MTH 254 STUDY GUIDE

Summary of Topics

HRP, September 2010

Summary of Suggested Problems

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Lesson 1—Coordinate Systems

Polar coordinates

Briggs–Cochran: Section 10.2, pages 645–646

The polar coordinate system is illustrated in Figure 1. A point P is specified by its distance r from the **origin** O (also called the **pole**) and the oriented angle θ formed by the ray $\overrightarrow{OP}$ and the **polar axis**, the ray labeled $\overrightarrow{OX}$ in the figure. A counterclockwise rotation from $\overrightarrow{OX}$ to $\overrightarrow{OP}$ gives a positive angle, and a clockwise rotation gives a negative angle. Every point corresponds to infinitely many pairs of polar coordinates, because the point is unchanged when the angle is altered by adding or subtracting a multiple of 2π.

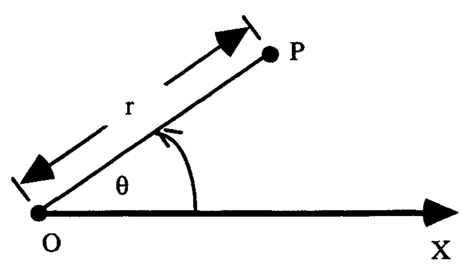

Figure 1: The Polar Coordinate System.

Points and sets of points can be plotted in polar coordinates using polar graph paper like that illustrated in Figure 2.

Usually when using polar coordinates, you will also be using a rectangular Cartesian coordinate system. In that case, the origins of the two coordinate systems are assumed to be the same point and the polar axis is assumed to be the positive x-axis. It is essential that you be able to switch back and forth between those two coordinate systems. Some simple trigonometry tells us

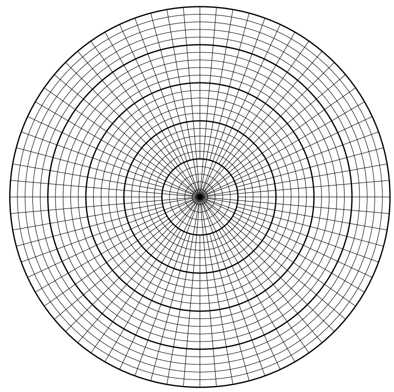

Figure 2: Polar Graph Paper.

that the conversion from polar coordinates to Cartesian coordinates is given by

$$\begin{aligned} x &= r\cos\theta\,, \\ y &= r\sin\theta\,. \end{aligned}$$

To convert in the other direction, note that the Pythagorean theorem tells us

$$r = \sqrt{x^2 + y^2}\,.$$

Obtaining θ from x and y is not as clean and clear, because of the possibility of accidentally dividing by zero. As long as $x \neq 0$, we have $\tan\theta = y/x$, so

$$\theta = \begin{cases} \arctan(y/x)\,, & \text{if } (x, y) \text{ is in the first or fourth quadrant,} \\ \arctan(y/x) + \pi\,, & \text{if } (x, y) \text{ is in the second or third quadrant.} \end{cases}$$

The equations $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$ are important enough and used enough that you should memorize them, even if you hate memorization.

Cartesian coordinates in 3-space

Briggs–Cochran: pages 693–695

In the three-dimensional rectangular Cartesian coordinate system each point in space is represented by a triple of numbers (x, y, z) where the value of x is the signed distance to the yz-plane, the value of y is the signed distance to the xz-plane, and the value of z is the signed distance to the xy-plane. This is illustrated in Figure 3 for the point $(1, 2, 3)$.

Unless something else is specified, assume that the xy-plane is horizontal and the positive z axis points upward. It is customary to use a right-handed coordinate system like the one shown in Figure 3. With the positive z-axis pointing upward as usual and with you looking down on the xy-plane, in a right-handed coordinate system it will require a counterclockwise rotation to rotate the positive x-axis onto the positive y-axis.

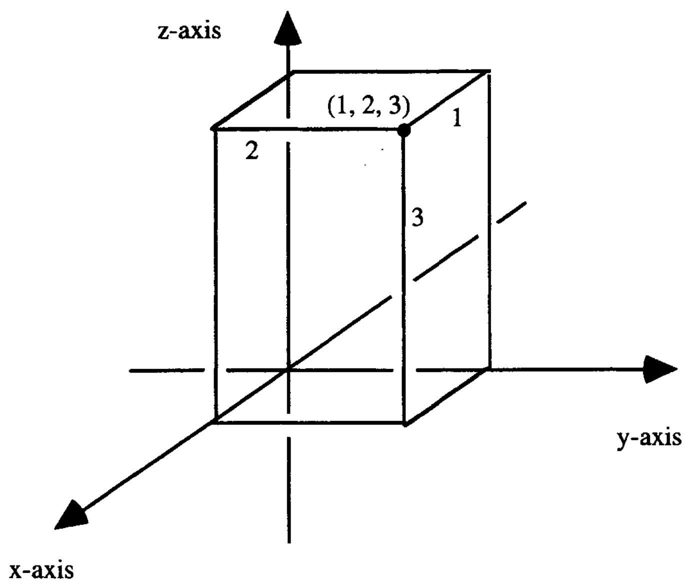

Figure 3: The Rectangular Cartesian Coordinate System.

Cylindrical coordinates

Briggs–Cochran: pages 918–920

The cylindrical coordinate system in 3-space uses polar coordinates in place of the x and y coordinates and leaves the third (usually vertical) coordinate z unaltered. This coordinate system is illustrated in Figure 4.

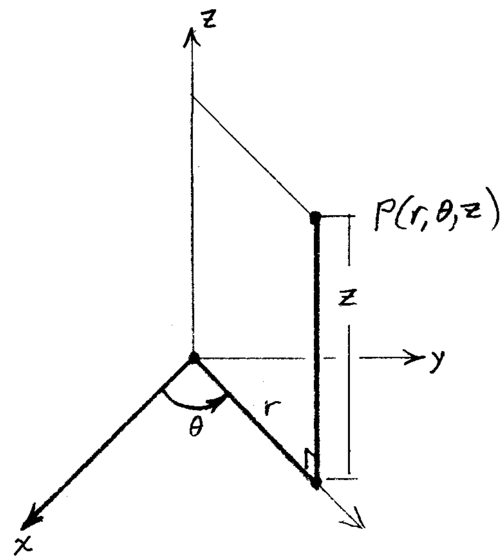

Figure 4: The Cylindrical Coordinate System.

Spherical coordinates

Briggs–Cochran: pages 924–926

The spherical coordinate system is often useful when dealing with a situation in which there is symmetry about a point in space. The rectangular coordinate system should be chosen so that the origin is the point of symmetry. One then converts between rectangular coordinates (x, y, z) and spherical coordinates (ρ, ϕ, θ) using the following formulas:

$$\begin{aligned} x &= \rho \, \sin\phi \, \cos\theta \, , \\ y &= \rho \, \sin\phi \, \sin\theta \, , \\ z &= \rho \, \cos\phi \, , \end{aligned}$$

and

$$
\begin{aligned}
\rho &= \sqrt{x^2 + y^2 + z^2}\,, \\
\tan\phi &= \sqrt{x^2 + y^2}/z \quad \text{if } z \neq 0\,, \\
\tan\theta &= y/x \quad \text{if } x \neq 0\,.
\end{aligned}
$$

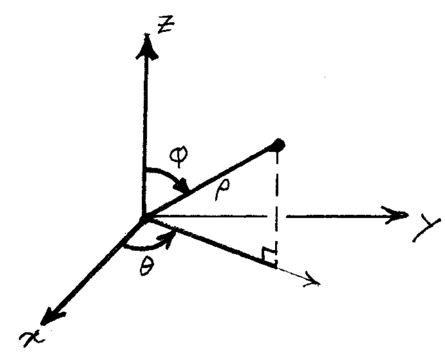

Figure 5: The Spherical Coordinate System.

The spherical coordinate ρ is the distance from the point to the origin. The angular coordinate ϕ in the spherical coordinate system is like the geographic coordinate of latitude, but ϕ is the angle measured downward from the North Pole, whereas the latitude is the angle north or south of the Equator. The angular coordinate θ in the spherical coordinate system agrees with the coordinate θ in the cylindrical coordinate system (so using the same symbol should be helpful rather than confusing). The spherical coordinate system is illustrated in Figure 5.

Listing the spherical coordinates in the order ρ, ϕ, θ is the convention used by American mathematicians, but it is not universal. More disturbingly, the names ϕ and θ are sometimes reversed, so check the definitions when using other sources.

