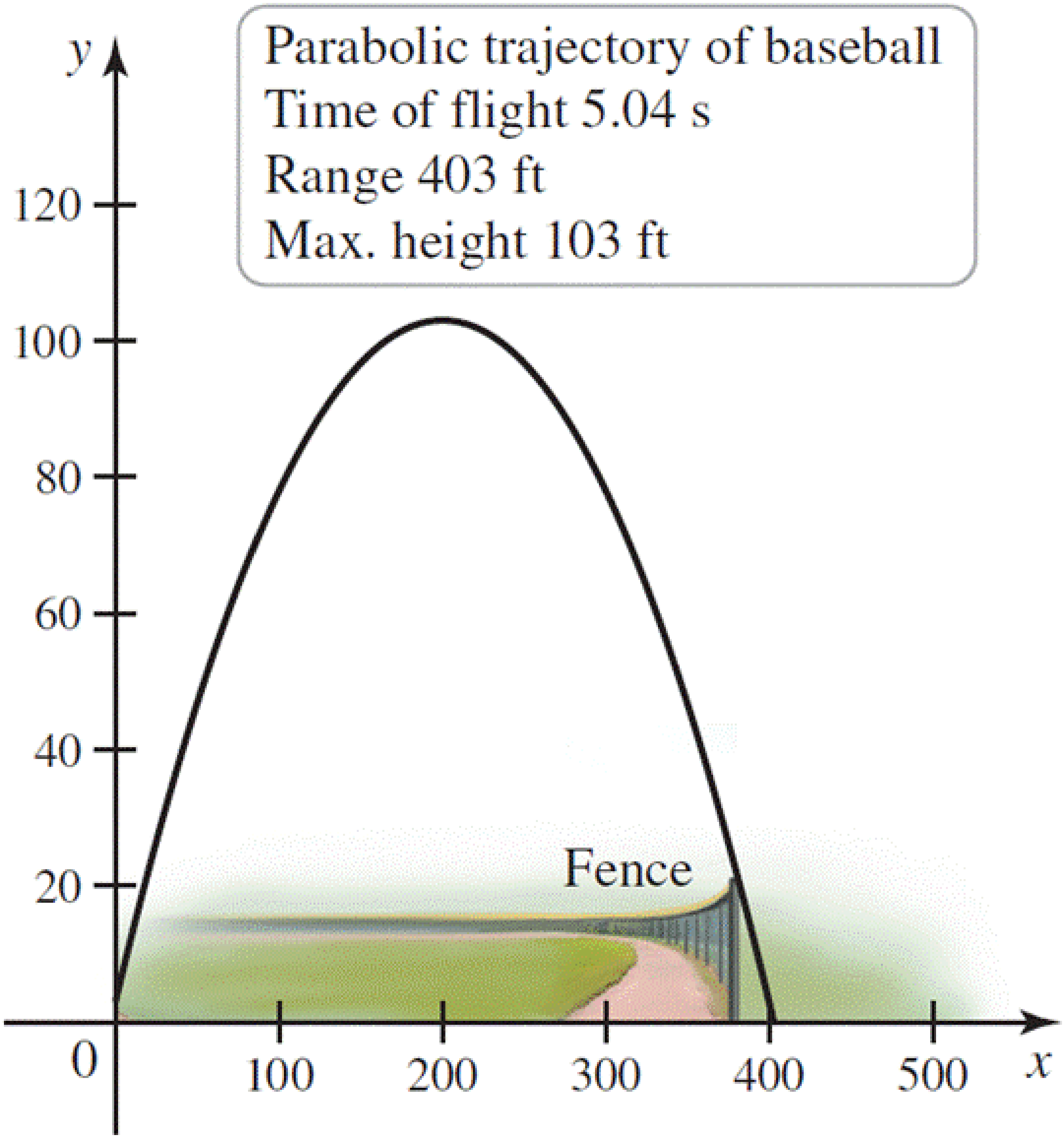
