An insurance model for the optimal allocation of hydropower flexibility in renewable energy markets

H. Jiang*[1], N. L. Gibson†[2], and Y. Chen‡[3]

[1,2]*Department of Mathematics, Oregon State University, Corvallis, OR, USA*
[3]*Department of Applied Economics, Oregon State University, Corvallis, OR, USA*

Abstract

This paper considers the revenue maximization problem for a hydropower company. The company can generate excess electricity by releasing water from a reservoir and then sell it to the energy market. On the other hand, the company has an obligation to keep the reservoir level above a pre-determined level, which may require the company to purchase electricity in order to fulfill the customers' power demand. The electricity price and reservoir level are both represented by diffusion processes. Two models for the stochastic control problem are considered, including one which assumes drifted Brownian motion for noise. For the first model, it turns out that the optimal timing and quantity of electricity generation or purchase can be determined using a simple strategy. In the second model, we refer to a one-factor diffusion model for electricity price, which is known to fit the data well. The existence and uniqueness of the value function is verified through techniques of viscosity solutions.

Keywords: Stochastic control, hydropower flexibility, renewable energy market, one-factor diffusion, Hamilton-Jacobi-Bellman equation, viscosity solution
Mathematics Subject Classification: 91B02, 91B24, 91B70, 91B74, 91B76, 91G80, 93E20

1 Introduction

Due to a high consumption rate of limited energy sources, the question of properly and efficiently utilizing hydroelectric power, in conjunction with other renewable energy sources, is an important and complex problem. Due to differing objectives and priorities, various proposed strategies have been considered. Additionally, many mathematical questions have been raised by the study of these approaches.

*jiangh@oregonstate.edu
†Corresponding author, gibsonn@oregonstate.edu
‡yong.chen@oregonstate.edu

Since water in reservoirs can either be released to generate hydropower in the current period, or withheld to generate hydropower in the future, one fascinating research question is about the best control strategy to maximize the profit from selling excess hydropower in the electricity market. However these actions should also mitigate the environmental impacts and avoid violation of legal and administrative regulations.

This paper is motivated by considerations of the Bonneville Power Administration (BPA) that operates in Pacific Northwest and markets electrical power from dams located in the Columbia River Basin. Significant uncertainties preclude deterministic optimization methods ([6, 10]). Under this consideration, several papers have made contributions through different approaches. The first attempt to model and assess the operational flexibility of the hydropower is due to [14], in which the model of flexibility is assessed for the system efficiency. The work [20] uses a different type of quantification and evaluates this flexibility by the method of option pricing. They believe that the flexibilities of the water storage and the inflow could be sold like options in the market. A direct optimization, a bi-level robust optimization approach, for the flexibility allocation problem was presented in [5].

This paper introduces a novel stochastic optimal control model which has been extensively studied in insurance ([1, 11, 13, 15, 17]). The reason that the stochastic control model fits this scenario is that we have a general assumption on the relation between demand and hydropower generating potential due to a continuous inflow. That is to say the contracts' or customers' demand should not exceed the supply ability in general except possibly during the peak electricity demand. Therefore, the reservoir will have excessive water in storage normally and the company should find a way to utilize this additional energy source properly. This is important for the future development of the renewable energy market. As market uncertainty tends to increase due to the increasing share of the wind and solar energy, the trading of energy flexibility will become more critical.

On the other hand, as mentioned above, the demand peak occurs randomly and it may cause a shortage of the reservoir storage for electricty generation. While meeting the regular demand from its customers, the company also needs to avoid the violation of a minimal reservoir level set by law (e.g., for wildlife considerations). Then the company needs to cease releasing the reservoir water, and instead purchase the electricity from other power suppliers in the energy market. This mechanism indeed matches that of the model with reflecting barrier ([21]) and optimal dividend problem with capital injection ([15]). These models do not apply here directly, but their extensions considered in [7, 8, 11] will help us to reveal the adaptability of this stochastic control here, where the energy commodity is traded with a stochastic electricty price.

The outline of this paper is as follows. In Section 2, we provide a classical stochastic model for this system. We model both the electricity price and reservoir level as a drifted Brownian motion. Although there is significant evidence that the model assumption on electricity price is not reasonable ([2]), we are interested in this model since here we can determine an analytical solution with an explicit simple strategy to implement. This model formulation has been considered in [7], but there it is mainly for the purpose of including some random interest rate into the stochastic control system. Here we take a different point of view and try to explore the applicability of this model in the energy market. In Section 2, after the problem formulation, we establish the associated Hamilton-Jacobi-Bellman (HJB) equation and point out that the value function will be the solution of this system. It needs to be mentioned that there are several constraints forced by the legitimate policies for the reservoir system, so the model here is somewhat different from that in [7]. We will solve this system and obtain an analytical solution. As mentioned above, this baseline model lacks the ability to resemble the realistic data specifically on price data.

A more convincing model from data analysis, the well-known one-factor diffusion model for electricity price, is described in [18]. In Section 4, we study the case when the electricity price follows this one-factor diffusion model which incorporates the seasonal change and mean reverting property of the data. Based on that, we give the objective value function and derive HJB equation (see p.539 of [3]). The corresponding numerical solution to this system is not known. In this paper, we address the issues regarding existence and uniqueness of solutions to the HJB equation for this problem via the method of viscosity solutions. With some standard techniques, we verify that the solution of an HJB equation is the value function and then prove that the value function is the viscosity solution of the HJB equation. This result implies that the numerical solution from the applicable method would generate the optimal executive "trigger price" strategy to control the dams.

2 General model

First we lay out our assumptions on the quantities of interest, and then we represent each with appropriate models. There should be sufficient electricity in the market to purchase at anytime. In particular, we assume that the electricity market is complete, which is most frequently assumed for stock prices in finance. The reservoir level is mainly influenced by the inflow and usage of the water by power company (we assume no spilling of water or evaporation). The inflow will add water into the reservoir which will raise the reservoir level, but the usage of water to produce the electricity will cause the reservoir level to go down. Considering that there is randomness inside these processes, we assume them to be stochastic processes. Here we make the following assumptions on the electricity price and the reservoir level. The price is denoted by P_t and described by the following stochastic differential equation (SDE)

$$dP_t = \theta(t, P_t)dt + \eta(t, P_t)dW_t, \tag{2.1}$$

where W_t is Brownian motion. The underlying natural reservoir level is modeled by,

$$dX_t = a(t, X_t)dt + \sigma(t, X_t)dB_t, \tag{2.2}$$

where B_t is Brownian motion and it is used here to account for the overall aggregate randomness in inflow, usage of water for electricity and other environmental consumption. The function $a(t, X_t)$ is the drift term and it can estimate the pattern that the reservoir level should behave like in the future.