Key skills for Lesson 1:

- Be able to describe sets of points using polar, cylindrical and spherical coordinates.

Exercises for Lesson 1

Textbook section 10.2, page 654

- (polar) # 27, 29, 31, 33, 35

Textbook section 13.5, pages 930–931

- (cylindrical) # 11, 13
- (spherical) # 35, 37

Lesson 2—Vectors in the Plane and in 3-Space
Briggs–Cochran: Sections 11.1 & 11.2, pages 680–703

This lesson is fundamental for the course, so total mastery is well worth the effort.

Vectors are introduced as quantities that have both a length (also called magnitude) and a direction. The book represents vectors with boldface letters like **v**, but when hand writing vectors we usually put a bar or arrow on top of the letter like $\overline{\mathrm{v}}$ or $\overrightarrow{\mathrm{v}}$.

- Two vectors with the same length and same direction are equal.

Vectors in the plane or in 3-space can be thought of as directed line segments that go from a starting point P to an ending point Q. The starting point is also called the "tail" and the ending point is also called the "head". This directed line segment is written $\overrightarrow{PQ}$. When thought of in terms of movement from P to Q, the vector $\overrightarrow{PQ}$ is called a **displacement vector**. Considering vectors to be displacement vector can be a useful metaphor when trying to understand the vector operations.

Other important examples of vectors for applications are **velocity**, **acceleration**, and **force**.

"Scalar" means "real number" in this book and in this course. The first operation on vectors is **scalar multiplication** of a vector by a real number. Scalar multiplication of the vector **v** by a positive real number r leaves the direction of the vector unchanged and multiplies the length by r. It might help you remember this if you think of 2**v** as scaling up the vector **v** by a factor of 2. When the scalar r is negative, the length is multiplied by $|r|$ (which equals $-r$ because we are considering the case when r is negative) and the direction is reversed.

If two vectors have the same direction or opposite directions, then the vectors are said to be **parallel**. The zero vector $\mathbf{0}$ is the vector with length 0. The convention in this book is that $\mathbf{0}$ has no direction, but is parallel to every vector.

- Two vectors are parallel if one is a scalar multiple of the other. As long as neither of the two parallel vectors is $\mathbf{0}$, then each is a scalar multiple of the other.

Vector addition is an operation on two vectors $\mathbf{u}$ and $\mathbf{v}$. The result of adding those two vectors is written $\mathbf{u}+\mathbf{v}$. The displacement vector metaphor helps you remember how to add vectors geometrically: If you put the starting point of $\mathbf{v}$ at the ending point of u, then $\mathbf{u}+\mathbf{v}$ goes from the starting point of $\mathbf{u}$ to the ending point of $\mathbf{v}$. The book calls this the **Triangle Rule**.

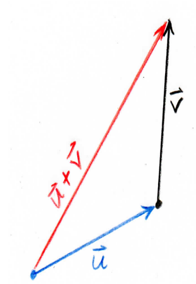

Figure 6: The Triangle Rule for vector addition.

To do calculations with numbers instead of geometrically, we need to write vectors using **components**. Assume you have a Cartesian coordinate system. Any vector $\mathbf{v}$ can be translated so that its starting point is at the origin O of the coordinate system. This is called the "standard position" for the vector. Once the starting point of $\mathbf{v}$ is at the origin, the ending point of the vector is at a point that we will call P. Then we have $\mathbf{v} = \overrightarrow{OP}$.

If you are working in the plane and the point P has Cartesian representation $(a,\, b)$, then we call the numbers a and b the x- and y-components of $\mathbf{v}$ and we write

$$\mathbf{v} = \langle a,\, b \rangle \,.$$

- $(a,\, b)$ is a point and $\langle\, a,\, b\,\rangle$ is a vector.

Now that we can write vectors using components, the operations become easy:

$$\begin{aligned} r\,\langle a,\, b\rangle &= \langle r\,a,\, r\,b\rangle \,, \\ \langle u_1,\, u_2\rangle + \langle v_1,\, v_2\rangle &= \langle u_1 + v_1,\, u_2 + v_2\rangle \,, \\ \langle u_1,\, u_2\rangle - \langle v_1,\, v_2\rangle &= \langle u_1 - v_1,\, u_2 - v_2\rangle \,. \end{aligned}$$

If you are working in 3-space and the point P has Cartesian representation $(a,\, b,\, c)$, then we call the numbers a, b, and c the x-, y-, and z-components of $\mathbf{v}$ and we write

$$\mathbf{v} = \langle\, a,\, b,\, c\,\rangle \,.$$

- $(a,\, b,\, c)$ is a point and $\langle\, a,\, b,\, c\,\rangle$ is a vector.

The operations become:

$$\begin{aligned} r\,\langle a,\, b,\, c\rangle &= \langle r\,a,\, r\,b,\, rc\rangle \,, \\ \langle u_1,\, u_2,\, u_3\rangle + \langle v_1,\, v_2,\, v_3\rangle &= \langle u_1 + v_1,\, u_2 + v_2,\, u_3 + v_3\rangle \,, \\ \langle u_1,\, u_2,\, u_3\rangle - \langle v_1,\, v_2,\, v_3\rangle &= \langle u_1 - v_1,\, u_2 - v_2,\, u_3 - v_3\rangle \,. \end{aligned}$$

The algebra of vectors is conceptually the same in two dimensions and in three dimensions—you simply have different numbers of coordinates to keep track of. Geometry and visualization are more challenging in 3-space.

A convenient way to make reasonably good pictures is to use the **isometric** method of drawing. This method is illustrated in the textbook in Figure 11.27 on page 694 and in Figure 7 below. Isometric means "having the same measure" and refers to the fact that a measurement in three space made along a line parallel to one of the coordinate axes is represented in the figure by the same distance (or the same scaled distance) in the direction parallel to the line representing that coordinate axis.

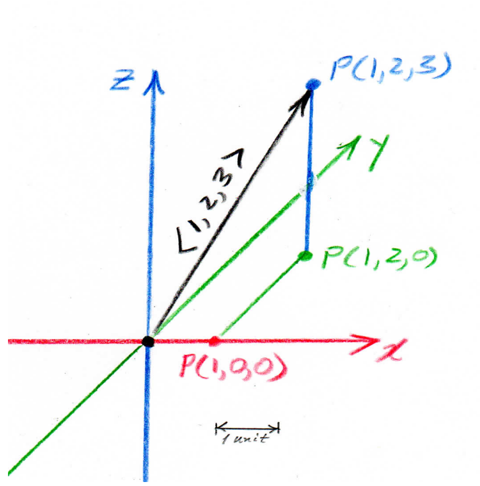

Figure 7: Isometric Drawing.

The **magnitude** or **length** of the vector **v** is written $|\mathbf{v}|$. The Pythagorean theorem tells us that

- $|\langle v_1, v_2\rangle| = \sqrt{v_1^2 + v_2^2}$ in the plane,
- $|\langle v_1, v_2, v_3\rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ in 3-space

The term **unit vector** means a vector with length 1. Given any vector **v**, as long as **v** is not the zero vector, you can create a unit vector with the same direction by dividing $\mathbf{v}$ by its length

$$\frac{1}{|\mathbf{v}|}\,\mathbf{v} \text{ is a unit vector in the direction of } \mathbf{v} \neq \mathbf{0}$$

$$-\frac{1}{|\mathbf{v}|}\,\mathbf{v} \text{ is a unit vector in the direction opposite to that of } \mathbf{v} \neq \mathbf{0}$$

- $\mathbf{i} = \langle 1,\, 0\rangle$ and $\mathbf{j} = \langle 0,\, 1\rangle$ are the **coordinate unit vectors** in the plane. These vectors are also called the **standard basis vectors**.
- $\langle v_1,\, v_2\rangle = v_1\,\mathbf{i} + v_2\,\mathbf{j}$ in the plane.
- $\mathbf{i} = \langle 1,\, 0,\, 0\rangle, \quad \mathbf{j} = \langle 0,\, 1,\, 0\rangle, \quad \mathbf{k} = \langle 0,\, 0,\, 1\rangle$ are the **coordinate unit vectors** in three dimensions; they are also called the **standard basis vectors** in $\mathbb{R}^3$.
- $\langle v_1,\, v_2,\, v_3\rangle = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ in 3-space.

All of the facts in the box on page 687 are easily verified and knowing that they are true can simplify and speed up calculations.

It is important to know the formula for the distance between two points in space and the equation of a sphere, which is a consequence of the distance formula.