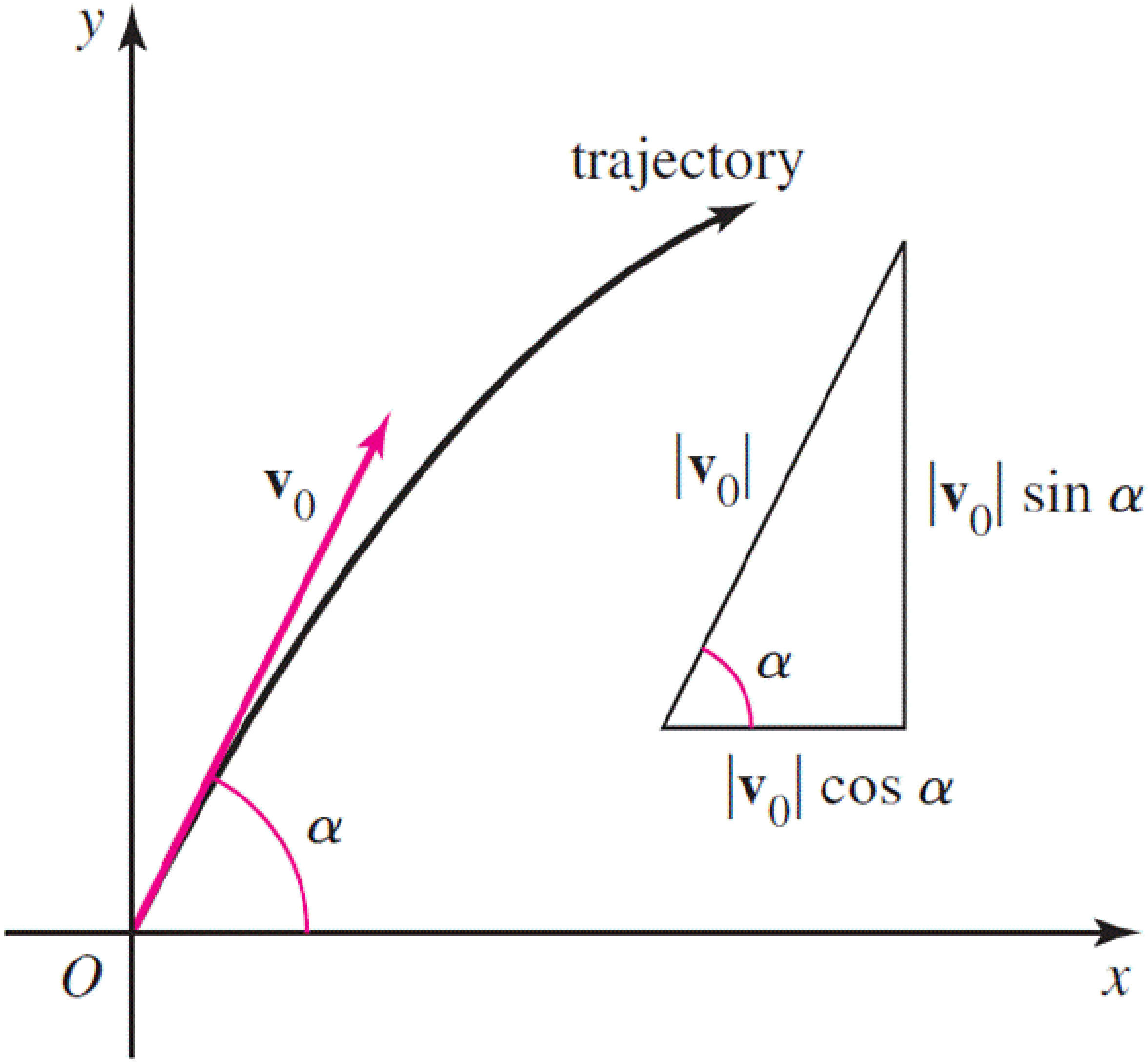
