Inverse Problems for Distributions of Parameters in PDE Systems

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Reed College Mathematics Colloquium
Nov 4, 2010

Outline

1. **Inverse Problems**
   - Preliminaries
   - Distributions

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2. **Maxwell's Equations**
   - Description
   - Simplifications
   - Discretization

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2. **Maxwell's Equations**
   - Description
   - Simplifications
   - Discretization
3. **Polarization**
   - Description
   - Random Polarization
   - Polynomial Chaos

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A General First Order Linear PDE System

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$$\mathcal{A} = \begin{bmatrix} 0 & \frac{1}{\mu}\frac{\partial}{\partial x} \\ \frac{1}{\epsilon}\frac{\partial}{\partial x} & 0 \end{bmatrix}$$

  yields the wave equation with speed $c = \sqrt{(}1/\epsilon\mu)$.

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- $u = [H, E, P]^T$ and $c = \sqrt{(}1/\epsilon\mu)$

  $$\mathcal{A} = \frac{1}{\tau}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1-\epsilon & c \\ 0 & \frac{\epsilon-1}{c} & -1 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{\mu}\frac{\partial}{\partial x} & 0 \\ \frac{1}{\epsilon}\frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  yields 1D Maxwell's equations with Debye polarization.

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An example of a numerical method is to replace $\frac{\partial u}{\partial x}$ at (t_j, x_i) with

$$\frac{U_{i,j} - U_{i-1,j}}{\Delta x}$$

for some fixed $\Delta x = x_i - x_{i-1}$. Called a finite difference.

Inverse Problems

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For example, a parameter estimation inverse problem attempts to determine values of a parameter set given (discrete) observations of (some) state variables.

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- Mathematically, find

$$\min_{q \in Q_{ad}} \left\| error\left(E(q), \hat{E}\right) \right\|.$$

For example, with data measured at fixed x and discrete times t_j

$$\min_{q \in Q_{ad}} \frac{1}{N} \sum_{j=1}^{N} \left(E(t_j; q) - \hat{E}_j\right)^2$$

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- Need a (fast) method for computing E.

Distributions of Parameters

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In these cases it not sufficient to use the average value of the parameters, rather one must compute all possible solutions and take the average of those.

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Expected value of solutions is given by

$$u(t, x; F) = \int_{\mathcal{Q}} \mathcal{U}(t, x; q) dF(q),$$

where $\mathcal{Q}$ is some admissible set and $F \in \mathfrak{P}(\mathcal{Q})$.

Inverse Problem for F

- Given data $\{\hat{E}\}_j$ we seek to determine a probability distribution F^*, such that

$$F^* = \min_{F \in \mathfrak{P}(\mathcal{Q})} \mathcal{J}(F),$$

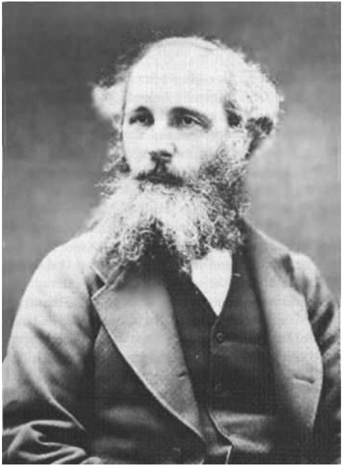
where, for example,

$$\mathcal{J}(F) = \sum_j \left(E(t_j; F) - \hat{E}_j \right)^2 .$$

- Given a trial distribution F_k we compute $E(t_j; F_k)$ and test $\mathcal{J}(F_k)$, then update F_{k+1} as necessary to find a minimum.
- Need either a parametrization or a discretization of F_k to have a finite dimensional problem.
- Need a (fast) method for computing $E(x, t; F)$.

Outline

Maxwell's Equations



- Maxwell's Equations were formulated circa 1870.
- They represent a fundamental unification of electric and magnetic fields predicting electromagnetic wave phenomenon.

Maxwell's Equations

$$\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} = \nabla \times \mathbf{H} \qquad \text{(Ampere)}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \text{(Faraday)}$$

$$\nabla \cdot \mathbf{D} = \rho \qquad \text{(Poisson)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss)}$$

$\mathbf{E} =$	Electric field vector	$\mathbf{D} =$	Electric displacement
$\mathbf{H} =$	Magnetic field vector	$\mathbf{B} =$	Magnetic flux density
$\rho =$	Electric charge density	$\mathbf{J} =$	Current density

Note: Need initial conditions and boundary conditions.