- **Distance from** $P(x_1,\, y_1,\, z_1)$ **to** $P(x_2,\, y_2,\, z_2)$ **equals**

$$\sqrt{[x_2 - x_1]^2 + [y_2 - y_1]^2 + [z_2 - z_1]^2}$$

Don't overlook the formula (in the middle of page 696 of the textbook) for the midpoint of the line segment joining $P(x_1,\, y_1,\, z_1)$ to $P(x_2,\, y_2,\, z_2)$.

Fortunately, it is easy to remember that the midpoint is the average of the points and is the point whose coordinates are the averages of the coordinates.

- **Sphere of radius r about** (a, b, c)

$$[x-a]^2 + [y-b]^2 + [z-c]^2 = r^2$$

A fun type of problem (at least some people think they are fun) presents you with the equation of a sphere *after* it has be rearranged algebraically and asks you to identify the radius and center. The key to solving these problems is, in turn for each variable x, y and z, to collect together the terms involving that variable and complete the square for those terms.

Key skills for Lesson 2:

- Be able to interpret directed line segments as vectors and vice-versa.
- Be able to perform algebraic operations with vectors and to interpret the meaning of those operations.
- Be able to use the coordinate unit vectors.
- Be able to apply vectors to solve word problems by using both the geometric interpretation of vectors and the algebraic interpretation of vectors.
- Be able to go back and forth between the algebraic and geometric descriptions of a sphere.
- Be able to find the midpoint of the line segment connecting two points.

Exercises for Lesson 2

Textbook section 11.1, pages 690–691

- (plane) # 17–35 odd numbered, 39, 45, 51

Textbook section 11.2, pages 700–701

- (3-space) # 11, 13, 19, 23, 27, 29, 31, 33, 37, 39, 45, 47

Lesson 3—Dot Products
Briggs–Cochran: Section 11.3, pages 703–713

For a pair of vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, you typically will compute the dot product $\mathbf{u}\cdot\mathbf{v}$ using the formula

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3\,. \tag{$\star$}$$

From ($\star$), we see that

$$|\mathbf{u}|^2 = \mathbf{u}\cdot\mathbf{u}\,.$$

The crucial geometric fact is that, provided neither $\mathbf{u}$ nor $\mathbf{v}$ is the zero vector,

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta\,, \tag{$\star\star$}$$

where θ is the angle between the two vectors.

- You need to know both ($\star$) and ($\star\star$), because in many problems it is easy to apply one of them, but it is information in the other that must be found.

Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be **orthogonal** or **perpendicular** if the angle between them is $\pi/2$. Using the dot product, we see that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u}\cdot\mathbf{v} = 0$ (we also make it a convention that the zero vector is orthogonal to every vector).

The properties of the dot product are

- $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$,
- $c(\mathbf{u}\cdot\mathbf{v}) = (c\,\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\,\mathbf{v})$,
- $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$,

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors and c is a scalar. You should convince yourself that these properties follow from ($\star$).

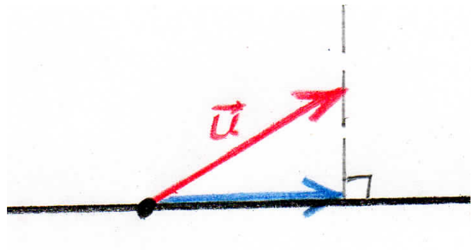

Figure 8: The orthogonal projection of **u** onto a line.

Often one needs to find the orthogonal projection of a vector **u** on a given line. This is illustrated in Figure 8 in which the blue horizontal vector is the orthogonal projection of the red vector **u** onto the black horizontal line. The line onto which **u** is to be projected is itself defined by a vector **v**, which can be any non-zero vector parallel to the line. The result of the projection is called the **orthogonal projection of u onto v** and it is denoted by $\text{proj}_{\mathbf{v}}\mathbf{u}$. It is important to remember that $\text{proj}_{\mathbf{v}}\mathbf{u}$ is a vector.

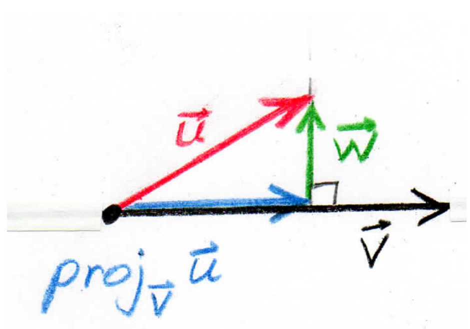

Figure 9: Finding the orthogonal projection of **u** onto a line.

Formulas for the orthogonal projection of **u** onto **v** are given on page 708 of the textbook, but this is a case in which it is worth the effort to know and use the derivation. The derivation is based on Figure 9 in which **u** is seen to be the sum of two vectors, one of which is parallel to **v** and the second of which is perpendicular to **v**. The vector that is parallel to **v** is $\text{proj}_{\mathbf{v}}\mathbf{u}$. The vector that is perpendicular to $\mathbf{v}$ is shown in green in the figure; we have called it $\mathbf{w}$, just so it has a name that we can use in what follows.

Since $\text{proj}_{\mathbf{v}}\mathbf{u}$ is parallel to $\mathbf{v}$, it equals $c\,\mathbf{v}$ for some scalar c. Since $\mathbf{w}$ is perpendicular to $\mathbf{v}$, it must satisfy $\mathbf{w}\cdot\mathbf{v} = 0$. We have the following set of equations to solve:

$$\begin{aligned} \mathbf{u} &= \text{proj}_{\mathbf{v}}\mathbf{u} + \mathbf{w} \\ \text{proj}_{\mathbf{v}}\mathbf{u} &= c\,\mathbf{v} \\ \mathbf{w}\cdot\mathbf{v} &= 0\,. \end{aligned}$$

We use the second equation to substitute for $\text{proj}_{\mathbf{v}}\mathbf{u}$ in the first equation giving us

$$\begin{aligned} \mathbf{u} &= c\,\mathbf{v} + \mathbf{w} \\ \mathbf{w}\cdot\mathbf{v} &= 0\,. \end{aligned}$$

Taking the dot product of $\mathbf{v}$ with both sides the the first equation, we obtain

$$\mathbf{u}\cdot\mathbf{v} = c\,\mathbf{v}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{v}\,.$$

The equation $\mathbf{w}\cdot\mathbf{v} = 0$ allows us to conclude that

$$\mathbf{u}\cdot\mathbf{v} = c\,\mathbf{v}\cdot\mathbf{v}\,,$$

so we find

$$c = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\,.$$

Thus we have

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\,.$$

If $\mathbf{v}$ is a unit vector, then the multiplier c that we found above is called the **scalar component of u in the direction of v** and it is written $\text{scal}_{\mathbf{v}}\mathbf{u}$.

For $\mathbf{v}$ that is not necessarily a unit vector, $\text{scal}_{\mathbf{v}}\mathbf{u} = |u| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. We have

$$\text{scal}_{\mathbf{v}}\mathbf{u} = \begin{cases} |\,\text{proj}_{\mathbf{v}}\mathbf{u}\,|, & \text{if } \ 0 \le \theta < \pi/2, \\ 0, & \text{if } \ \theta = \pi/2, \\ -|\,\text{proj}_{\mathbf{v}}\mathbf{u}\,|, & \text{if } \ 0 \le \theta < \pi/2. \end{cases}$$

If we need to know $\mathbf{w}$, we easily see that

$$\mathbf{w} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\,.$$

Key skills for Lesson 3:

- Be able to compute dot products and know the algebraic rules the dot product satisfies.
- Be able to relate angles between vectors to dot products.
- Be able to find the orthogonal projection of one vector on another and the scalar component of the projection.

Exercises for Lesson 3

Textbook section 11.3, pages 710–711

- # 9, 11, 15, 21, 23, 25, 27, 29, 33

Lesson 4—Cross Products
Briggs–Cochran: Section 11.4, pages 713–720

Like the dot product, the cross product can be found using a geometric rule or by a computation using the components of the vectors. In problems, often one is easier to compute, but it is information from the other that is needed. The cross product only applies to pairs of vectors in three dimensional space.

The geometric definition of the cross product is complicated:

> $\mathbf{u} \times \mathbf{v}$ is the vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, pointing in the direction given by the right-hand rule, and having length equal to the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$. The area of that parallelogram equals $|\mathbf{u}|\ |\mathbf{v}|\ \sin\theta$, where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

- The cross product is **anti-commutative**, i.e. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

- Quite often the cross product is used to obtain a vector perpendicular to two given vectors.