There are two controls in our approach. One control is to use some portion of the water to generate the electricity to sell. The other one is to buy the electricity from market to meet any unmet needs of consumers, and/or government regulations, at the same time. We allow for the controls to depend on the states of the reservoir level, thus these two controls will be denoted by two stochastic processes: $S(t, X_t)$ and $Z(t, X_t)$. The quantity $S(t, X_t)$

represents the amount of water to be used to generate electricity to sell on the market. The quantity $Z(t, X_t)$ is the amount of electricity purchased from the market, converted to an equivalent amount of water. Both will be determined by the choice of a particular strategy, which we will denote with index τ. The resulting controlled reservoir level is indicated by

$$X^\tau = X - S^\tau + Z^\tau. \tag{2.3}$$

Any control strategy must ensure that the reservoir level stays above the minimum level m to avoid violating legal constraints.

The ultimate goal for the power company is to maximize the total expected discounted net profit, namely the quantity defined below. Suppose that at time t_c, $X^\tau_{t_c} = x, P_{t_c} = p$, then

$$V^\tau(t_c; p, x) = E_{p,x}\left\{\int_{t_c}^{\infty} e^{-\delta s} P_s (dS^\tau_s - \phi dZ^\tau_s)\right\}$$

where p and x are the current market electricity price and reservoir level respectively. The interest rate, $\delta > 0$, is assumed to be constant. The purchase cost $Z(t, X_t)P_t$ is penalized by the factor $\phi > 1$. The optimization problem is to find the optimal strategy τ^*, in the set $\mathcal{A}$ of all admissible strategies for τ (e.g., such that $X^\tau \geqslant m$, [7]), that maximizes $V^\tau(p, x)$. We call

$$V(t_c; p, x) = \sup_{\tau\in\mathcal{A}} E_{p,x}\left\{\int_{t_c}^{\infty} e^{-\delta s} P_s (dS^\tau_s - \phi dZ^\tau_s)\right\}$$

the value function of this problem.

In the following, we will consider the problem of the optimization of the total expected discounted net profit, under two different assumptions for the form of the SDE representing the price.

3 Homogeneous model

In this section, we assume the simple scenario that the reservoir storage process satisfies $a(t, X_t) = a$ and $\sigma(t, X_t) = \sigma$, from (2.2), plus the effects due to the controls, thus

$$dX^\tau_t = a\ dt + \sigma\ dB_t - dS^\tau_t + dZ^\tau_t. \tag{3.1}$$

The market electricity price satisfies $P_t = \exp(r_t)$ where $dr_t = b\ dt + \eta\ dW_t$. In other words, P_t is represented with Geometric Brownian motion. Then the value function can be rewritten as

$$V(r, x) = \sup_{\tau\in\mathcal{A}} E_{r,x}[\int_0^{\infty} e^{-\delta s + r_s}(dS^\tau_s - \phi dZ^\tau_s)].$$

Through the process of applying the dynamic programming principle, assuming independence between X_t and r_t (this is condition is relaxed in Section 3.3 below), we can derive the following HJB equation:

$$bV_r + \frac{\eta^2}{2}V_{rr} + aV_x + \frac{\sigma^2}{2}V_{xx} - \delta V + \sup_\tau \{(e^r - V_x)g^\tau - (\phi e^r - V_x)l^\tau\} = 0 \tag{3.2}$$

where g^τ and l^τ are the time derivatives of S_t^τ and Z_t^τ at the differentiable points. Note that S_t^τ and Z_t^τ are differentiable almost everywhere. In the following, we will provide the way to find the analytical solution to this HJB equation. Note that we need not explicitly construct g^τ or l^τ in order to determine the optimal strategy τ^* and the corresponding value function.

3.1 Analytical solutions

We solve (3.2) by the separation $V(r,x) = e^r F(x)$. Then it can be reduced to:

$$aF_x + \frac{\sigma^2}{2}F_{xx} + (b - \delta + \frac{\eta^2}{2})F + \sup_\tau \{(1 - F_x)g^\tau - (\phi - F_x)l^\tau\} = 0. \tag{3.3}$$

Suppose that $0 \leqslant l^\tau \leqslant L$ and $0 \leqslant g^\tau \leqslant G$. Then we have the following lemma.

Lemma 3.1. *The function $F(x)$ satisfies the following properties when both G and L are unbounded:*

(a) $F'(m) = \phi$, $F'(x) \leqslant \phi$, where m is the minimal reservoir level.

(b) There exists a point x_0 such that $F'(x_0) = 1$.

(c) $F(x)$ is continuously differentiable.

(d) $F(x)$ is positive, increasing, and concave.

(e) $F(x)$ is bounded above if G is finite.

The proof of this Lemma is given in Appendix A. We then divide the real line $\{x \geqslant 0\}$ into two sets $S_1 = \{x : 1 \leqslant F_x \leqslant \phi\}$ and $S_2 = \{x : 0 \leqslant F_x < 1\}$ which we consider separately.

(1) Let $F_x < 1$. Suppose G and L are unbounded, then (3.3) becomes

$$(a - G)F_x + \frac{\sigma^2}{2}F_{xx} + (b - \delta + \frac{\eta^2}{2})F + G = 0.$$

Dividing both sides by G and letting $G \to \infty$, we have

$$F_x - 1 = 0,$$

and therefore

$$F(x) = x + C, \quad x \in S_2. \tag{3.4}$$

For clarity, let $F_2(x) = x + C$.

(2) Let $1 \leqslant F_x \leqslant \phi$. Then, 3.3 becomes

$$aF_x + \frac{\sigma^2}{2}F_{xx} + (b - \delta + \frac{\eta^2}{2})F = 0$$

and therefore

$$F(x) = Ae^{\alpha x} + Be^{\beta x}, \quad x \in S_1 \tag{3.5}$$

where $\alpha = \frac{-a-\sqrt{a^2-2\sigma^2(b-\delta+\frac{\eta^2}{2})}}{\sigma^2}$ and $\beta = \frac{-a+\sqrt{a^2-2\sigma^2(b-\delta+\frac{\eta^2}{2})}}{\sigma^2}$.
For clarity, let $F_1(x) = Ae^{\alpha x} + Be^{\beta x}$.