Constitutive Laws

Maxwell's equations are completed by constitutive laws that describe the response of the medium to the electromagnetic field.

$$\boxed{\begin{aligned} \mathbf{D} &= \epsilon\mathbf{E} + \mathbf{P} \\ \mathbf{B} &= \mu\mathbf{H} + \mathbf{M} \\ \mathbf{J} &= \sigma\mathbf{E} + \mathbf{J}_s \end{aligned}}$$

$\mathbf{P} =$	Polarization	$\epsilon =$	Electric permittivity
$\mathbf{M} =$	Magnetization	$\mu =$	Magnetic permeability
$\mathbf{J}_s =$	Source Current	$\sigma =$	Electric Conductivity

Linear, Isotropic, Non-dispersive and Non-conductive media

Assume no material dispersion, i.e., speed of propagation is not frequency dependent.

$$\boxed{\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{B} &= \mu \mathbf{H} \end{aligned}}$$

$$\epsilon = \epsilon_0 \epsilon_r \qquad \epsilon_r = \text{Relative Permittivity}$$

$$\mu = \mu_0 \mu_r \qquad \mu_r = \text{Relative Permeability}$$

Maxwell's Equations in One Space Dimension

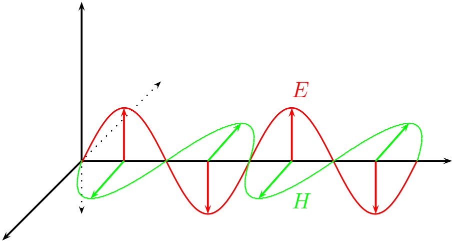
- The time evolution of the fields is thus completely specified by the curl equations

$$\begin{aligned} \epsilon \frac{\partial \mathbf{E}}{\partial t} &= \nabla \times \mathbf{H} \\ \mu \frac{\partial \mathbf{H}}{\partial t} &= -\nabla \times \mathbf{E} \end{aligned}$$

- Assuming that the electric field is polarized to oscillate only in the y direction, propagate in the x direction, and there is uniformity in the z direction:

Equations involving E_y and H_z.

$$\begin{aligned} \epsilon \frac{\partial E_y}{\partial t} &= -\frac{\partial H_z}{\partial x} \\ \mu \frac{\partial H_z}{\partial t} &= -\frac{\partial E_y}{\partial x} \end{aligned}$$



The Yee Scheme

In 1966 Kane Yee originated a set of finite-difference equations for the time dependent Maxwell's curl equations **(finite difference time domain or FDTD)**

- **Staggered Grids**: Choose E components on integer points in space and time, and H components on the half-grids in both variables.
- **Idea**:First order derivatives are much more accurately evaluated on staggered grids, such that if a variable is located on the integer grid, its first derivative is best evaluated on the half-grid and vice-versa.

Yee Scheme in One Space Dimension

$$\frac{H_z\big|_{r+\frac{1}{2}}^{n+\frac{1}{2}} - H_z\big|_{r+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta t} = -\frac{1}{\mu}\frac{E_y\big|_{r+1}^{n} - E_y\big|_{r}^{n}}{\Delta x}$$

$$\frac{E_y\big|_{r}^{n+1} - E_y\big|_{r}^{n}}{\Delta t} = -\frac{1}{\epsilon}\frac{H_z\big|_{r+\frac{1}{2}}^{n+\frac{1}{2}} - H_z\big|_{r-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x}$$

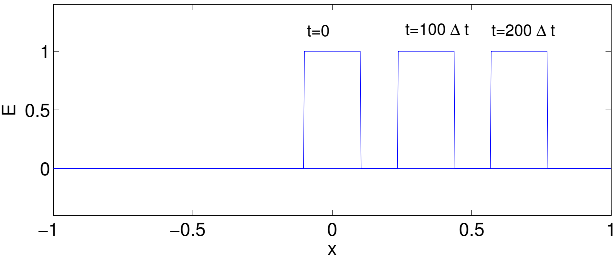
- This method is an explicit second order scheme in both space and time.
- It is conditionally stable with the CFL condition

$$\nu = \frac{c\Delta t}{\Delta x} \leq 1$$

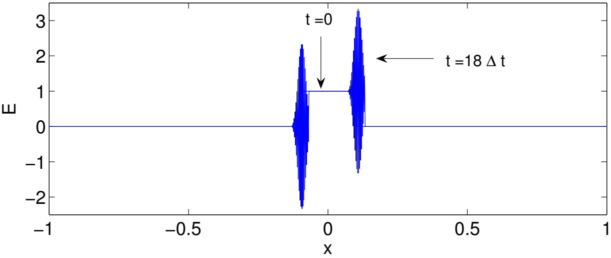
where ν is called the Courant number and $c = 1/\sqrt{\epsilon\mu}$.