The computation of the cross product using the components of $\mathbf{u}$ and $\mathbf{v}$ is also complicated. Using determinants as a memory aid, we have the following:

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\
&= (u_2\, v_3 - u_3\, v_2)\, \mathbf{i} - (u_1\, v_3 - u_3\, v_1)\, \mathbf{j} + (u_1\, v_2 - u_2\, v_1)\, \mathbf{k}\,.
\end{aligned}$$

Key skills for Lesson 4:

- Be able to compute the cross product of two vectors.
- Be able to use the cross product to obtain areas, normal vectors, torques, and forces.

Exercises for Lesson 4

Textbook section 11.4, pages 718–719

- # 13–21 odd numbered, 25, 27, 31, 35, 37, 39

Lesson 5—Lines & Curves; Calculus on Curves
Briggs–Cochran: Sections 11.5 & 11.6, pages 720–737

The textbook emphasizes the vector equation for a line:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}\,,$$

where

- $\mathbf{r}_0$ is the position vector of a point on the line,
- $\mathbf{v}$ is a vector parallel to the line, and
- t is a parameter that is allowed to vary through the real numbers.

You can also describe a line with the three parametric equations

$$\begin{aligned} x(t) &= x_0 + a\,t \\ y(t) &= y_0 + b\,t \\ z(t) &= z_0 + c\,t \end{aligned}$$

which are simply the components of the vector equation. You go back and forth between these two forms using

$$\begin{aligned} \mathbf{r}_0 &= \langle\, x_0, y_0, z_0 \,\rangle \\ \mathbf{v} &= \langle\, a, b, c \,\rangle\,. \end{aligned}$$

The vector equation for a line is a special case of a vector-valued function. A vector function is a function of one independent real variable that takes vectors as its values: For example, the function $\mathbf{v}(t)$ given by the following formula

$$\mathbf{v}(t) = \langle\, t,\, \cos t,\, t^2 \,\rangle$$

is a vector function. The components of the function $\mathbf{v}(t)$ are the three functions

$$v_1(t) = t, \quad v_2(t) = \cos t, \quad v_3(t) = t^2 .$$

It is important to realize that the limit of a vector function is the vector formed by taking the limits of all the components. If any component does not have a limit, then the vector function does not have a limit. A consequence of this fact about limits is that a vector function is continuous if and only if its component functions are all continuous.

If you think of the values of a vector function as position vectors, then the set of values taken by a continuous vector function is a curve in space. If you also think of the position varying through space as the independent variable varies, then you have a parametric representation of a space curve. It can be very helpful to your intuition to think of that independent variable as time and to think of the value of the vector function as being the location of a particle at that time.

Your main goal in this section is to get comfortable with thinking of space curves in this way. Many of the most important applications of vector functions require you to interpret the vector function as a parametrization of a space curve.

In practice, derivatives and integrals of vector-valued functions are computed component by component. This means that no new methods are needed to solve problems, but each problem involves two or three sub-problems.

Computing derivatives and integrals component by component relies on the assumption that the basis vectors constant. The standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are constant, so working component by component is valid. Physicists and engineers sometimes work with basis vectors that are not constant, and in that case, a more general approach—beyond the scope of this course—is needed.

The derivative of a vector-valued function gives a vector tangent to the curve represented by the vector function. This way of finding a tangent vector is only applicable when the derivative is not the zero vector.

If the vector-valued function is

$$\mathbf{r}(t) = \langle\, x(t), y(t), z(t)\,\rangle$$

and it is differentiable, then

$$\mathbf{r}'(t) = \langle\, x'(t), y'(t), z'(t)\,\rangle\,.$$

If also $\mathbf{r}'(t_0) \neq \mathbf{0}$, then the **unit tangent vector** to the curve at $\mathbf{r}(t_0)$ is

$$\mathbf{T} = \frac{\mathbf{r}'(t_0)}{|\mathbf{r}'(t_0)|}\,.$$

Key skills for Lesson 5:

- Be able to find an equation for a line or a line segment.
- Be able to find limits of vector functions and be able to recognize whether or not a vector function is continuous.
- Be able to identify the space curve parametrized by a vector function.
- Be able to construct a vector function to parametrize a space curve.
- Be able to compute derivatives of vector functions.
- Be able to find the tangent line to a space curve.
- Be able to compute the integral of a vector function.

Exercises for Lesson 5

Textbook section 11.5, pages 726–727

- # 11, 13, 17–27 odd numbered, 35, 41, 43

Textbook section 11.6, pages 735–736

- # 9, 15, 17, 23, 25, 27, 33, 37, 43, 47, 51

Lesson 6—Motion in Space
Briggs–Cochran: Section 11.7, pages 737–750

If one thinks of a vector function as giving the position of a particle as a function of time, then the derivative of the vector function gives the velocity of the particle and the second derivative of the vector function gives the acceleration of the particle. The speed of the particle is defined to be the magnitude of the velocity vector.

- Velocity is a vector.

- Speed is the magnitude of the velocity vector, so it is a non-negative scalar.

Because integration undoes differentiation, you can start with information about the velocity or the acceleration and integrate to get back to the position of the particle. The roles of the constants of integration are played by the initial position and the initial velocity.

The acceleration is of great interest and importance because of the equation

$$\mathbf{F}(t) = m\,\mathbf{a}(t)$$

(Newton's Second Law of Motion) relating force $\mathbf{F}(t)$ and acceleration $\mathbf{a}(t)$ for a particle of constant mass m.

For ordinary objects near the surface of the Earth, the force of gravity on a particle of mass m is well-approximated by

$$\mathbf{F} = -m\,g\,\mathbf{k}\,,$$

where the positive z-axis of the coordinate system is assumed to point up and g is the gravitational constant. Combining this approximation with Newton's Second Law, we have

$$\mathbf{a}(t) = -g\,\mathbf{k}\,.$$

You should be prepared to integrate this last equation twice to obtain the trajectory of the particle.

Key skills for Lesson 6:

- Be able to compute velocity, acceleration, and speed.

- Be able to recognize when a trajectory lies on a circle or sphere.

- Be able to find position from information about velocity and acceleration.

Exercises for Lesson 6

Textbook section 11.7, pages 746–747

- # 7, 9, 11, 13, 21, 23, 27, 29, 35, 39

Lesson 7—Lengths of Curves
Briggs–Cochran: Section 11.8, pages 750–758

The arc length of a space curve is defined to be the limit of the lengths of inscribed polygons. Fortunately, for a space curve with a continuously differentiable parametrization, $\mathbf{r}(t)$, this limit can be shown to equal the following integral, where we assume the parametrization of the curve is over the interval from a to b.

$$\text{arc length} = \int_a^b |\mathbf{r}'(t)|\,dt\,.$$

Because the integrand involves a square root, it is often difficult to carry out the integration.

Key skills for Lesson 7:

- Be able to find the arc length of a curve given in rectangular Cartesian coordinates.
- Be able to find the arc length of a curve given in polar coordinates.

Exercises for Lesson 7

Textbook section 11.8, page 756

- # 7, 11, 15, 19, 21, 27, 29

Lesson 8—Curvature and Normal Vectors
Briggs–Cochran: Section 11.9, pages 758–770

As a theoretical principle, one can reparametrize a curve using arc length as the parameter. This would be done by expressing arc length s as a function of the given parameter t using the equation

$$s = \int_a^t |\mathbf{r}'(\tau)|\, d\tau$$

and then solving this equation for t as a function of s (as the inverse function theorem says you can). You can hardly ever carry this process through to completion, so it is mainly a theoretical device.

Curvature, denoted by κ, is defined to be the magnitude of the rate of change of the unit tangent vector, $\mathbf{T}$, with respect to arc length:

$$\kappa = \left| \frac{d\,\mathbf{T}}{ds} \right| .$$

Since it is usually impossible to reparametrize by arc length using elementary functions, you need a more effective way to compute the curvature. One way is to use the chain rule:

$$\kappa = \left| \frac{d\,\mathbf{T}}{ds} \right| = \left| \frac{d\,\mathbf{T}/dt}{ds/dt} \right| = \frac{1}{|\mathbf{v}|} \left| \frac{d\,\mathbf{T}}{dt} \right| .$$

All the quantities on the far right-hand side of this last equation can be computed without any reparametrization of the curve.

Many students prefer to memorize and apply the formula

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} .$$

The derivative of the unit tangent vector to a curve is automatically normal to the curve. The unit vector in the direction of the derivative $\mathbf{T}'(t)$

(assuming $|\mathbf{T}'(t)|$ is non-zero) is called the **principal unit normal vector**, written $\mathbf{N}(t)$.

For the record, we mention that a third unit vector, called the **binormal vector**, written $\mathbf{B}(t)$, is formed by taking the cross-product of the unit tangent vector and the principal unit normal vector. The textbook makes no use of the binormal vector.