We may now solve for the solution $F(x)$ by applying the properties in Lemma 3.1. Regarding (e), as it is unrealistic to allow the reservoir to fill to any arbitrary level, we assume that the release level G is unbounded so that the reservoir level can stop ascending if sufficient water is released. We now have the following:

$$\begin{aligned} F_1(x) &= Ae^{\alpha x} + Be^{\beta x} \\ F_2(x) &= x + C \\ F_1'(m) &= \phi \\ F_1'(x_0) &= 1 \\ F_1(x_0) &= F_2(x_0) \\ F_1''(x_0) &= 0 \end{aligned}$$

By some manipulations of these equations, we get the simplifications:

$$\begin{aligned} A\alpha e^{\alpha m} + B\beta e^{\beta m} &= \phi, \\ A\alpha e^{\alpha x_0} + B\beta e^{\beta x_0} &= 1, \\ Ae^{\alpha x_0} + Be^{\beta x_0} &= x_0 + C, \\ A\alpha^2 e^{\alpha x_0} + B\beta^2 e^{\beta x_0} &= 0. \end{aligned}$$

After some algebraic operations, we derive the following implicit equation that defines x_0,

$$(\alpha - \beta)\phi e^{(\alpha+\beta)x_0} = \alpha e^{\alpha x_0 + \beta m} - \beta e^{\beta x_0 + \alpha m}. \tag{3.6}$$

Once we obtain the answer of x_0, we can solve for A, B, C.

$$\begin{aligned} A &= -\frac{\beta}{(\alpha - \beta)\alpha e^{\alpha x_0}}, \\ B &= \frac{\alpha}{(\alpha - \beta)\beta e^{\beta x_0}}, \\ C &= \frac{\alpha + \beta}{\alpha\beta} - x_0. \end{aligned}$$

Therefore the value function is

$$V(x, r) = \begin{cases} e^r F_1(x) & \text{if } m \leqslant x \leqslant x_0 \\ e^r F_2(x) & \text{if } x \geqslant x_0. \end{cases}$$

A similar argument can be made if G is bounded. In either case, it is the value of x_0 that determines the strategy, i.e., the *barrier strategy* for this simple example, which says to sell if the reservoir level exceeds x_0 (as well as buy if the level falls below m, which itself is often given).

3.2 Optimal strategies

We emphasize that the stochastic process for the reservoir level, namely X_t satisfying SDE: $dX_t = adt + \sigma dB_t$, is the real reservoir level with electricity power generation to meet demand. The controlled reservoir level is indicated by $X^\tau = X - S^\tau + Z^\tau$. In the previous section, we show that the optimal timing to generate the extra electricity to sell or purchase electricity from the market depends on two constants: m and x_0. The quantity m is the level that should be calibrated with caution. It should be higher than the minimal level required by the law, but it should not be too high to be useless either. Here we assume that m has been pre-determined. Therefore, whenever the reservoir level X_t^τ falls down to the level m, the company will stop generating electricity from the water. Instead, they will buy as much as needed to meet demand. In any case, X_t^τ is not allowed to go down below m, because the company will then risk severe penalty. In fact, this will be avoided after they stop using water, since the continuous inflow will replenish the reservoir.

However when X_t^τ keeps climbing and is about to cross the level x_0, the company should make use of any water that is above x_0 to generate electricity. In reality, it is possible to implement since the inflow usually does not include too much variation (relative to price or demand variation). The picture in Figure 1 depicts the basic procedure for this type of barrier strategy as an example.

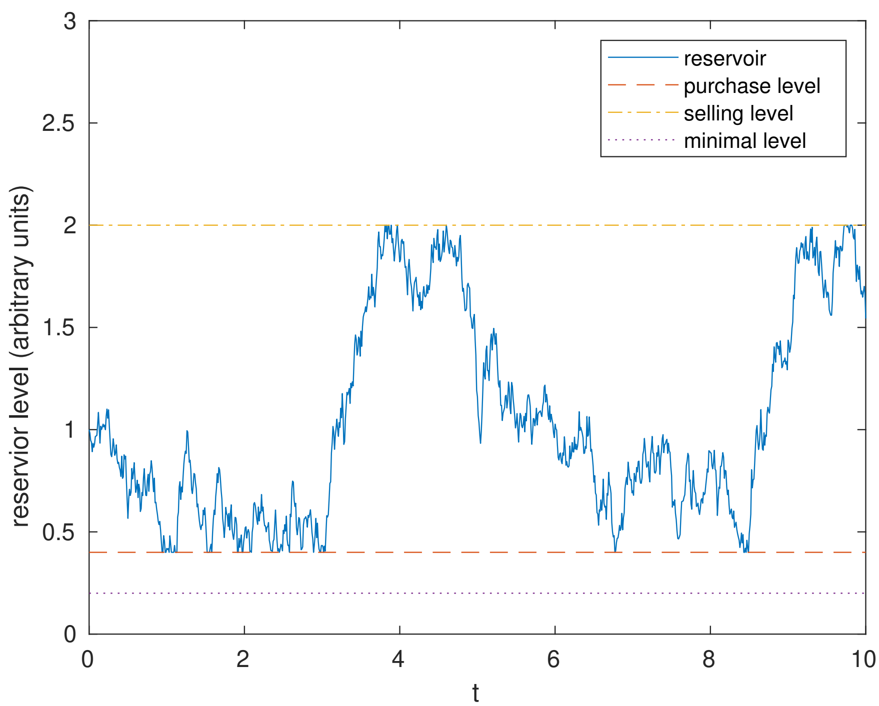

Figure 1: Simulation of an example of the optimal barrier strategy. The selling level is x_0.

3.3 Model with dependence

We make a slight but important improvement to the model in the previous section. It is unrealistic to assume independence between the electricity price and the power generation, since in economic theory, the supply and demand are closely related to each other. For example, if significant quantities of excess power are sold on to the market, the price will fall. Therefore it is reasonable to impose the dependence assumption on the model. Evidence shows that the price and generated power are negatively correlated. One way to measure the dependence is the covariance function.

Suppose that X and Y are two stochastic processes. The covariance between these two processes are defined as follows:

$$cov(X_t, Y_t) = E[(X_t - E[X_t])(Y_t - E[Y_t])].$$

From this, the correlation between X and Y is defined using covariance:

$$corr(X_t, Y_t) = \frac{cov(X_t, Y_t)}{\sigma_{X_t}\sigma_{Y_t}},$$

where $\sigma_{X_t}, \sigma_{Y_t}$ are standard variances. Here we assume that the correlation between Brownian motions B_t and W_t, which are introduced in (2.1) and (2.2), is a constant $k < 0$. It is negative since electricity price and generated power have negative correlation.

The quadratic variation between the differentials dr_t and dX_t, is then given as:

$$< dr_t, dX_t >= kdt.$$

Suppose $f \in \mathbb{C}^{2,2}$. Two-dimensional Itô's formula for continuous Markov processes can be given by:

Lemma 3.2 (Itô Lemma)**.**

$$df(X_t, Y_t) = f_x dX_t + f_y dY_t + f_{xy} < dX_t, dY_t > + \frac{1}{2} f_{xx} < dX_t, dX_t > + \frac{1}{2} f_{yy} < dY_t, dY_t > .$$

Applying this to the problem formulation we have,

$$dV(r_t, X_t) = V_r dr_t + V_x dX_t + V_{rx} < dr_t, dX_t > + \frac{1}{2} V_{rr} < dr_t, dr_t > + \frac{1}{2} V_{xx} < dX_t, dX_t > .$$

The value function for this problem is still the same as above:

$$V(r, x) = \sup_{\tau \in \mathcal{A}} E_{r,x}[\int_0^\infty e^{-\delta t + r_t}(dS_t^\tau - \phi dZ_t^\tau)].$$

The HJB equation becomes:

$$bV_r + \frac{\eta^2}{2} V_{rr} + k\eta\sigma V_{rx} + aV_x + \frac{\sigma^2}{2} V_{xx} - \delta V + \sup_\tau \{(e^r - V_x)g^\tau - (\phi e^r - V_x)l^\tau\} = 0. \quad (3.7)$$

As before, we may use the separation $V(r, x) = e^r F(x)$. Then (3.7) can be reduced to the following equation:

$$(a + k\eta\sigma)F_x + \frac{\sigma^2}{2} F_{xx} + (b - \delta + \frac{\eta^2}{2})F + \sup_\tau \{(1 - F_x)g^\tau - (\phi - F_x)l^\tau\} = 0.$$

According to Lemma 3.1, we can solve this equation by exactly the same way that is applied in the previous section. The process will not be repeated here.