Numerical Stability: A Square Wave

- Case $c\Delta t = \Delta x$

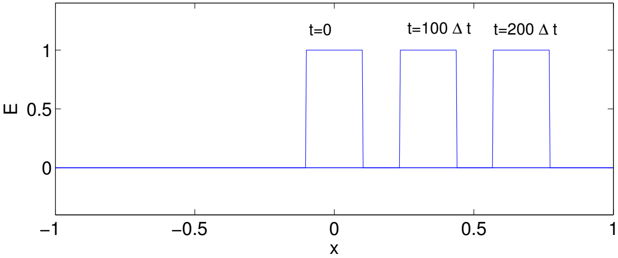


- Case $c\Delta t > \Delta x$



Numerical Dispersion: A Square Wave

- Case $c\Delta t = \Delta x$



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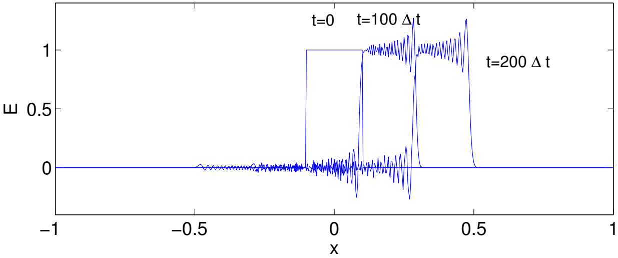

Outline

Dispersive Dielectrics

- Recall
$$\mathbf{D} = \epsilon \mathbf{E} + \mathbf{P}$$
where $\mathbf{P}$ is the dielectric polarization.
- Debye model
$$\tau \dot{\mathbf{P}} + \mathbf{P} = \epsilon_0(\epsilon_s - \epsilon_\infty)\mathbf{E}$$
where $q = \{\epsilon_\infty, \epsilon_s, \tau\}$ and, in particular, τ is called the relaxation time.

Frequency Domain

- Converting to frequency domain via Fourier transforms

$$\mathbf{D} = \epsilon\mathbf{E} + \mathbf{P}$$

becomes

$$\hat{\mathbf{D}} = \epsilon(\omega)\hat{\mathbf{E}}$$

where $\epsilon(\omega)$ is called the complex permittivity.

- Debye model gives

$$\epsilon(\omega) = \epsilon_\infty + \frac{\epsilon_s - \epsilon_\infty}{1 + i\omega\tau}$$

- Cole-Cole model (heuristic generalization)

$$\epsilon(\omega) = \epsilon_\infty + \frac{\epsilon_s - \epsilon_\infty}{1 + (i\omega\tau)^{1-\alpha}}$$

Unfortunately, the Cole-Cole model corresponds to a fractional order differential equation in the time domain, and simulation is not straight-forward.

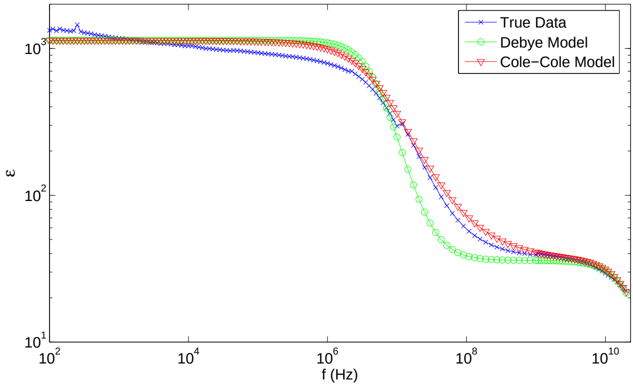

Figure: Real part of $\epsilon(\omega)$, ϵ, or the permittivity.

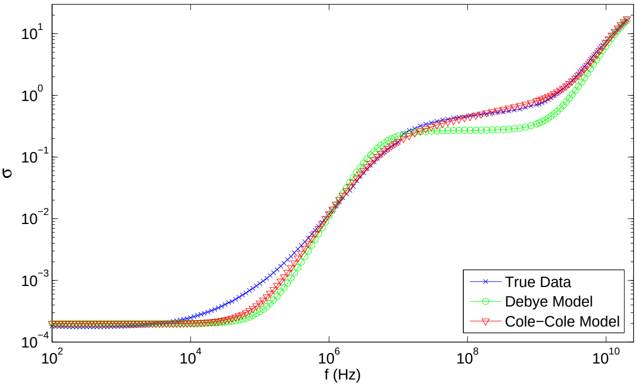

Figure: Imaginary part of $\epsilon(\omega)$, σ, or the conductivity.