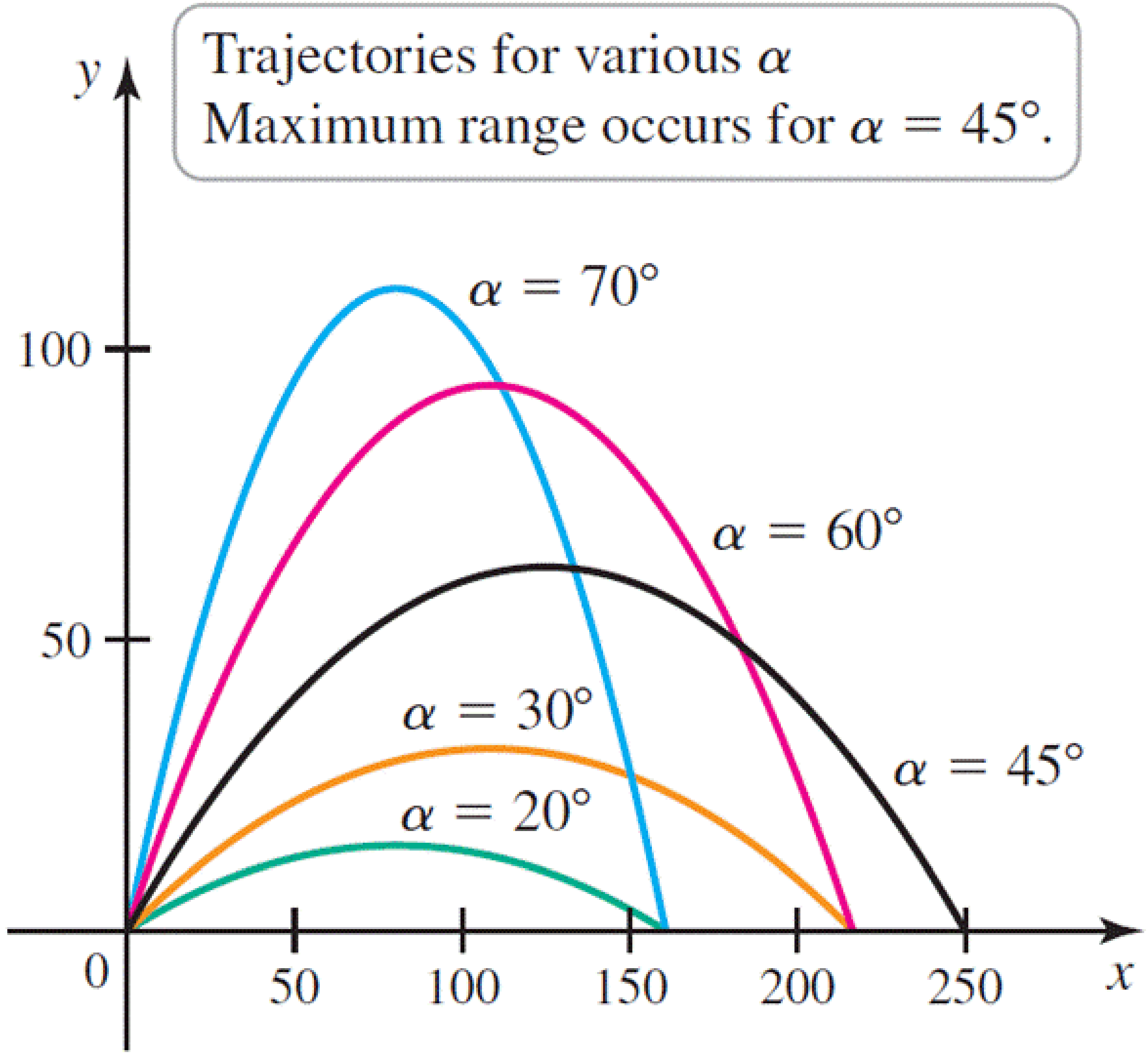