3.4 Example

For simplicity, suppose that the price and power produced are independent from each other. Also, consider the case when G is unbounded. Suppose that parameters are given as: $a = 6, b = 0.5, \eta = 1.2, \sigma = 3, \delta = 1.3, \phi = 1.8$ and $m = 3$. Solving the equation (3.6) numerically, we obtain that the selling level should be approximately $x_0 = 6.32$.

The optimal operation can be interpreted in the following way:

(1) When the reservoir level reaches the purchase level m=3, the company immediately stops using the water to generate the power for its customers because of the high risk in low level. Instead, it purchases the electricity from the market to meet the needs.

(2) When the reservoir level stays between the purchase level m=3 and the selling level $x_0 = 6.32$, the company only uses the amount of water that could generate the necessary electricity for its customers.

(3) When the reservoir level runs above the selling level $x_0 = 6.32$, the company will use the portion of water that is above x_0 to generate the electricity and then sell it in the market.

Figure 2 illustrates this strategy, which depends on the reservoir level. Figure 3 gives the computed value function for this system and the optimal "curve strategy". We call the optimal strategy here a "curve strategy" because the control is exerted whenever the state of this two-dimensional system hits the boundaries of some specific region.

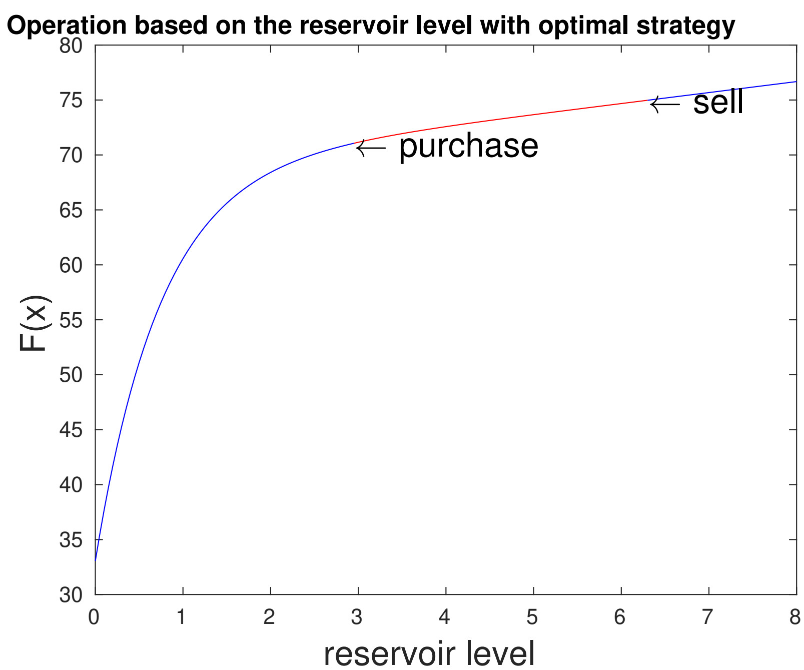

Figure 2: The optimal operation based on the reservoir level for parameters in the example. The plot shows $F(x)$, the solution to (3.3), e.g., $F_1(x)$ until x_0, and then switching to $F_2(x)$.

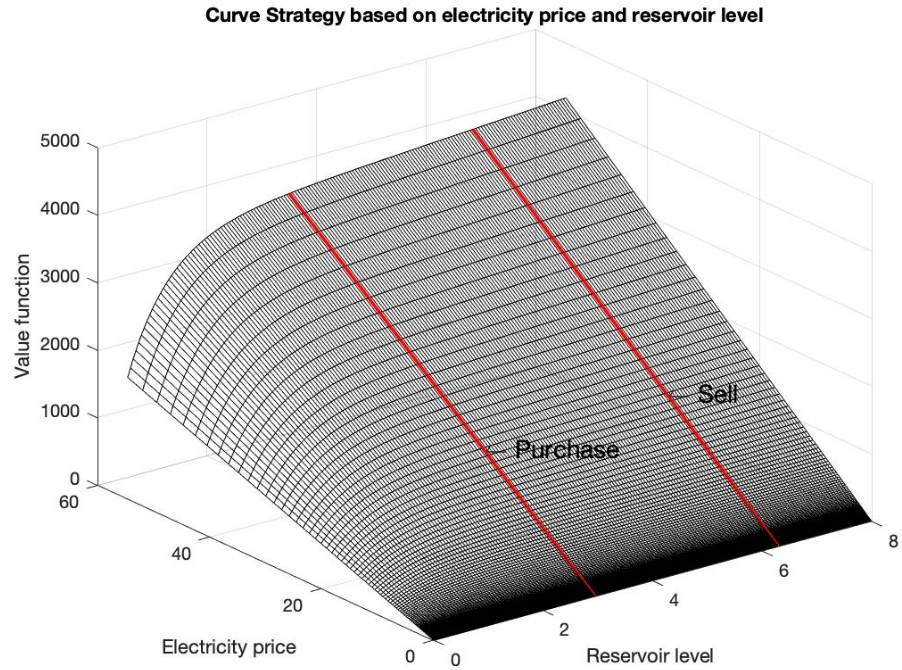

Figure 3: The value function and curve strategy in terms of electricity price and reservoir level for parameters in the example. The plot shows $V(r, x)$, the solution to (3.2).

4 One-factor diffusion model

4.1 Model assumption

In actual data of electricity price, there are features like mean-reversion and spike prices. However, the model proposed above for the price cannot capture these important features. To account for this, papers ([2, 18]) investigate a sophisticated model to incorporate these features. It is called the one-factor diffusion model and has the assumptions that:

$$
\begin{aligned}
dr_t &= (f'(t) + \lambda(b - r_t))dt + \eta dW_t \\
P_t &= \exp(r_t)
\end{aligned}
$$

where the function $f(t)$ will have a seasonal effects on the price and the stochastic process $R = (r_t)_{t \geqslant 0}$ has the mean-reverting property.