Motivation

- The Cole-Cole model corresponds to a fractional order ODE in the time-domain and is difficult to simulate.
- Debye is efficient to simulate, but does not represent permittivity well.
- Better fits to data are obtained by taking linear combinations of Debye models (multi-pole Debye), idea comes from the known existence of multiple physical mechanisms.
- An alternative approach is to consider the Debye model but with a (continuous) distribution of relaxation times.
- Empirical measurements suggest a log-normal distribution.

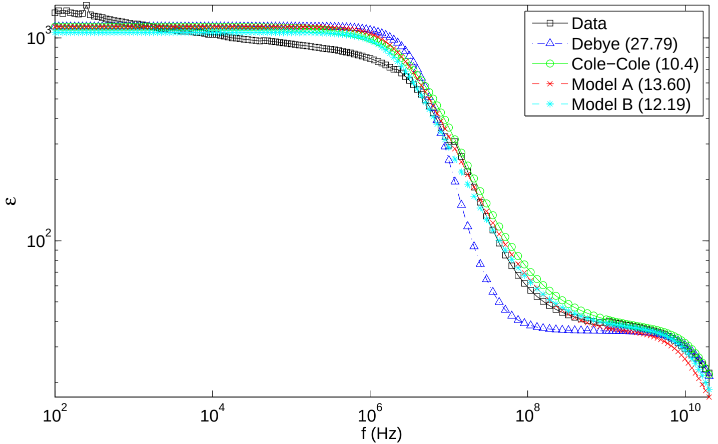

Figure: Real part of $\epsilon(\omega)$, called simply ϵ, or the permittivity. Model A refers to the Debye model with a uniform distribution on τ.

Random Polarization

We define the random polarization $\mathcal{P}(x, t; \tau)$ to be the solution to

$$\tau\dot{\mathcal{P}} + \mathcal{P} = \epsilon_0(\epsilon_s - \epsilon_\infty)E$$

where τ is a random variable with PDF $f(\tau)$, for example,

$$f(\tau) = \frac{1}{\tau_b - \tau_a}$$

for a uniform distribution.

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for a uniform distribution.

The electric field depends on the macroscopic polarization, which we take to be the expected value of the random polarization at each point (x, t)

$$P(x, t; F) = \int_{\tau_a}^{\tau_b} \mathcal{P}(x, t; \tau) f(\tau) d\tau.$$

Numerical Approximation of Random Polarization

Recall, to solve the inverse problem for the distribution of relaxation times, we need a method of accurately and efficiently simulating $P(x, t; F)$.

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Numerical Approximation of Random Polarization

Recall, to solve the inverse problem for the distribution of relaxation times, we need a method of accurately and efficiently simulating $P(x, t; F)$.

- Could apply a quadrature rule to the integral in the expected value. Results in a linear combination of individual Debye solves.
- Alternatively, we can use a method which separates the time derivative from the randomness and applies a truncated expansion in random space, called Polynomial Chaos. Results in a linear system.

Polynomial Chaos: Simple example

Consider the first order, constant coefficient, linear ODE

$$\dot{y} = -ky, \quad k = k(\xi) = \xi, \quad \xi \sim \mathcal{N}(0,1).$$

We apply a Polynomial Chaos expansion in terms of orthogonal Hermite polynomials H_j to the solution y:

$$y(t,\xi) = \sum_{j=0}^{\infty} \alpha_j(t)\phi_j(\xi), \quad \phi_j(\xi) = H_j(\xi)$$

then the ODE becomes

$$\sum_{j=0}^{\infty} \dot{\alpha}_j(t)\phi_j(\xi) = -\sum_{j=0}^{\infty} \alpha_j(t)\xi\phi_j(\xi),$$

Triple recursion formula

$$\sum_{j=0}^{\infty} \dot{\alpha}_j(t)\phi_j(\xi) = -\sum_{j=0}^{\infty} \alpha_j(t)\xi\phi_j(\xi),$$

We can eliminate the explicit dependence on ξ by using the triple recursion formula for Hermite polynomials

$$\xi H_j = jH_{j-1} + H_{j+1}.$$

Thus

$$\sum_{j=0}^{\infty} \dot{\alpha}_j(t)\phi_j + \alpha_j(t)(j\phi_{j-1} + \phi_{j+1}) = 0.$$

Galerkin Projection onto span($\{\phi_i\}_{i=0}^p$)