Key skills for Lesson 8:

- Be able to recognize an arc length parametrization.
- Be able to find the unit tangent and the principal unit normal.
- Be able to find the curvature of a curve.

Exercises for Lesson 8

Textbook section 11.9, pages 768–769

- # 11, 13, 15, 17, 23, 25, 31, 33, 39

Lesson 9—Planes and Surfaces
Briggs–Cochran: Section 12.1, pages 774–788

The simplest type of surface in 3-space is a plane. In terms of the cartesian coordinates x, y, z a plane is the set of points $P(x, y, z)$ that satisfy a scalar equation

$$Ax + By + Cz = D\,,$$

where A, B, C, and D are constants and at least one of A, B, and C is non-zero.

In terms of vectors, a plane is the set of points P such that

$$(\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = 0\,,$$

where $\mathbf{p}$ is the position vector for some fixed point in the plane and $\mathbf{n}$ is a fixed non-zero vector perpendicular to the plane. You can always replace $\mathbf{n}$ by $c\,\mathbf{n}$, as long as $c \neq 0$, without changing the plane. In particular,

- $\mathbf{n}$ and $-\mathbf{n}$ define the same plane.
- The constants A, B, and C in the scalar equation of a plane can be used as the components of $\mathbf{n}$ in the vector equation the plane.
- The angle between two planes is defined to be the angle between lines perpendicular to the planes.

Cylinders. Mathematically a cylinder is defined by a curve C and a line ℓ: The resulting cylinder is the union of all the lines that are parallel to ℓ and intersect C. The surfaces we call cylinders in everyday life are right circular cylinders: C is a circle and ℓ is a line perpendicular to the plane containing C.

Quadric surfaces. The scalar equation of a plane involves only first powers of coordinates and no products of coordinates. The next step up in complexity is to allow squares of coordinates and products of two coordinates. The general equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0\,.$$

Classifying such surfaces might appear to be a hopeless task, but in fact all qualitative possibilities are illustrated on page 785 of the textbook (assuming the expression on the left above is truly quadratic and that it does not factor into a product of linear terms).

In general, making the appropriate change of variables to convert a particular quadratic equation into standard form can be quite a bit of work. Completing the square is something you can do, so if faced with such a conversion problem, look for squares to complete.

Key skills for Lesson 9:

- Be able to form the equation of a plane from geometric information.
- Be able to deduce geometric information about a plane from its equation.
- Be able to determine whether two planes intersect and to find the line of intersection if they do intersect.
- Be able to identify a cylinder with axis parallel to a coordinate axis.
- Be able to recognize the various quadric surfaces in standard form.

Exercises for Lesson 9

Textbook section 12.1, pages 786–787

- # 11, 13, 15, 17, 19, 23, 25, 29, 33, 37, 41, 45, 49, 53, 57

Lesson 10—Graphs and Level Curves
Briggs–Cochran: Section 12.2, pages 789–800

A function f of two variables is a rule that assigns a real number $f(x, y)$ to each point $P(x, y)$ in the domain D. The graph of the function is defined to be the set of points $(x, y, f(x, y))$ obtained as (x, y) varies over the domain.

A set G is the graph of some function f with domain D if for each point (x, y) in D there exists exactly one number z such that $(x, y, z) \in G$ (this is the **vertical line test**), and in that case $z = f(x, y)$. Thus a function is completely determined by its graph, and you can identify which sets are graphs of functions. The point of view that a function *is* its graph saves us from considering the question of what kinds of rules are allowed in defining a function.

Visual representation of graphs of functions of two variables is an art that requires practice. Computer systems are available that will do the job quickly and accurately, allowing you to concentrate on choosing the view that emphasizes the relevant features.

Drawing the level curves of a function of two variables requires less skill than drawing the graph of the function, but the results are typically harder to interpret.

Key skills for Lesson 10:

- Be able to find the domain and range of a function of several variables.
- Be able to visually represent functions of two variables using graphs and level curves.

Exercises for Lesson 10

Textbook section 12.2, pages 797–799

- # 11, 15, 21, 25, 27, 31, 33, 43, 47

Lesson 11—Limits and Continuity
Briggs–Cochran: Section 12.3, pages 800–809

The definition of the limit of a function is fundamental. Continuity is then defined in terms of limits. It is generally difficult to work directly with the definition of continuity, but fortunately most problems you will encounter can be dealt with more simply.

Certain general classes of functions are well understood. For instance

- all polynomial functions in any number of variables are continuous everywhere,
- any ratio of polynomials (called a **rational function**) is automatically continuous everywhere that the denominator is non-zero.

Additional continuous functions are constructed using the fact that

- a continuous function of a continuous function is continuous.

The question of whether or not a rational function is continuous at a particular point where the denominator equals 0 can usually be settled by using polar (or spherical) coordinates about the point.

Example. Find the limit of

$$\frac{x^2 - 2y^2}{x^4 + y^4}$$

at the origin if it exists.

Solution. First, notice that the denominator equals 0 only at the origin. Then change to polar coordinates to rewrite the function as

$$\frac{r^2(\cos^2 q - 2\sin^2 q)}{r^4(\cos^4 q + \sin^4 q)} = r^{-2}\ \frac{\cos^2 q - 2\sin^2 q}{\cos^4 q + \sin^4 q}$$

Picking any angle for which $\cos^2\theta - 2\sin^2\theta$ is not zero, we see that if any sort of limit existed, it could not be finite because r^{-2} goes to infinity as r decreases to 0. But also there are angles for which $\cos^2\theta - 2\sin^2\theta$ equals zero (for instance $\sin^{-1}(1/\sqrt{3})$), so if a limit existed, it must be 0. Thus, there is no limit. $\square$

Key skills for Lesson 11:

- Be able to find limits of functions.
- Be able to show that a limit does not exist.
- Be able to identify where a function is continuous.

Exercises for Lesson 11

Textbook section 12.3, pages 808–809

- # 11-45 odd numbered

Lesson 12—Partial Derivatives
Briggs–Cochran: Section 12.4, pages 810–821

In practice, computing a partial derivative is exactly the same computation as you have done in MTH 251 for ordinary derivatives. The main difficulties are not allowing yourself to be distracted by the other variables and becoming accustomed to the various notations for partial derivatives.

Your geometric intuition should be focused on the interpretation of partial derivatives as slopes of cross sections of graphs. This is illustrated on page 811 of the textbook.

Differentiability

A function f is **differentiable at a point** $\mathbf{p}_0$ if there is a linear function L such that $f(\mathbf{p})$ is so well approximated by $f(\mathbf{p}_0) + L(\mathbf{p} - \mathbf{p}_0)$ that

$$\frac{|\ f(\mathbf{p}) - [f(\mathbf{p}_0) + L(\mathbf{p} - \mathbf{p}_0)]\ |}{|\mathbf{p} - \mathbf{p}_0|} \to 0 \quad \text{as} \quad \mathbf{p} \to \mathbf{p}_0\,.$$

It would be nice if the existence of the partial derivatives guaranteed that a function of several variable is differentiable, but unfortunately that is not the case. If the partial derivatives exist *and are continuous*, then the function is differentiable.

Key skills for Lesson 12:

- Be able to compute partial derivatives of all orders.

- Understand the definition and interpretation of partial derivatives.

Exercises for Lesson 12

Textbook section 12.4, pages 818–819

- # 7–15 odd numbered, 19, 21, 25, 27, 29, 35, 37, 41, 43, 45

Lesson 13—The Chain Rule
Briggs–Cochran: Section 12.5, pages 821–830

The Chain Rule in several variables causes difficulty because there are so many terms to remember to include. Suppose

$$w = f(x, y, z)$$

and

$$x = g(r, s, t) \quad y = h(r, s, t)\,, \quad z = k(r, s, t)\,.$$

Thus w is a function of x, y, and z and in turn x, y, and z are functions of r, s, and t. This is also sometimes written more compactly as

$$w = f\Big(x(r, s, t), y(r, s, t), z(r, s, t)\Big).$$

We call w the **dependent variable**, and we call r, s, and t the **independent variables**. You think of r, s, and t as the quantities that may be varied, and w is the quantity that changes as a result. But here there are other variables involved—namely x, y, and z. You think of the change in r, s, and t resulting in changes in x, y, and z that lead to the change in w. The variables x, y, and z are called **intermediate variables**.