In [21], the optimal control of a diffusion model for the reservoir has been considered. This is known as the storage problem in general. The model we describe in Section 2 is in this classification. It is worth mentioning that we can consider the general diffusion model for the reservior as suggested by [21]. However, here we keep the simplified model of drifted Brownian motion for the underlying reservoir system, plus the effects of the controls, (3.1).

There is not much difference compared with the system we introduced previously except the assumption on electricity price. Therefore we have the following formulation for the value function: suppose that at time t_c, $X^\tau_{t_c} = x, r_{t_c} = r$,

$$V(t_c; x, r) = \sup_\tau E_{x,r}[\int_{t_c}^\infty e^{-\delta(s-t_c)} P_s(g^\tau_s - \phi l^\tau_s) ds]. \tag{4.1}$$

After Bellman Dynamic Programming (c.f., [7, 19]), we arrive with the following Hamilton-Jacobi-Bellman equation:

$$V_t - \delta V + (f'(t_c) + \lambda(b-r))V_r + \frac{\eta^2}{2} V_{rr} + aV_x + \frac{\sigma^2}{2} V_{xx} + \sup_\tau \{(e^r - V_x)g^\tau - (\phi e^r - V_r) l^\tau\} = 0. \tag{4.2}$$

However, there is no analytical solution, at least known to us, so here we emphasize the analysis for this model and its solution.

4.2 Verification theorem

First, we want to show that the solution of HJB (4.2) indeed is the value function (4.1) we are looking for. We answer this with the following verification theorem and its proof is quite standard, as suggested by other numerous research work (c.f., [7, 9]). Before the proof, we introduce the bi-variate extended generator for the two-dimensional stochastic process (X, R):

Definition 4.1. *Suppose that $S(x, r) \in C^{1,2}(\mathbb{R}^+, \mathbb{R})$. The bi-variate extended generator for two-dimensional stochastic process (X,R) is:*

$$GS(x, r) = \lim_{t \to 0^+} \frac{E_{x,r}[S(X_t, r_t)] - S(x, r)}{dt}.$$

In particular, if $X = (X^\tau_t), R = (r^\tau_t)$ and the current state is $(X^\tau_{t_c}, r^\tau_{t_c}) = (x, r)$, then

$$G^\tau S(x, r) = (f'(t_c) + \lambda(b - r))S_r + \frac{\eta^2}{2} S_{rr} + aS_x + \frac{\sigma^2}{2} S_{xx} - g^\tau S_x + l^\tau S_x.$$

If $X = (X_t), R = (r_t)$, then the bivariate extended generator for (X, R) is:

$$AS(x, r) = (f'(t_c) + \lambda(b - r))S_r + \frac{\eta^2}{2} S_{rr} + aS_x + \frac{\sigma^2}{2} S_{xx}.$$

Below we will give the estimation of $V_x(t_c; x, r)$ for any $r \in \mathbb{R}$.

Lemma 4.1. *For any $r \in \mathbb{R}$ and time $t_c \geqslant 0$, $e^r \leqslant V_x(t_c; x, r) \leqslant \phi e^r$ for $x \geqslant 0$.*

Proof. In the proof, we denote the left derivative by V^- and right derivative by V^+. It can be divided into two steps here:

(A) Denote a stopping time $\tau_h = \inf\{s > t_c | X_s = x, X_{t_c} = x + h\}$. Let h be small enough and $x \geqslant 0$. Then

$$\begin{aligned} V(t_c; x + h, r) &= \sup_\tau E_{x+h,r} \left[\int_0^\infty e^{-\delta(s-t_c)+r_s} (g^\tau_s - \phi l^\tau_s) ds \right] \\ &= \sup_\tau E \left[\int_0^{\tau_h} e^{-\delta(s-t_c)+r_s} (g^\tau_s - \phi l^\tau_s) ds + \int_{\tau_h}^\infty e^{-\delta(s-t_c)+r_s} (g^\tau_s - \phi l^\tau_s) ds \right] \\ &\leqslant \sup_\tau E \left[\int_0^{\tau_h} e^{-\delta(s-t_c)+r_s} (g^\tau_s - \phi l^\tau_s) ds \right] + e^{-\delta(\tau_h - t_c)} E_r \left[V(\tau_h; x, r_{\tau_h}) \right]. \end{aligned}$$

Note that when $h \to 0$, the inequality above becomes equality. On the other hand,

$$V(t_c; x+h, r) - e^{-\delta(\tau_h - t_c)} E_r\left[V(\tau_h; x, r_{\tau_h})\right] = (V(t_c; x+h, r) - V(\tau_h; x+h, r)) + (V(\tau_h; x+h, r) - \dots$$
$$- e^{-\delta(\tau_h - t_c)} V(x+h, r)) + (e^{-\delta(\tau_h - t_c)} V(x+h, r) - e^{-\delta(\tau_h - t_c)} E_r\left[V(x+h, r_{\tau_h}))\right] + \dots$$
$$+ (e^{-\delta(\tau_h - t_c)} E_r\left[V(x+h, r_{\tau_h})\right] - e^{-\delta(\tau_h - t_c)} E_r\left[V(x, r_{\tau_h}))\right]$$

Here we claim that $\lim_{h\to 0} \frac{\tau_h - t_c}{h} = 0$ as the consequence of Law of the Iterated Logarithm (P 143 of [4]). With this and the above decomposition of $V(t_c; x+h, r) - e^{-\delta(\tau_h - t_c)} V(t_h; x, r_{\tau_h})$, we have

$$\frac{V(t_c; x+h, r) - e^{-\delta(\tau_h - t_c)} E_r\left[V(\tau_h; x, r_{\tau_h})\right]}{h} \approx \frac{\tau_h - t_c}{h}(V - V_t) + V_r \frac{E[r - r_{\tau_h}]}{h} + V_x.$$

The first term will vanish as $h \to 0$ because $\lim_{h\to 0} \frac{\tau_h - t_c}{h} = 0$. For the second term, notice that

$$\frac{E\left[r - r_{\tau_h}\right]}{h} \approx -f'(t_c)\frac{\tau_h - t_c}{h} - \lambda(b-r)\frac{\tau_h - t_c}{h} - \frac{E\left[\eta dW_{\tau_h}\right]}{h} \to 0 \text{ as } h \to 0$$

since $\frac{\tau_h - t_c}{h} \to 0$. Then $\frac{V(t_c; x+h, r) - e^{-\delta(\tau_h - t_c)} E_r\left[V(\tau_h; x, r_{\tau_h})\right]}{h} = V_x(t_c; x, r)$ when h is small enough.