Taking the weighted inner product with each basis gives

$$\sum_{j=0}^{\infty} \dot{\alpha}_j(t)\langle \phi_j, \phi_i \rangle_W + \alpha_j(t)\big(j\langle \phi_{j-1}, \phi_i \rangle_W + \langle \phi_{j+1}, \phi_i \rangle_W\big) = 0,$$

$$i = 0, \ldots, p.$$

Where

$$\langle f(\xi), g(\xi) \rangle_W = \int f(\xi) g(\xi) W(\xi) d\xi.$$

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Where

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Using orthogonality, $\langle\phi_j,\phi_i\rangle_W = \langle\phi_i,\phi_i\rangle_W\delta_{ij}$, we have

$$\dot{\alpha}_i\langle\phi_i,\phi_i\rangle_W + (i+1)\alpha_{i+1}\langle\phi_i,\phi_i\rangle_W + \alpha_{i-1}\langle\phi_i,\phi_i\rangle_W = 0, \quad i = 0,\ldots,p,$$

Deterministic ODE system

Letting $\vec{\alpha}$ represent the vector containing $\alpha_0(t), \ldots, \alpha_p(t)$ (and assuming $\alpha_{p+1}(t)$, etc. are identically zero) the system of ODEs can be written

$$\dot{\vec{\alpha}} + M\vec{\alpha} = \vec{0},$$

with

$$M = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & p \\ & & & 1 & 0 \end{bmatrix}$$

The mean value of $y(t, \xi)$ is $\alpha_0(t)$.

Generalizations

For any choice of family of orthogonal polynomials, there exists a triple recursion formula. Given the arbitrary relation

$$\xi\phi_j = a_j\phi_{j-1} + b_j\phi_j + c_j\phi_{j+1}$$

(with $\phi_{-1} = 0$) then the matrix above becomes

$$M = \begin{bmatrix} b_0 & a_1 & & & \\ c_0 & b_1 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_p \\ & & & c_{p-1} & b_p \end{bmatrix}$$

Generalizations

Consider the non-homogeneous ODE

$$\dot{y} + ky = g(t), \quad k = k(\xi) = \sigma\xi + \mu, \quad \xi \sim \mathcal{N}(0,1).$$

then

$$\dot{\alpha}_i + \sigma\left[(i+1)\alpha_{i+1} + \alpha_{i-1}\right] + \mu\alpha_i = g(t)\delta_{0i}, \quad i = 0, \ldots, p,$$

or the deterministic ODE system

$$\dot{\vec{\alpha}} + (\sigma M + \mu I)\vec{\alpha} = g(t)\vec{e_1}.$$

Exponential convergence

- Any set of orthogonal polynomials can be used in the truncated expansion, but there may be an optimal choice.
- If the polynomials are orthogonal with respect to weighting function $f(\xi)$, and k has PDF $f(k)$, then it is known that the PC solution converges exponentially in terms of p.
- In practice, approximately 4 are generally sufficient.

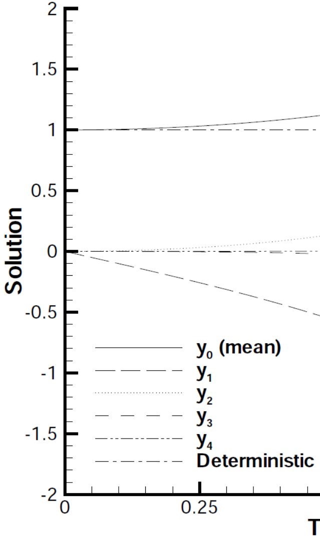

Figure: Solution of each mode with Gaussian random coefficient by fourth-order Hermitian-chaos.

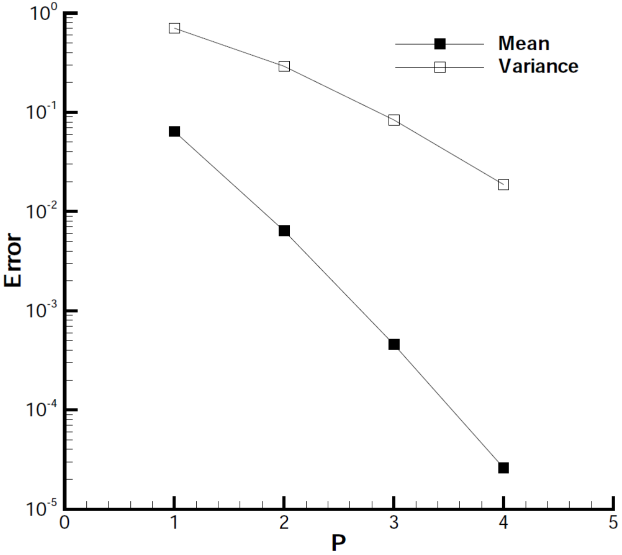

Figure: Convergence of error with Gaussian random coefficient by fourth-order Hermitian-chaos.