Once you have identified the dependent, independent, and intermediate variables, then you can recall the terms to include in the Chain Rule through the use of a tree diagram such as in Figure 10. The tree diagram is constructed with the dependent variable at the top, the independent variables at the bottom, and the intermediate variables in the middle. A branch is drawn from each variable to any other variable upon which it directly depends, and the branch is labeled with the corresponding partial derivative.

Each path through the tree represents how the dependent variable depends on an independent variable. In finding a partial derivative, say $\partial w/\partial t$,

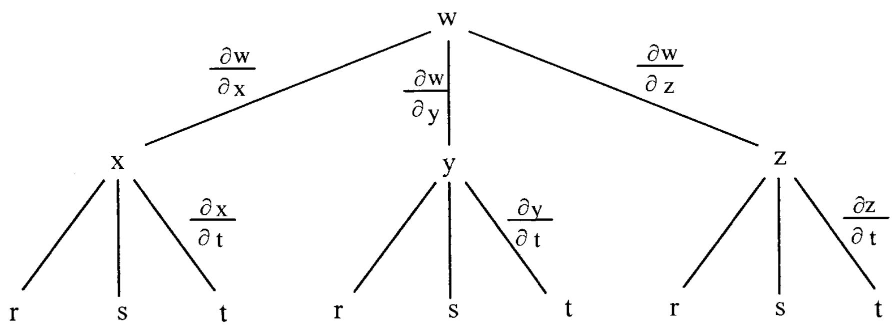

Figure 10: The Chain Rule.

all the ways that w is affected by t must be included, so we sum the products of partial derivatives on all branches ending with t:

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial t}\,.$$

One often occurring case of the chain rule is that in which a function f is evaluated along a curve $\mathbf{r}(t)$. In that case, we have

$$\frac{d}{dt}\left(f(\mathbf{r}(t)\right) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,,$$

where

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

is the gradient vector (which is formally introduced in the next section of the textbook).

Key skills for Lesson 13:

- Be able to apply the chain rule to find total and partial derivatives.
- Be able to apply Theorem 12.9 (page 825) on implicit differentiation.

Exercises for Lesson 13

Textbook section 12.5, page 827

- # 7–21 odd numbered, 27, 33

Lesson 14—Directional Derivatives and the Gradient Briggs–Cochran: Section 12.6, pages 830–842

The **directional derivative** of a function at a point is the instantaneous rate of change of the function (at the point) as you move along a line in the given direction that passes through the point. That is what the definition on page 831 expresses in technical language. To compute a directional derivative, you must be given a direction and "direction" always means a unit vector.

The **gradient vector** of a function at a point is the vector whose components are the partial derivatives of the function evaluated at the point. Note that at each point the gradient vector has numerical entries, but these numerical entries will change from point to point. Therefore, we think of the gradient of a function as a vector whose entries are themselves functions, namely, the partial derivatives of the original function.

The connection between the directional derivative and the gradient is that

- the directional derivative equals the dot product of the gradient vector and the direction vector. This is valid for differentiable functions.

Because the directional derivative of a function in the direction $\mathbf{u}$ equals the dot product of $\mathbf{u}$ and the gradient of the function, it follows that the gradient points in the direction of most rapid increase of the function. Likewise, for a function of two variables, the gradient of the function at a point is orthogonal to the level curve of the function through that point. For a function of three variables, the gradient is orthogonal to the level surface of the function.

- The gradient points in the direction of most rapid increase of a differentiable function.

- In 2-variables, the gradient of a differentiable function at a point is perpendicular to the level curve of the function that passes through the point.

Key skills for Lesson 14:

- Know and understand the definition of the directional derivative.
- Be able to compute the gradient of a function.
- Be able to compute directional derivatives using the gradient vector.
- Be able to find the direction of most rapid increase or decrease of a function at a point.
- Be able to use the gradient vector to find the tangent to a level curve.

Exercises for Lesson 14

Textbook section 12.6, pages 840–841

- # 9–19 odd numbered, 23, 27, 33, 35, 37, 45–51 odd numbered

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Lesson 15—Tangent Planes, Linear Approximation
Briggs–Cochran: Section 12.7, pages 842–853

Surfaces in the form $\boldsymbol{F(x,y,z)=0}$. The textbook begins its discussion of tangent planes by considering the tangent plane to a surface defined in 3-space by the equation $F(x,y,z)=0$. Assuming the function F is differentiable and that (a,b,c) is a point on the surface, i.e., $F(a,b,c)=0$ holds, then the key fact is that

- the gradient vector $\nabla F(a,b,c)$ is perpendicular to the surface $F(x,y,z)=0$ at the point (a,b,c).

Thus you know a normal to the tangent plane, namely, $\nabla F(a,b,c)$, and a point on the tangent plane, (a,b,c), and those two items are precisely what you need to construct the equation of the plane:

$$\nabla F(a,b,c)\cdot\langle\, x-a,\ y-b,\ z-c\,\rangle = 0\,. \tag{1}$$

For (1) to define a plane, the variables x, y, and z must occur to at most the first power, so a, b, and c must be constants.

Surfaces in the form of a graph $\boldsymbol{z=f(x,y)}$. Recall that the tangent line to the graph of a function $y=f(x)$ at the point (a,b) with $b=f(a)$ can be found using the point-slope form of the equation of a line

$$y-b=m(x-a)\,, \tag{2}$$

where the slope m is simply the derivative of f at a, that is,

$$m=f'(a)\,. \tag{3}$$

The tangent line given in (2) is a very good approximation to the graph if and only if the derivative in (3) exists.

There is a similar "point-slope" form for the equation of a plane in three-dimensional space:

$$z - c = m_1(x - a) + m_2(y - b)\,, \tag{4}$$

in which (a, b, c) is a point in the plane and m_1 and m_2 are slopes in the x and y directions. For a graph $z = f(x, y)$ the partial derivatives give you the slopes in the x and y directions, thus

$$m_1 = \frac{\partial f}{\partial x}\,, \qquad m_2 = \frac{\partial f}{\partial y}\,.$$

- It is important to remember that to define a plane the slopes in (4) must be constants.

For the tangent plane defined in (4) to be a good approximation to the graph, the function must be differentiable at the point (a, b, c). That is a stronger requirement than simply the existence of the partial derivatives at the point.

Key skills for Lesson 15:

- Be able to find the tangent plane to a surface.
- Be able to find the best linear approximation to a function.
- Be able to find the differential of a function.
- Be able to use differentials to approximate function changes.

Exercises for Lesson 15

Textbook section 12.7, pages 850–851

- # 9–25 odd numbered, 29, 35, 39

Lesson 16—Maximum/Minimum Problems
Briggs–Cochran: Section 12.8, pages 853–863

Finding maxima and minima of functions of two or more variables is conceptually very much like the process for functions of one variable, but it is more complex to carry out. For a function of one variable, if a maximum or minimum occurs in the interior of the interval under consideration and if the function is differentiable at that point, then the derivative must equal 0. The situation is similar for functions of two or more variables, but the condition is that all the partial derivatives must equal 0 at the point, that is, $\nabla f = \mathbf{0}$ holds at the point.

For a function of one variable, the second derivative often tells you whether a given critical point is a local maximum or minimum. A positive second derivative means a local minimum, a negative second derivative means a local maximum, and when the second derivative is zero you are warned that no conclusion can be drawn.

For a function of two variables, the second derivative test is as follows:

- If
$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\,\partial y} \\ \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix} \overset{\text{def}}{=} \frac{\partial^2 f}{\partial x^2}\ \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 > 0$$
holds, then the function has **either** a maximum or a minimum at the point. In this case,
  - if $\partial^2 f/\partial x^2$ is positive, then there is a minimum at the point,
  - if $\partial^2 f/\partial x^2$ is negative, then there is a maximum at the point.

- If
$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\,\partial y} \\ \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix} = \frac{\partial^2 f}{\partial x^2}\,\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 < 0$$
holds, then the function has neither a maximum nor a minimum—the point is a **saddle point**.

- If
$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\,\partial y} \\ \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix} = \frac{\partial^2 f}{\partial x^2}\,\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 = 0$$
holds, then the second derivative test gives no information.

On a closed, bounded subset of Euclidean space, a continuous function will attain its absolute maximum and its absolute minimum. Each extremum occurs either at a critical point in the interior of the given subset or at a point on the boundary of the given subset.

To fully explore the possibility of an extreme value on the boundary you may need to parametrize the boundary, maybe even in several pieces if the boundary has corners. An alternative to parametrization of the boundary is the use of the method of Lagrange multipliers as discussed in the next lesson.

Key skills for Lesson 16:

- Be able to find critical points.
- Be able to apply the second derivative test.
- Be able to find absolute maxima and minima of functions on closed, bounded domains.
- On open and unbounded domains, be able to determine whether absolute extrema exist and be able to find them if they do exist.