Suppose $h > 0$. In this problem formulation, $\{g_s^\tau\}$ is the amount that can be taken out of process X but here X only changes from $x+h$ to x and that means $\int_{t_c}^{\tau_h} g_s^\tau ds \geqslant h$ in this case. Then

$$\sup_\tau E\left[\int_{t_c}^{\tau_h} e^{-\delta(s-t_c)+r_s}(g_s^\tau - \phi l_s^\tau)ds\right] \geqslant E\left[\int_{t_c}^{\tau_h} e^{-\delta(s-t_c)+r_s} h ds\right] \approx e^r h.$$

Combining all the results in (1), when $h \to 0$,

$$V(t_c; x+h, r) - e^{-\delta(\tau_h - t_c)} E_r\left[V(\tau_h; x, r_{\tau_h})\right] = \sup_\tau E\left[\int_{t_c}^{\tau_h} e^{-\delta(s-t_c)+r_s}(g_s^\tau - \phi l_s^\tau)ds\right] \geqslant e^r h,$$

$$\frac{V(t_c; x+h, r) - e^{-\delta(\tau_h - t_c)} E_r\left[V(\tau_h; x, r_{\tau_h})\right]}{h} \geqslant e^r,$$

$$V_x^+(x, r) \geqslant e^r.$$

Suppose on the other hand that $h < 0$. Then we have the estimation that $\int_{t_c}^{\tau_h} l_s ds \geqslant -h$. So $-\int_{t_c}^{\tau_h} \phi l_s ds \leqslant -\phi h$. Thus

$$V(t_c; x+h, r) - e^{-\delta(\tau_h - t_c)} E_r\left[V(\tau_h; x, r_{\tau_h})\right] = \sup_\tau E[\int_{t_c}^{\tau_h} e^{-\delta(s-t_c)+r_s}(g_s^\tau - \phi l_s^\tau)ds] \geqslant \phi e^r h,$$

$$\frac{V(t_c; x+h, r) - e^{-\delta(\tau_h - t_c)} E_r\left[V(\tau_h; x, r_{\tau_h})\right]}{h} \leqslant \phi e^r,$$

$$V_x^-(x, r) \leqslant \phi e^r.$$

(B) Denote $\tau_h = \inf\{s \geqslant t_c | X_s = x+h, X_{t_c} = x\}$. Notice that

$$V(t_c; x, r) = \sup_\tau E_{x,r}\left[\int_{t_c}^{\infty} e^{-\delta(s-t_c)+r_s}(g_s^\tau - \phi l_s^\tau)ds\right]$$
$$\approx \sup_\tau E\left[\int_{t_c}^{\tau_h} e^{-\delta(s-t_c)+r_s}(g_s^\tau - \phi l_s^\tau)ds\right] + e^{-\delta(\tau_h - t_c)} E_r\left[V(\tau_h; x, r_{\tau_h})\right].$$

Suppose that $h > 0$. Similarly to the arguments above, here X moves upwards from x to $x + h$ so we have bound $\int_{t_c}^{\tau_h} l_s^\tau ds \geqslant h$, thus

$$V(t_c; x, r) - e^{-\delta(\tau_h - t_c)} E_r \left[V(\tau_h; x + h, r_{\tau_h}) \right] \geqslant -\phi \frac{h}{\tau_h} e^{r_{\tau_h}} \tau_h = -\phi h e^{r_{\tau_h}}.$$

Dividing both sides by h and letting $h \to 0$, with $\lim_{h \to 0} \frac{\tau_h - t_c}{h} = 0$, then

$$-V_x^+(t_c; x, r) \geqslant -\phi e^r \Rightarrow V_x^+(t_c; x, r) \leqslant \phi e^r.$$

Now suppose that $h < 0$. Then we have the estimation $\int_{t_c}^{\tau_h} g_s ds \geqslant -h$, and

$$V(t_c; x, r) - e^{-\delta(\tau_h - t_c)} E_r \left[V(\tau_h; x + h, r_{\tau_h}) \right] \geqslant -\frac{h}{\tau_h - t_c} e^{r_{\tau_h}} (\tau_h - t_c) = -h e^{r_{\tau_h}},$$
$$V_x^-(t_c; x, r) \geqslant e^r.$$

Both (A) and (B) show that $e^r \leqslant V_x(t_c; x, r) \leqslant \phi e^r$ for $r \in \mathbb{R}, x \geqslant 0$ and any time $t_c \geqslant 0$. $\square$

Theorem 4.1 (Verification theorem)**.** *Suppose that the function $V(x, r)$ solves HJB equation (4.2). Then we have*

$$V(t_c; x, r) = \sup_\tau E_{x,r} \left[\int_{t_c}^\infty e^{-\delta(s - t_c) + r_s} (g_s^\tau - \phi l_s^\tau) ds \right].$$

Proof.

$$\lim_{t \to 0} \frac{e^{-\delta(t+s-t_c)} E_{X_s, r_s} \left[V(t_c; X_{t+s}, r_{t+s}) \right] - e^{-\delta(s-t_c)} V(t_c; X_s, r_s)}{dt} = e^{-\delta s}(-\delta V + AV(t_c; X_s, r_s)(l_s^\tau - g_s^\tau) V_x),$$
$$= e^{-\delta(s-t_c)}(-\delta V + AV(t_c; X_s, r_s) + g_s^\tau(e^{r_s} - V_x) + l_s^\tau(V_x - e^{r_s}\phi)) - e^{-\delta(s-t_c)+r_s}[g_s^\tau - l_s^\tau \phi].$$

Then we take the expectation and integration from t_c to T on both sides,

$$e^{-\delta(T-t_c)} V(T; X_T, r_T) - V(t_c; x, r) = E_{x,r}[\int_{t_c}^T e^{-\delta(s-t_c)}(-\delta V + AV(t_c; X_s, r_s) + g_s^\tau(e^{r_s} - V_x) + ...$$
$$l_s^\tau(V_x - \phi e^{r_s}))ds] - E_{x,r} \left[\int_{t_c}^T e^{-\delta(s-t_c)+r_s}(g_s^\tau - l_s^\tau \phi) ds \right].$$

Next, notice that after we take the supremum over all admissible strategies for τ, the first expectation vanishes to zero since its integrand is then identical to the left-hand side of system (4.2). Now we have,

$$e^{-\delta(T-t_c)} V(T; X_T, r_T) - V(t_c; x, r) = -\sup_\tau E_{x,r} \left[\int_{t_c}^T e^{-\delta(s-t_c)+r_s}(g_s^\tau - l_s^\tau \phi) ds \right].$$

In the end, we take the limit $T \to \infty$. Here we need the following estimation and it is justified afterwards:

$$\lim_{T \to \infty} e^{-\delta(T-t_c)} V(T; X_T, r_T) = \lim_{T \to \infty} \frac{V(T; x_T, r_T)}{e^{\delta(T-t_c)}} = 0.$$

Since we have proved that $V_x(t_c; x, r)$ is bounded above by ϕe^r and bounded below by e^r in Lemma 4.1,

$$\frac{e^{r_T} X_T}{e^{\delta(T-t_c)}} \leqslant \frac{V(T; x_T, r_T)}{e^{\delta(T-t_c)}} \leqslant \frac{\phi e^{r_T} X_T}{e^{\delta(T-t_c)}}.$$