Parabolic trajectory of baseball
Time of flight 5.04 s
Range 403 ft
Max. height 103 ft
y
120
100
80
60
40
20
0
100
200
300
400
500
x
Fence

FIGURE 11.90



y
trajectory
$\mathbf{v}_0$
$|\mathbf{v}_0|$
$|\mathbf{v}_0| \sin \alpha$
α
$|\mathbf{v}_0| \cos \alpha$
α
O
x

FIGURE 11.91

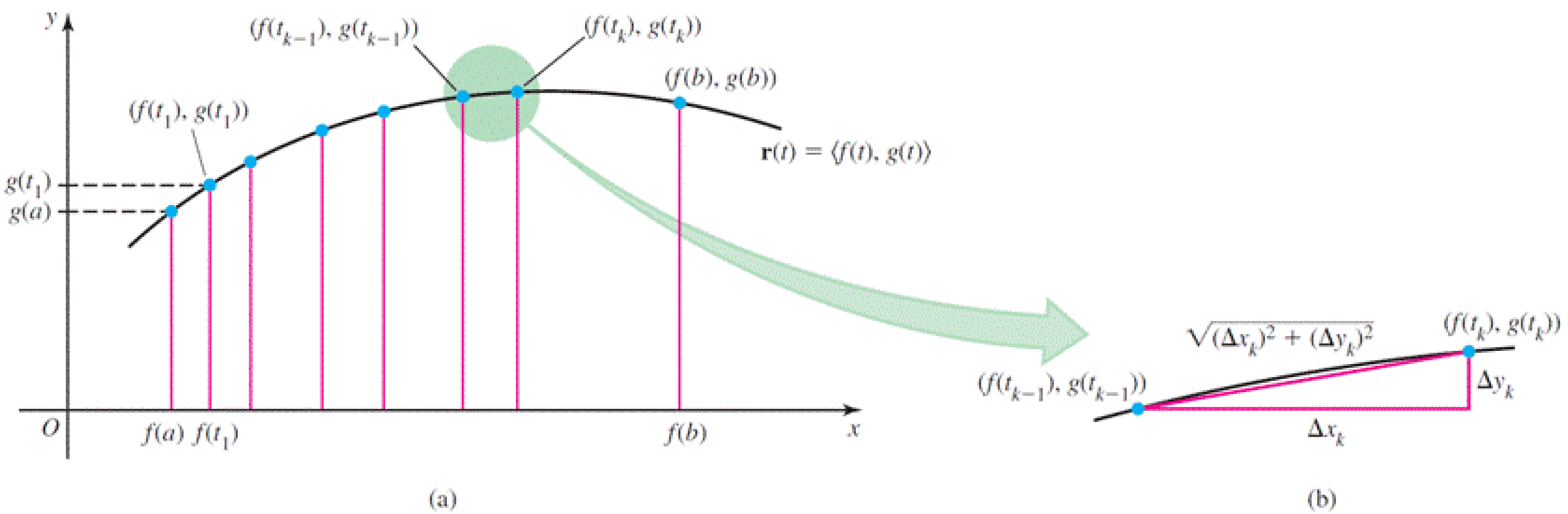


Trajectories for various α
Maximum range occurs for $\alpha = 45°$.
y
100
50
0
$\alpha = 70°$
$\alpha = 60°$
$\alpha = 45°$
$\alpha = 30°$
$\alpha = 20°$
50
100
150
200
250
x

FIGURE 11.92

11.8

Length of Curves



y
$(f(t_{k-1}), g(t_{k-1}))$
$(f(t_k), g(t_k))$
$(f(b), g(b))$
$(f(t_1), g(t_1))$
$g(t_1)$
$g(a)$
$\mathbf{r}(t) = \langle f(t), g(t) \rangle$
O
$f(a)$ $f(t_1)$
$f(b)$
x
(a)
$\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$
$(f(t_k), g(t_k))$
$(f(t_{k-1}), g(t_{k-1}))$
Δy_k
Δx_k
(b)

FIGURE 11.95

DEFINITION **Arc Length for Vector Functions**

Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f', g', and h' are continuous, and the curve is traversed once for $a \le t \le b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt.$$

11.9

Curvature and Normal Vectors

THEOREM 11.9 Arc Length as a Function of a Parameter

Let $\mathbf{r}(t)$ describe a smooth curve for $t \geq a$. The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du,$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\dfrac{ds}{dt} = |\mathbf{v}(t)| > 0$. If $|\mathbf{v}(t)| = 1$ for all $t \geq a$, then the parameter t is the arc length.

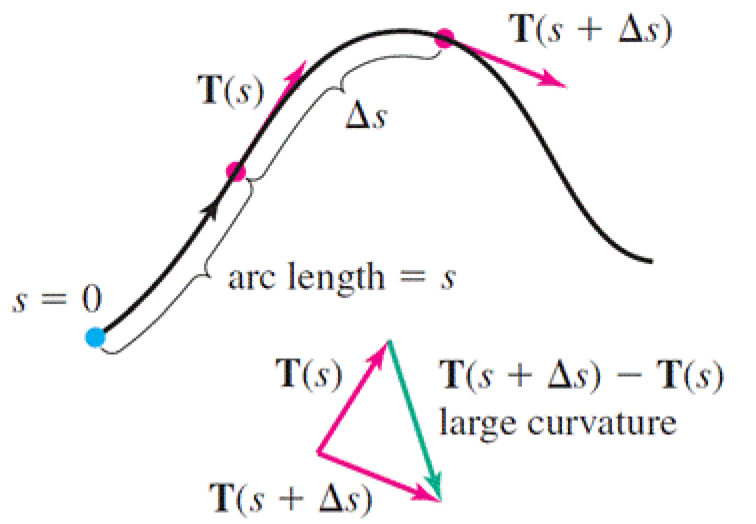


FIGURE 11.102 (a) (b)

DEFINITION **Curvature**

Let $\mathbf{r}$ describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ is the unit tangent vector, the **curvature** is $\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right|$.