Exercises for Lesson 16

Textbook section 12.8, page 861

- # 9–19 odd numbered, 29, 33, 37, 39, 47, 51

Lesson 17—Lagrange Multipliers
Briggs–Cochran: Section 12.9, pages 864–871

In the preceding lesson, it was suggested that the way to find the maximum or minimum of a function on the boundary of a domain is to parametrize the boundary and thus reduce the problem to finding the maximum or minimum of a function in a setting one dimension lower. The method of Lagrange multipliers is an alternative approach that is of both theoretical and practical significance.

Why the method of Lagrange multipliers works:

Suppose we want to find the maximum of the function $f(x, y)$ on the curve defined by $g(x, y) = 0$, and suppose that the gradient of g is not the zero vector anywhere on the curve, that is, $\nabla g(x, y) \neq (0, 0)$ holds for all (x, y) on the curve.

It is an unlikely special case that the maximum of f over all (x, y) happens to occur at a point on the given curve. It is more reasonable to expect that the maximum on the curve will occur at a point (x_0, y_0) that is not a critical point of f.

We now consider what happens in the typical case when the maximum of f on the curve $g(x, y) = 0$ occurs at a point (x_0, y_0) that is not a critical point of f. At such a point, the gradient of f at (x_0, y_0) is not the zero vector. We have both

$$\nabla f(x_0, y_0) \neq (0, 0) \quad \text{and} \quad \nabla g(x_0, y_0) \neq (0, 0)\,.$$

Let $\mathbf{u}$ be a unit vector tangent to the curve $g = 0$ at the point (x_0, y_0). Since f has its maximum along the curve at (x_0, y_0), the directional derivative of f in the direction $\mathbf{u}$ at the point (x_0, y_0) must be zero. Since the gradient of f is not the zero vector, the only way for that directional derivative to equal 0 is for the gradient of f at (x_0, y_0) to be orthogonal to $\mathbf{u}$. But we also know that the gradient of g at (x_0, y_0) is orthogonal to $\mathbf{u}$.

Since $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are both perpendicular to $\mathbf{u}$, they must be parallel. Since $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel and neither is the zero vector, each can be written as a multiple of the other.

Traditionally, we express $\nabla f(x_0, y_0)$ as a multiple of $\nabla g(x_0, y_0)$, because that will also be true in the special case when $\nabla f(x_0, y_0) = (0, 0)$ [by using the multiplier 0].

Key skills for Lesson 17:

- Understand why the method of Lagrange multipliers works.

- Be able to apply the method of Lagrange multipliers.

Exercises for Lesson 17

Textbook section 12.9, pages 869–870

- # 5, 7, 9, 15, 25[1], 27

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[1]See page 68 of this study guide for a correction to the textbook's answer for this problem.

Lesson 18—Double Integrals on Rectangular Regions Briggs–Cochran: Section 13.1, pages 876–886

Imagine you would like to find the volume of an irregular solid made from modeling clay or Playdoh. Imagine further that this solid is a model of the region under the graph of a function of two variables, and for simplicity assume that the base of this object is a rectangle. If you were very good with a knife you could cut this object into slices with parallel sides. Then you could cut each slice into sticks. One end of each stick would be flat and the other end would be a small part of the graph and so would be nearly flat. The volume of each stick could be estimated by estimating its length and multiplying that length estimate times the cross-sectional area of the stick. The total of all the estimates for the volumes of the sticks would be the estimate for the volume of the solid.

A Riemann sum is an estimate for the volume under a graph obtained just as we imagined doing by slicing up a model into sticks. But in a Riemann sum you just use the numbers without the model. Figure 13.1, page 877, shows you the picture.

The double integral of a non-negative function over a rectangular region is equal to the volume under the graph of the function, and both the double integral and the volume under the graph are defined to be the limit of Riemann sums obtained for the function as the maximum size of the subregions goes to zero.

When the graph of the function is a plane, the volume under the graph can be computed by more elementary methods. It can be shown that the values obtained by elementary geometry and by limits of Riemann sums agree.

Sometimes when the graph of the function is a curved shaped you may know a formula for the volume, for example, if the graph is a hemisphere you can use the formula for the volume of a sphere to find the volume under the graph. In such a case, it may seem like there are two ways to find the volume under the graph, but, in fact, the formula you use was found as a special case of taking a limit of Riemann sums. Of course, the classical formulas for volumes were discovered millennia before Riemann lived, but as far as we can tell essentially the same idea was used.

As a computational method for finding volumes, computing limits of Riemann sums is very difficult to apply. A more streamlined process for finding volumes and double integrals is to use iterated integrals as described next.

Iterated Integrals

An important result about double integrals is the following theorem.

THEOREM. If $f(x,y)$ is a continuous function defined on the rectangle

$$R = \left\{ \ (x,y) : a \leq x \leq b, \quad c \leq y \leq d \ \right\},$$

then $\int\int_R f(x,y)\,dA$ exists and

$$\int\int_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right)\,dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right)\,dy\,.$$

EXAMPLE. Integrate

$$f(x,y) = x + y^2$$

over the rectangle

$$R = \left\{ \ (x,y) : 0 \leq x \leq 1, \quad 2 \leq y \leq 3 \ \right\}.$$

SOLUTION. The theorem tells you to use an iterated integral to evaluate the double integral, and that you can do the iterated integration in either order. So first you must pick an order of integration. It is usually best to do the integral that produces the simpler result first. When integrating polynomials, a limit of integration equal to 0 leads to a simple result. So you should probably do the x-integral first:

$$\int\int_R x + y^2\, dA = \int_2^3 \left(\int_0^1 x + y^2\, dx \right) dy\,.$$

We compute

$$\int_0^1 x + y^2\, dx = \left(x^2/2 + xy^2\right)\Big|_0^1 = 1/2 + y^2\,.$$

So

$$\begin{aligned}
\int\int_R x + y^2\, dA &= \int_2^3 \left(\int_0^1 x + y^2\, dx \right) dy \\
&= \int_2^3 (1/2 + y^2)\, dy = \left(y/2 + y^3/3\right)\Big|_2^3 \\
&= (3/2 + 9) - (2/2 + 8/3) = (9 + 54 - 6 - 16)/6 = 41/6\,.
\end{aligned}$$

□

If getting the correct answer is really important (say, on an exam), then do the integration in the other order as a check—the answers must be equal.

Key skills for Lesson 18:

- Know the definition of the double integral.
- Be able to interpret a double integral as a volume.
- Be able to evaluate iterated integrals over rectangular regions.
- Be able to find the average value of a function over a plane region.

Exercises for Lesson 18

Textbook section 13.1, pages 883–884

- # 5–23 odd numbered, 27

Lesson 19—Double Integrals over General Regions
Briggs–Cochran: Section 13.2, pages 886–896

Double integrals over general regions are done in the same manner as double integrals over rectangles, but the process is more complicated. One imagines slicing the solid region between the graph $z = f(x, y)$ and the x, y-plane into slabs using planes parallel to either the x, z-plane or parallel to the y, z-plane. This results in cutting the region of integration into strips as in Figure 11.

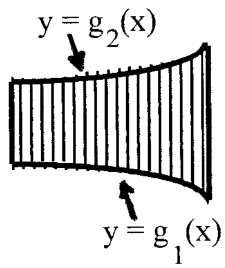

Figure 11: Cutting the region into strips parallel to the y-axis.

Notice that in Figure 11 the strips are parallel to the y-axis and the top and bottom of the strips are formed by the graphs of functions of x. The double integral can be found using an iterated integral with the x-integration done first:

$$\int\int_D f(x, y)\, dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right) dx\,.$$

The limits on the x-integration are the smallest and largest values of x occurring in the region, so the left and right side of the regions are $x = a$ and $x = b$, respectively.

Not every region D will slice nicely into strips parallel to the y-axis. Sometimes it is necessary to use strips parallel to the x-axis as in Figure 12.

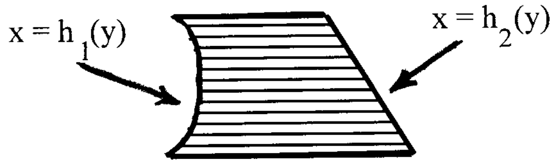

Figure 12: Cutting the region into strips parallel to the x-axis.

In this case the integral is computed by

$$\int\int_D f(x,y)\,dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right)\,dy\,.$$

Decomposition of regions. Some regions of integration must be split into subregions to allow the calculation of a double integral. This is called "Decomposition of Regions" (see Example 6 (page 893) in the textbook).