Generalized Polynomial Chaos

Table: Popular distributions and corresponding orthogonal polynomials.

Distribution	Polynomial	Support
Gaussian	Hermite	$(-\infty, \infty)$
gamma	Laguerre	$[0, \infty)$
beta	Jacobi	$[a, b]$
uniform	Legendre	$[a, b]$

Note: lognormal random variables may be handled as a non-linear function (e.g., Taylor expansion) of a normal random variable.

Random Polarization

We can apply Polynomial Chaos method to our random polarization

$$\tau\dot{\mathcal{P}} + \mathcal{P} = \epsilon_0(\epsilon_s - \epsilon_\infty)E, \quad \tau = \tau(\xi) = r\xi + r$$

resulting in

$$(rM + mI)\dot{\vec{\alpha}} + \vec{\alpha} = \epsilon_0(\epsilon_s - \epsilon_\infty)E\vec{e_1} =: \vec{g}$$

or

$$A\dot{\vec{\alpha}} + \vec{\alpha} = \vec{g}.$$

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or

$$A\dot{\vec{\alpha}} + \vec{\alpha} = \vec{g}.$$

The macroscopic polarization, the expected value of the random polarization at each point (t, x), is simply

$$P(t, x; F) = \alpha_0(t, x).$$

Applying the central difference approximation, based on the Yee scheme, Maxwell's equations with conductivity and polarization included

$$\epsilon \frac{\partial E}{\partial t} = -\frac{\partial H}{\partial z} - \sigma E - \frac{\partial P}{\partial t}$$

and

$$\mu \frac{\partial H}{\partial t} = -\frac{\partial E}{\partial z}$$

become

$$\frac{E_k^{n+\frac{1}{2}} - E_k^{n-\frac{1}{2}}}{\Delta t} = -\frac{1}{\epsilon}\frac{H_{k+\frac{1}{2}}^{n} - H_{k-\frac{1}{2}}^{n}}{\Delta z} - \frac{\sigma}{\epsilon}\frac{E_k^{n+\frac{1}{2}} + E_k^{n-\frac{1}{2}}}{2} - \frac{1}{\epsilon}\frac{P_k^{n+\frac{1}{2}} - P_k^{n-\frac{1}{2}}}{\Delta t}$$

and

$$\frac{H_{k+\frac{1}{2}}^{n+1} - H_{k+\frac{1}{2}}^{n}}{\Delta t} = -\frac{1}{\mu}\frac{E_{k+1}^{n+\frac{1}{2}} - E_k^{n+\frac{1}{2}}}{\Delta z}.$$

Note that while the electric field and magnetic field are staggered in time, the polarization updates simultaneously with the electric field.

Need a similar approach for discretizing the PC system

$$A\dot{\vec{\alpha}} + \vec{\alpha} = \vec{g}.$$

Applying second order central differences, as before, to $\vec{\alpha} = \vec{\alpha}(z_k)$:

$$A\frac{\vec{\alpha}^{n+\frac{1}{2}} - \vec{\alpha}^{n-\frac{1}{2}}}{\Delta t} + \frac{\vec{\alpha}^{n+\frac{1}{2}} + \vec{\alpha}^{n-\frac{1}{2}}}{2} = \frac{\vec{g}^{n+\frac{1}{2}} + \vec{g}^{n-\frac{1}{2}}}{2}.$$

Combining like terms gives

$$(2A + \Delta t I)\vec{\alpha}^{n+\frac{1}{2}} = (2A - \Delta t I)\vec{\alpha}^{n-\frac{1}{2}} + \Delta t \left(\vec{g}^{n+\frac{1}{2}} + \vec{g}^{n-\frac{1}{2}}\right)$$

Note that we first solve the discrete electric field equation for $E_k^{n+\frac{1}{2}}$ and plug in here (in $\vec{g}^{n+\frac{1}{2}}$) to update $\vec{\alpha}$.

Comments on Polynomial Chaos

- Gives a simple and efficient method to simulate systems involving distributions of parameters.
- Works equally well in three spatial dimensions.
- Limitation: choice of polynomials depends on type of distribution.
- Need error estimates to be sure that a sufficient number of polynomials is used in the expansion.