THEOREM 11.10 Formula for Curvature

Let $\mathbf{r}(t)$ describe a smooth parameterized curve, where t is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{T}$ is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

THEOREM 11.11 Alternative Curvature Formula

Let $\mathbf{r}$ be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3},$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

DEFINITION **Principal Unit Normal Vector**

Let $\mathbf{r}$ describe a smooth parameterized curve. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

In practice, we use the equivalent formula

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

evaluated at the value of t corresponding to P.

THEOREM 11.12 Properties of the Principal Unit Normal Vector

Let $\mathbf{r}$ describe a smooth parameterized curve with unit tangent vector $\mathbf{T}$ and principal unit normal vector $\mathbf{N}$.

1. $\mathbf{T}$ and $\mathbf{N}$ are orthogonal at all points of the curve; that is, $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0$ at all points where $\mathbf{N}$ is defined.
2. The principal unit normal vector points to the inside of the curve—in the direction that the curve is turning.

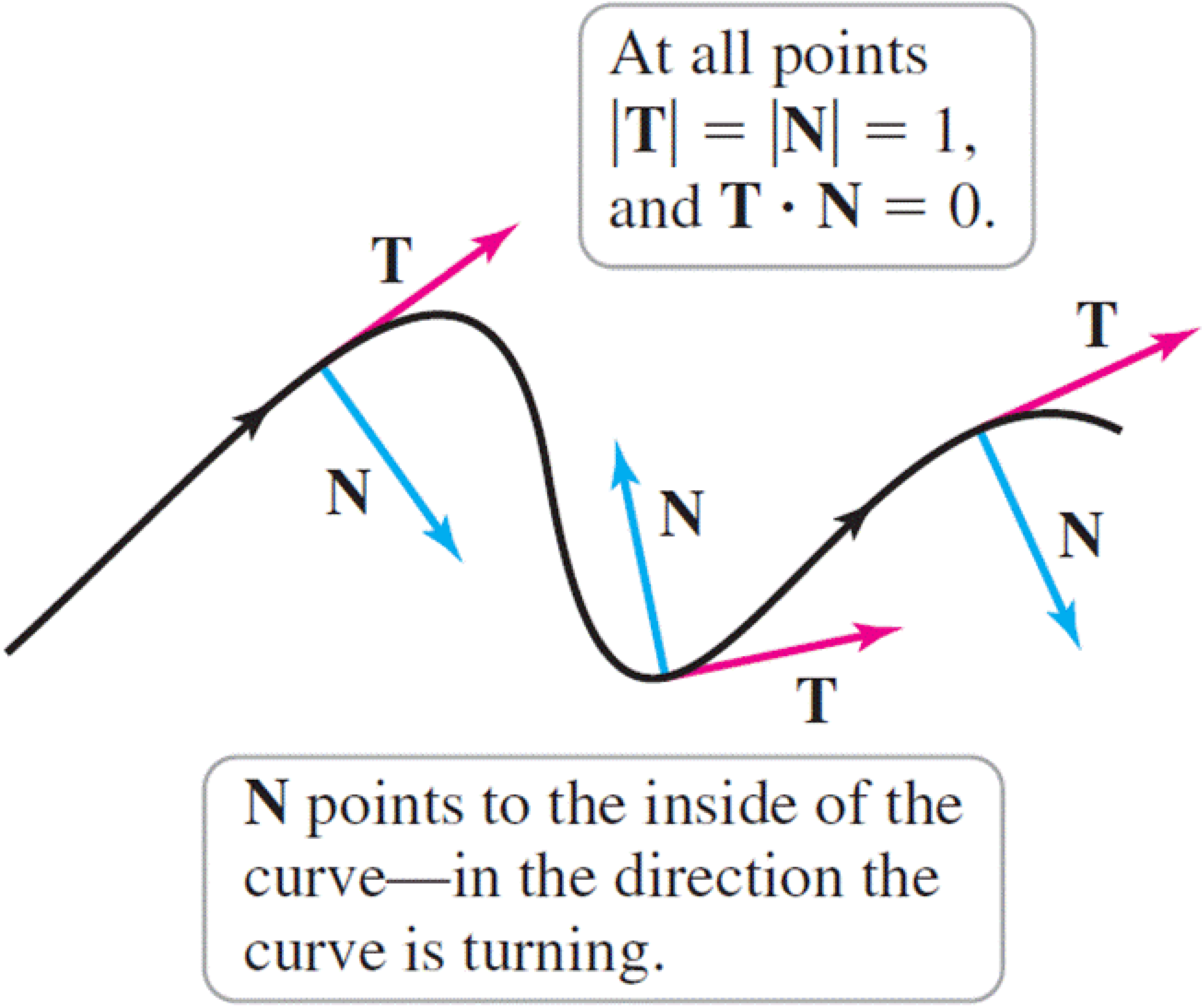


FIGURE 11.104

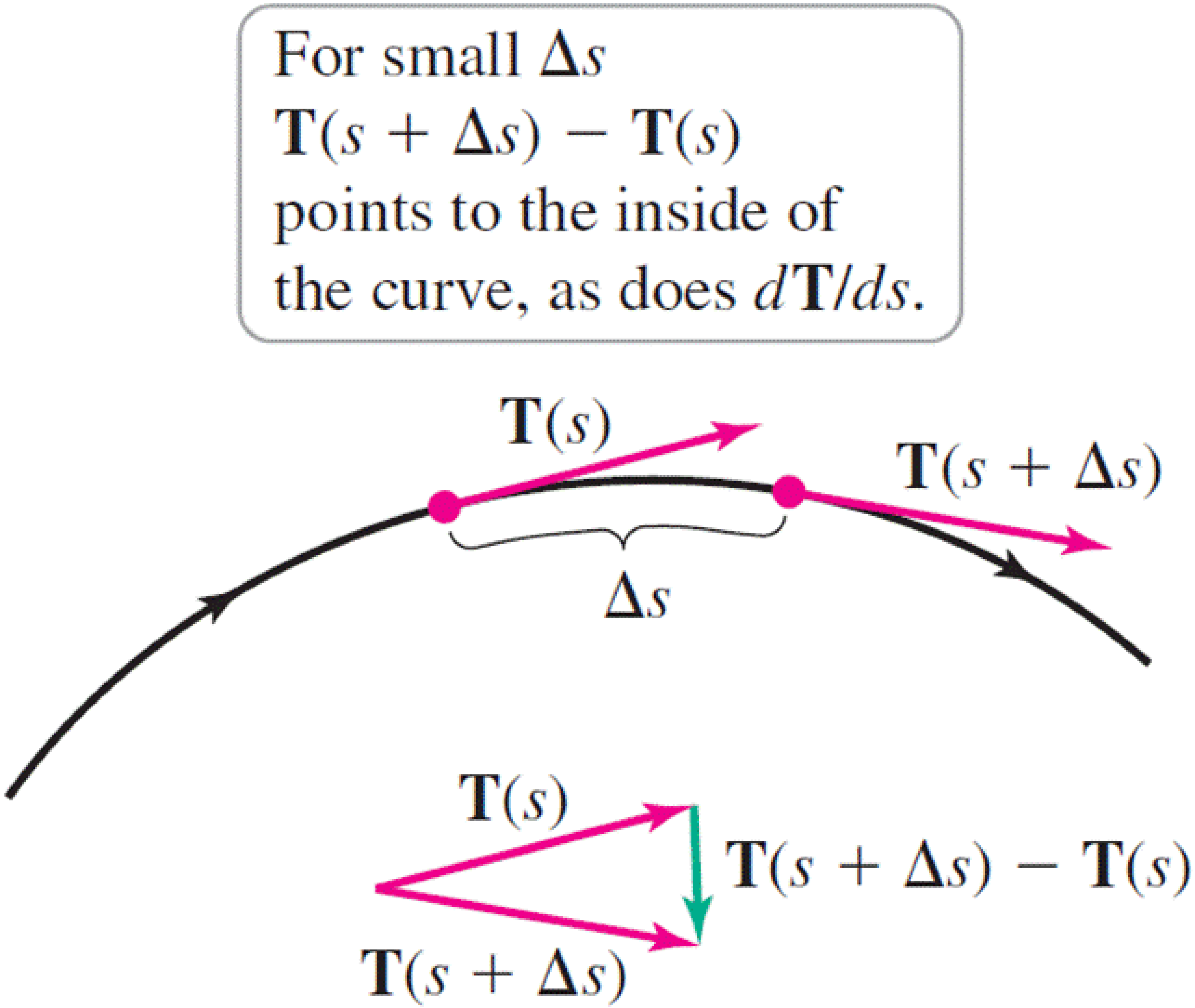


FIGURE 11.105

THEOREM 11.13 Tangential and Normal Components of the Acceleration

The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of $\mathbf{T}$) and its **normal component** a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N \mathbf{N} + a_T \mathbf{T},$$