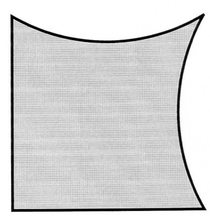

Figure 13: A region that must be decomposed.

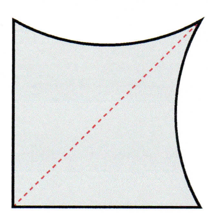

Figure 14: A decomposition that will allow integration.

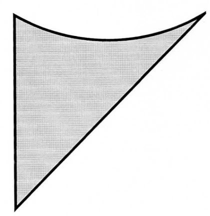

Figure 15: This region can be cut into strips parallel to the y-axis.

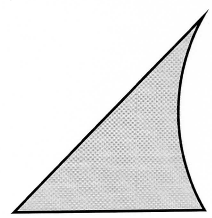

Figure 16: This region can be cut into strips parallel to the x-axis.

One example of such a region is shown in Figure 13. A decomposition that can be used is shown in Figure 14. The two pieces are shown in Figures 15 and 16.

Key skills for Lesson 19:

- Be able to evaluate iterated integrals.
- Be able to evaluate double integrals using iterated integrals.
- Be able to find volumes using double integrals.
- Be able to reverse the order of integration in an iterated integral.
- Know and be able to decompose regions when necessary to compute double integrals.

Exercises for Lesson 19

Textbook section 13.2, pages 894–895

- # 7, 13, 19, 23, 25, 29, 35, 39, 43, 49, 51, 53, 57, 63

Lesson 20—Double Integrals in Polar Coordinates
Briggs–Cochran: Section 13.3, pages 897–906

The main thing to remember in changing a double integral from rectangular coordinates to polar coordinates is that

$$dA = dx\ dy = r\ dr\ d\theta\,.$$

Typical situations in which you should change to polar coordinates are when the region is more easily described in terms of polar coordinates than in rectangular coordinates, that is, when the boundary of the region is formed by circular arcs and segments of rays radiating from the origin.

There are also cases in which the integrand simplifies when converted to polar coordinates.

Finally, it is a classical trick to use a change to polar coordinates so that the r in $r\,dr\,d\theta$ makes it possible to do the integration in closed form, even though in rectangular coordinates the iterated integration could not be done in closed form (see Problem 65, page 906, for example).

Key skills for Lesson 20:

- Be able to convert double integrals from rectangular coordinates to polar coordinates.
- Be able to recognize when areas and volumes are more easily computed using polar coordinates.
- Be able to recognize when changing to polar coordinates will simplify the evaluation of an iterated integral.

Exercises for Lesson 20

Textbook section 13.3, pages 903–904

- # 7, 9, 13, 17, 19, 21, 23, 27, 31, 37, 43

Lesson 21—Triple Integrals
Briggs–Cochran: Section 13.4, pages 906–918

There are no new ideas in this section, but the fact that a triple integral must be computed by iterating three single integrals increases the complexity of the calculations. Further, the regions of integration are solids in three-space, so it is hard to draw and/or visualize them when determining the limits of integration.

Key skills for Lesson 21:

- Be able to evaluate triple integrals using iterated integrals.
- Be able to change the order of integration in a triple integral.
- Be able to find volume and masses.
- Be able to find the average value of a function.

Exercises for Lesson 21

Textbook section 13.4, pages 914–916

- # 7, 11, 15–37 odd numbered, 41

Lesson 22—Triple Integrals in Cylindrical and Spherical Coordinates
Briggs–Cochran: Section 13.5, pages 918–934

When the region of integration in a triple integral is a figure of rotation, it may be easier to set up the integral in cylindrical coordinates instead of rectangular coordinates. When you use cylindrical coordinates, it is important to remember that

$$dV = dx\ dy\ dz = r\ dr\ d\theta\ dz\,.$$

When the region of integration in a triple integral is bounded by a sphere (or part of a sphere), it may be easier to set up the integral in spherical coordinates instead of rectangular or cylindrical coordinates. When you use spherical coordinates, it is important to remember that

$$dV = dx\ dy\ dz = \rho^2\ \sin\phi\ d\rho\ d\theta\ d\phi\,.$$

Key skills for Lesson 22:

- Be able to sketch figures given in cylindrical coordinates.
- Be able to change a triple integral from rectangular to cylindrical coordinates.
- Be able to set up and evaluate triple integrals in cylindrical coordinates.
- Be able to sketch figures given in spherical coordinates.
- Be able to set up and evaluate triple integrals in spherical coordinates.

Exercises for Lesson 22

Textbook section 13.5, pages 930–932

- (cylindrical) # 11, 13, 17, 19, 21, 23, 29, 31, 33
- (spherical) # 35, 37, 39, 41, 43, 45, 47, 49, 51

Lesson 23—Integrals for Mass Calculations
Briggs–Cochran: Section 13.6, pages 934–944

Finding the mass and center of mass of an object are standard physical applications of integration. Density is integrated to find the mass. For solid objects the density is measured as mass per unit volume.

Thin rods are idealized to be *infinitely* thin, and the density is measured as mass per unit length. Thin flat plates are also idealized to be *infinitely* thin, and the density is measured as mass per unit area. A thin flat plate is sometimes called a **lamina**.

To find the x-, y-, or z-coordinate of the center of mass, integrate the product of the variable in question and the density, then divide the result of that integration by the total mass.

Key skills for Lesson 23:

- Be able to find the center of mass of a set of point masses.
- Be able to find the mass and center of mass of a thin rod with varying density.
- Be able to find the mass and center of mass of a thin flat plate with varying density.
- Be able to find the mass and center of mass of a solid body with varying density.

Exercises for Lesson 23

Textbook section 13.6, pages 942–943

- # 7, 9, 11, 15, 17, 23, 29, 33, 35, 37

Lesson 24—Change of Variables in Multiple Integrals Briggs–Cochran: Section 13.7, pages 945–956

The 2-dimensional change of variables formula is

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\;du\,dv\,,$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}\;.$$

The absolute value around the Jacobian determinant is needed because our integrals are with respect to unoriented area, so the orientation of the mapping is irrelevant.

The change of variables formula is based on the fact that the area of the parallelogram determined by $a\,\mathbf{i} + b\,\mathbf{j}$ and $c\,\mathbf{i} + d\,\mathbf{j}$ equals the absolute value of the determinant

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix}\;.$$

Assuming that f is differentiable, the best linear approximation at a point to the mapping $(u,v) \overset{f}{\longmapsto} (x,y)$ is given by

$$\mathbf{i} \longmapsto \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}\,,$$

and

$$\mathbf{j} \longmapsto \frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial y}{\partial v}\,\mathbf{j}\,.$$

The area of the parallelogram determined by $\mathbf{i}$ and $\mathbf{j}$ equals 1, while the area of the image parallelogram, which is determined by the vectors $(\partial x/\partial u)\,\mathbf{i} +$

$(\partial y/\partial u)\,\mathbf{j}$ and $(\partial x/\partial v)\,\mathbf{i} + (\partial y/\partial v)\,\mathbf{j}$, equals

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} .$$

Thus the linear approximation to f scales area by the factor $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$, and that is the factor to include in the integral formula.

The parallelepiped determined by

$$a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}\,,$$
$$d\,\mathbf{i} + e\,\mathbf{j} + f\,\mathbf{k}\,,$$
$$g\,\mathbf{i} + h\,\mathbf{j} + i\,\mathbf{k}$$

has volume equal to the absolute value of the determinant

$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} .$$

An argument similar to that given above for area leads one to conclude that the 3-dimensional change of variables formula is

$$dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\; du\,dv\,dw\,,$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} .$$

The textbook also uses the notations

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)},$$

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)}.$$

It may be easier to remember how to use and how to compute the Jacobians when the notations $\frac{\partial(x,y)}{\partial(u,v)}$ and $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ are employed.

Key skills for Lesson 24:

- Be able to determine the image of a region under a transformation.
- Be able to compute a Jacobian in both two and three variables.
- Be able to invert a transformation.
- Be able to apply the change of variables formula to both double and triple integrals.

Exercises for Lesson 24

Textbook section 13.7, pages 954–955

- # 7, 11, 15, 17, 19, 23, 27, 33, 37, 41

Corrections to Textbook Answers

- § 12.9, # 25:

$$\text{Minimum distance} = \sqrt{38 - 6\sqrt{29}} = \sqrt{29} - 3,$$

$$\text{Maximum distance} = \sqrt{38 + 6\sqrt{29}} = \sqrt{29} + 3.$$