Outline

Inverse Problem for RTD

Now that we have a numerical method for simulating Maxwell's equations with random polarization

$$P(x,t;F) = \int_{\tau_a}^{\tau_b} \mathcal{P}(x,t;\tau)dF(\tau)$$

we address the inverse problem for the relaxation time distribution F.

- Given data $\{\hat{E}\}_j$ we seek to determine a probability distribution F^*, such that

$$F^* = \min_{F \in \mathfrak{P}(\mathcal{Q})} \mathcal{J}(F),$$

where

$$\mathcal{J}(F) = \sum_j k\left(E(t_j;F) - \hat{E}_j\right)^2.$$

Discrete Distribution Example

- Mixture of two Debye materials with τ_1 and τ_2
- Total polarization a weighted average

$$P = \alpha_1 P_1(\tau_1) + \alpha_2 P_2(\tau_2)$$

- Corresponds to the discrete probability distribution

$$dF(\tau) = [\alpha_1 \delta(\tau_1) + \alpha_2 \delta(\tau_2)]\, d\tau$$

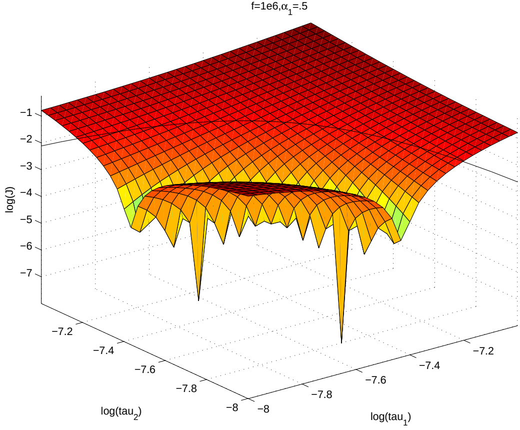
Discrete Distribution Inverse Problem

- Assume the proportions α_1 and $\alpha_2 = 1 - \alpha_1$ are known.
- Define the following least squares optimization problem:

$$\min_{(\tau_1,\tau_2)} \mathcal{J} = \min_{(\tau_1,\tau_2)} \sum_j \left| E(t_j, 0; (\tau_1, \tau_2)) - \hat{E}_j \right|^2,$$

where $\hat{E}_j$ is *synthetic* data generated using (τ_1^*, τ_2^*) in our simulation routine.

Discrete Distribution J using $10^6 Hz$



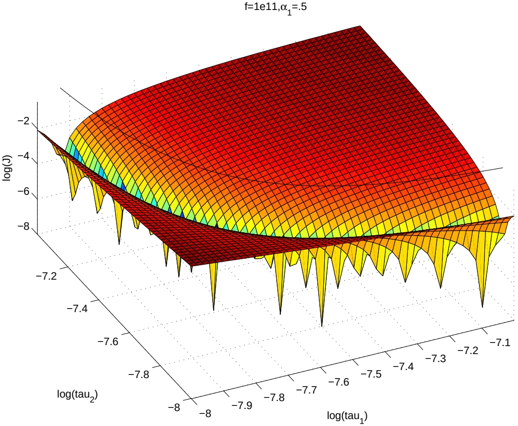
The solid line above the surface represents the curve of constant $\tilde{\tau} := \alpha_1\tau_1 + (1 - \alpha_1)\tau_2$. Note: $\omega\tilde{\tau} \approx .15 < 1$.

Inverse Problem Results $10^6 Hz$

	τ_1	τ_2	$\tilde{\tau}$
Initial	3.95000e-8	1.26400e-8	2.60700e-8
LM	3.19001e-8	1.55032e-8	2.37016e-8
Final	3.16039e-8	1.55744e-8	2.37016e-8
Exact	3.16000e-8	1.58000e-8	2.37000e-8

- Levenberg-Marquardt converges to curve of constant $\tilde{\tau}$
- Traversing curve results in accurate final estimates

Discrete Distribution J using $10^{11}Hz$



The solid line above the surface represents the curve of constant $\tilde{\lambda} := \frac{1}{c\tilde{\tau}} = \frac{\alpha_1}{c\tau_1} + \frac{\alpha_2}{c\tau_2}$. Note: $\omega\tilde{\tau} \approx 15000 > 1$.