where $a_N = \kappa |\mathbf{v}|^2 = \dfrac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|}$ and $a_T = \dfrac{d^2 s}{dt^2}$.

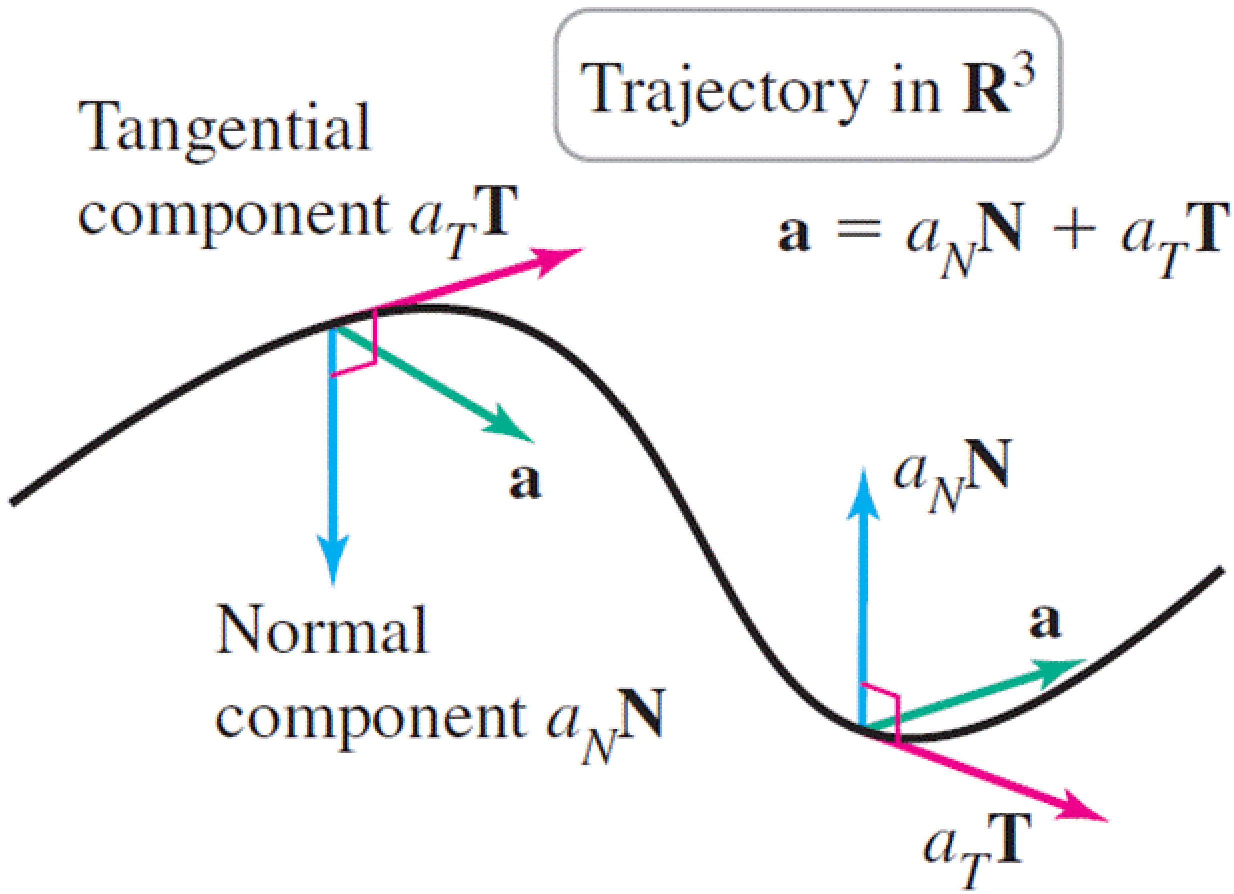


FIGURE 11.107