By Itô's Lemma,

$$\begin{aligned} X_T &= x + \int_{t_c}^{T} a ds + \int_{t_c}^{T} \sigma dB_s \\ e^{r_T} &= e^r + \int_{t_c}^{T} e^{r_s}(f'(s) + \lambda(b - r_s)) + \frac{\eta^2}{2} e^{r_s} ds + \int_{t_c}^{T} e^{r_s} \eta^2 dW_s. \end{aligned}$$

It is obvious that $\frac{\phi e^{r_T} X_T}{e^{\delta(T-t_c)}} \to 0$ and $\frac{e^{r_T} x_T}{e^{\delta(T-t_c)}} \to 0$ as $T \to \infty$, which then imply that $\frac{V(T; X_T, r_T)}{e^{\delta(T-t_c)}} \to 0$ when $T \to \infty$. Then everything is simplified to the following:

$$V(t_c; x, r) = \sup_{\tau} E_{x,r} \left[\int_{t_c}^{\infty} e^{-\delta(s-t_c)+r_s} (g_s^\tau - l_s^\tau \phi) ds \right].$$

□

Theorem 4.1 verifies that the solution of HJB equation (4.2), if it exists, will be the value function of the formulated problem at the beginning of this section. Hence, we need to show the existence and uniqueness of solution of (4.2).

4.3 Viscosity solutions

In the following, we will address this issue with a technical argument: viscosity solution. This problem is one of singular stochastic control problems and the approach of viscosity solution has been well explored in literature. The book [9] has a complete exposition on this direction. We will use the idea from this reference to justify that the solution of (4.2) is the viscosity solution. It needs to be mentioned that paper [7] has proved the viscosity solution in a very similar model and some of the proof below is close to that. We will begin with the definition of viscosity solution.

Definition 4.2. *We say that a function $V(t; x, r) \in C(\mathbb{R}^+; \mathbb{R}^+, \mathbb{R})$ is the* ***viscosity supersolution*** *of (4.2) if for any point $(\bar{t}, \bar{x}, \bar{r})$ where there exists a function $w(t; x, r) \in C^{1,1,2}(\mathbb{R}^+; \mathbb{R}^+, \mathbb{R})$ such that $V(t; \bar{x}, \bar{r}) = w(\bar{t}; \bar{x}, \bar{r})$ and $V(t; x, r) - w(t; x, r)$ has a local minimum at $(\bar{t}, \bar{x}, \bar{r})$, we have*

$$w_t - \delta w + (f'(\bar{t}) + \lambda(b - r))w_r + \frac{\eta^2}{2} w_{rr} + a w_x + \frac{\sigma^2}{2} w_{xx} + \sup_{\tau}(g^\tau(e^r - w_x) - l^\tau(\phi e^r - w_x)) \leqslant 0.$$

We say that a function $V(t; x, r) \in C(\mathbb{R}^+; \mathbb{R}^+, \mathbb{R})$ is the ***viscosity subsolution*** *of (4.2) if for any point $(\bar{t}, \bar{x}, \bar{r})$ where there exists a function $w(t; x, r) \in C^{1,1,2}(\mathbb{R}^+; \mathbb{R}^+, \mathbb{R})$ such that $V(\bar{t}; \bar{x}, \bar{r}) = w(\bar{t}; \bar{x}, \bar{r})$ and $V(t; x, r) - w(t; x, r)$ has a local maximum at $(\bar{t}, \bar{x}, \bar{r})$, we have*

$$w_t - \delta w + (f'(\bar{t}) + \lambda(b - r))w_r + \frac{\eta^2}{2} w_{rr} + a w_x + \frac{\sigma^2}{2} w_{xx} + \sup_{\tau}(g^\tau(e^r - w_x) - l^\tau(\phi e^r - w_x)) \geqslant 0.$$

The function $V(t; x, r)$ is a ***viscosity solution*** *of problem (4.2) if it is both a viscosity subsolution and a viscosity supersolution of (4.2).*

Theorem 4.2 (Viscosity solution)**.** *The objective function $V(t;x,r)$ defined in (4.1) is the viscosity solution of HJB equation (4.2).*

Proof. Here we will prove the viscosity subsolution for $V(t;x,r)$. The approach for viscosity supersolution could be referred to (Chapt 8 of [9]). Suppose that at the point $(\bar{t},\bar{x},\bar{r})$, which is in the domain, we have a function $w\in C^{1,1,2}(\mathbb{R}^+;\mathbb{R}^+,\mathbb{R})$ such that $V(t;x,r)-w(t;x,r)$ has a local maximum at $(\bar{t};\bar{x},\bar{r})$ and $V(\bar{t};\bar{x},\bar{r})=w(\bar{t};\bar{x},\bar{r})$. Then,

$$\begin{aligned} w(\bar{t};\bar{x},\bar{r}) = V(\bar{t};\bar{x},\bar{r}) &= \sup_{\tau}\left(E_{\bar{x},\bar{r}}\left[\int_{\bar{t}}^{\infty} e^{-\delta(s-\bar{t})+r_s}(g_s^\tau-\phi l_s^\tau)ds\right]\right)\\ &= \sup_{\tau}\left(E_{\bar{x},\bar{r}}\left[\int_{\bar{t}}^{h} e^{-\delta(s-\bar{t})+r_s}(g_s^\tau-\phi l_s^\tau)ds\right]+E_{\bar{x},\bar{r}}\left[\int_{h}^{\infty} e^{-\delta(s-\bar{t})+r_s}(g_s^\tau-\phi l_s^\tau)ds\right]\right)\\ &= \sup_{\tau}\left(E_{\bar{x},\bar{r}}\left[\int_{\bar{t}}^{h} e^{-\delta(s-\bar{t})+r_s}(g_s^\tau-\phi l_s^\tau)ds\right]+e^{-\delta(h-\bar{t})}E_{\bar{x},\bar{r}}\left[V(h;X_h,r_h)\right]\right)\\ &\leqslant \sup_{\tau}\left(E_{\bar{x},\bar{r}}\left[\int_{\bar{t}}^{h} e^{-\delta(s-\bar{t})+r_s}(g_s^\tau-\phi l_s^\tau)ds\right]+e^{-\delta(h-\bar{t})}E_{\bar{x},\bar{r}}\left[w(h;X_h,r_h)\right]\right). \end{aligned}$$