Inverse Problem Results $10^{11} Hz$

	τ_1	τ_2	$\tilde{\lambda}$
Initial	3.95000e-8	1.26400e-8	0.174167
LM	4.08413e-8	1.41942e-8	0.158333
Final	3.16038e-8	1.57991e-8	0.158333
Exact	3.16000e-8	1.58000e-8	0.158333

- Levenberg-Marquardt converges to curve of constant $\tilde{\lambda}$
- Traversing curve results in accurate final estimates

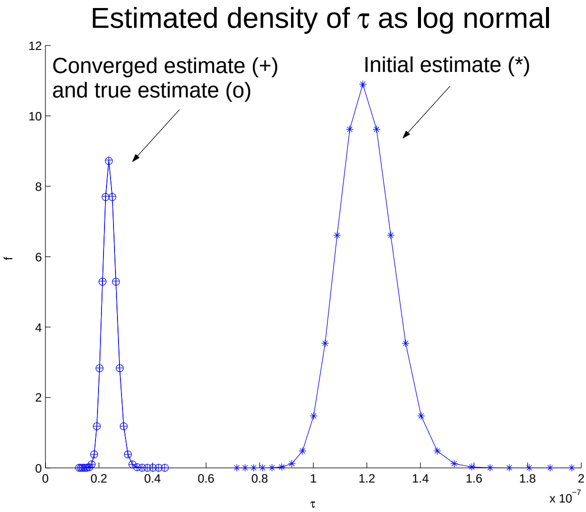
Log-Normal Distribution of τ

- Gaussian distribution of $\log(\tau)$ with mean μ and with standard deviation σ:

$$dF(\tau; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\ln 10} \frac{1}{\tau} \exp\left(-\frac{(\log \tau - \mu)^2}{2\sigma^2}\right) d\tau,$$

- Corresponding inverse problem:

$$\min_{q=(\mu,\sigma)} \sum_j \left| E(t_j, 0; (\mu, \sigma)) - \hat{E}_j \right|^2 .$$



Shown are the initial density function, the minimizing density function and the true density function (the latter two being practically identical).

Bi-Gaussian Distribution of $\log \tau$

- Bi-Gaussian distribution with means μ_1 and μ_2 and with standard deviations σ_1 and σ_2:

$$dF(\tau) = \alpha_1 d\hat{F}(\tau; \mu_1, \sigma_1) + (1 - \alpha_1) d\hat{F}(\tau; \mu_2, \sigma_2),$$

  where

$$d\hat{F}(\tau; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\ln 10} \frac{1}{\tau} \exp\left(-\frac{(\log \tau - \mu)^2}{2\sigma^2} \right) d\tau,$$

- Corresponding inverse problem:

$$\min_{q=(\mu_1,\sigma_1,\mu_2,\sigma_2)} \sum_j \left| |E(t_j, 0; q)| - |\hat{E}_j| \right|^2.$$

Bi-Gaussian Results with $10^6 Hz$

case	μ_1	σ_1	μ_2	σ_2	$\tilde{\tau}$
Initial	1.58001e-7	0.036606	3.16002e-9	0.0571969	8.1201e-8
μ_1,μ_2	4.27129e-8	0.036606	4.24844e-9	0.0571969	2.36499e-8
Final	3.09079e-8	0.0136811	1.63897e-8	0.0663628	2.37978e-8
Exact	3.16000e-8	0.0457575	1.58000e-8	0.0457575	2.37957e-8

- Levenberg-Marquardt converges to curve of constant $\tilde{\tau}$
- Traversing curve results in reasonable final estimates for μ_k but worse for σ_k.

Note: for this continuous distribution,

$$\tilde{\tau} = \int_{\mathcal{T}} \tau dF(\tau).$$

Bi-Gaussian Results with $10^{11} Hz$

case	μ_1	σ_1	μ_2	σ_2	$\tilde{\lambda}$
Initial	1.58001e-7	0.036606	3.16002e-9	0.0571969	0.538786
μ_1,μ_2	1.58001e-7	0.036606	1.12595e-8	0.0571969	0.158863
Final	3.23914e-8	0.0366059	1.56020e-8	0.0571968	0.158863
Exact	3.16000e-8	0.0457575	1.58000e-8	0.0457575	0.158863

- Levenberg-Marquardt converges to curve of constant $\tilde{\lambda}$
- Traversing curve results in reasonable final estimates for μ_k but no change in σ_k.

Note: for this continuous distribution,

$$\tilde{\lambda} = \int_{\mathcal{T}} \frac{1}{c\tau} dF(\tau).$$

Comments on Time-domain Inverse Problems

- Our estimation methods worked well for discrete distributions
- Our estimation methods worked well for the continuous uniform distribution and Gaussian distributions
- We are currently only able to determine the means in the bi-Gaussian distributions, this data is relatively insensitive to the standard deviations