Then, subtracting w on both sides

$$\begin{aligned} w(\bar{t};\bar{x},\bar{r}) &\leqslant \sup_{\tau}\left(E_{\bar{x},\bar{r}}\left[\int_{\bar{t}}^{h} e^{-\delta(s-\bar{t})+r_s}(g_s^\tau-\phi l_s^\tau)ds\right]+e^{-\delta(h-\bar{t})}E_{\bar{x},\bar{r}}\left[w(h;X_h,r_h)\right]\right)\\ 0 &\leqslant \sup_{\tau}\left(E_{\bar{x},\bar{r}}\left[\int_{\bar{t}}^{h} e^{-\delta(s-\bar{t})+r_s}(g_s^\tau-\phi l_s^\tau)ds\right]+e^{-\delta(h-\bar{t})}E_{\bar{x},\bar{r}}\left[w(h;X_h,r_h)\right]-w(\bar{t};\bar{x},\bar{r})\right)\\ 0 &\leqslant \frac{\sup_{\tau}\left(E_{\bar{x},\bar{r}}\left[\int_{\bar{t}}^{h} e^{-\delta(s-\bar{t})+r_s}(g_s^\tau-\phi l_s^\tau)ds\right]\right)}{h-\bar{t}}+\frac{e^{-\delta(h-\bar{t})}E_{\bar{x},\bar{r}}\left[w(h;X_h,r_h)\right]-w(\bar{t};\bar{x},\bar{r})}{h-\bar{t}}. \end{aligned} \tag{4.3}$$

Letting $h\to\bar{t}$, the second term of (4.3) will be $w_t-\delta w+Gw(\bar{t};\bar{x},\bar{r})$ and the first term becomes $e^r(g^\tau-\phi l^\tau)$. Here the extended generator G is defined for stochastic process (X^τ,R^τ). Therefore,

$$e^r(g^\tau-\phi l^\tau)+w_t-\delta w+Gw(\bar{t};\bar{x},\bar{r})\geqslant 0,$$

$$w_t-\delta w+(f'(\bar{t})+\lambda(b-r))w_r+\frac{\eta^2}{2}w_{rr}+aw_x+\frac{\sigma^2}{2}w_{xx}+\sup_{\tau}(g^\tau(e^r-w_x)-l^\tau(\phi e^r-w_x))\geqslant 0.$$

So $V(t;x,r)$ is the viscosity subsolution to (4.2). Similarly, we can show that it is the viscosity supersolution to (4.2) and therefore it is the viscosity solution. □

5 Conclusion

In this paper, we show that the management of flexibility in hydropower can be formulated into a stochastic control problem. The novelty of this model is that the flexibility of hydropower is realized through purchases and sales on electricity market, which itself possesses randomness. Initially we present an intuitive model for which the noise on reservoir level and electricity price are represented using drifted Brownian motions. With that, we could propose a simple strategy to decide the amount and moment to generate or purchase electricity.

However it is unsatisfactory in the sense that it does not use a reasonable representation of realistic data. Based on that, we consider an advanced model, one-factor diffusion, for which the seasonal effects and mean-reversion features are explained. The corresponding numerical solution to this system is not known. A Markov chain approximation has been suggested ([16]) as an efficient method to find the numerical solution, but numerical experiments on this two-variable HJB equation are not available to our best knowledge. On the other hand, [12] proposes a classical numerical methodology, best known as the semi-smooth projected-Newton method, which may be applicable given boundary data. Development of numerical methods for this problem is part of our future work. Here we provide a standard analysis of HJB equation system associated with this optimization problem. Specifically, we show that the value function is the viscosity solution of HJB equation for this problem, which justifies the uniqueness and existence of the optimal strategy.

6 Acknowledgements

We would like to thank Bonneville Power Administration (BPA) for the funding support on the research project (TIP342).

References

[1] Benjamin Avanzi, Vincent Tu, and Bernard Wong, *On optimal periodic dividend strategies in the dual model with diffusion*, Insurance: Mathematics and Economics **55** (2014), 210–224.

[2] Martin T. Barlow, *A diffusion model for electricity prices*, Mathematical finance **12** (2002), no. 4, 287–298.

[3] Rabi N Bhattacharya and Edward C Waymire, *Stochastic processes with applications*, Vol. 61, Siam, 2009.

[4] Rabindra Nath Bhattacharya and Edward C Waymire, *A basic course in probability theory*, Vol. 69, Springer, 2007.

[5] Arpan Biswas, Yong Chen, Nathan Gibson, and Christopher Hoyle, *Bilevel flexible-robust optimization for energy allocation problems*, ASCE-ASME J Risk and Uncert in Engrg Sys Part B Mech Engrg **6** (2020), no. 3.

[6] Duan Chen, Arturo S Leon, Nathan L Gibson, and Parnian Hosseini, *Dimension reduction of decision variables for multireservoir operation: A spectral optimization model*, Water Resources Research **52** (2016), no. 1, 36–51.

[7] Julia Eisenberg, *Optimal dividends under a stochastic interest rate*, Insurance: Mathematics and Economics **65** (2015), 259–266.

[8] Julia Eisenberg and Paul Krhner, *A note on the optimal dividends paid in a foreign currency*, Annals of Actuarial Science **11** (2017), no. 1, 6773.

[9] Wendell H. Fleming and Halil Mete Soner, *Controlled Markov processes and viscosity solutions*, Vol. 25, Springer Science & Business Media, 2006.

[10] Nathan L. Gibson, Christopher Gifford-Miears, Arturo S. Leon, and Veronika S. Vasylkivska, *Efficient computation of unsteady flow in complex river systems with uncertain inputs*, International Journal of Computer Mathematics **91** (2014), no. 4, 781–797.

[11] Huanqun Jiang, *Optimal barrier strategy for spectrally negative Lévy process discounted by a class of exponential Lévy processes*, Annals of Actuarial Science **12** (2018), no. 2, 326–337.

[12] Huanqun Jiang and Nathan L. Gibson, *A numerical approach for a constrained Hamiltion-Jacobi-Bellman equation arising from financial insurance*, Submitted (2019).

[13] Zhengjun Jiang and Martijn Pistorius, *Optimal dividend distribution under Markov regime switching*, Finance and Stochastics **16** (2012), no. 3, 449–476.

[14] Divas Karimanzira, Dirk Schwanenberg, Christopher Allen, and Steven Barton, *Short-term hydropower optimization and assessment of operational flexibility*, Journal of Water Resources Planning and Management **142** (2015), no. 2, 04015048.

[15] Natalie Kulenko and Hanspeter Schmidli, *Optimal dividend strategies in a Cramér Lundberg model with capital injections*, Insurance: Mathematics and Economics **43** (2008), no. 2, 270 –278.

[16] Harold Kushner and Paul G. Dupuis, *Numerical methods for stochastic control problems in continuous time*, Vol. 24, Springer Science & Business Media, 2013.

[17] R. L. Loeffen, *An optimal dividends problem with transaction costs for spectrally negative Lévy processes*, Insurance: Mathematics and Economics **45** (2009), no. 1, 41–48.

[18] Julio J. Lucia and Eduardo S. Schwartz, *Electricity prices and power derivatives: Evidence from the nordic power exchange*, Review of derivatives research **5** (2002), no. 1, 5–50.

[19] Hanspeter Schmidli, *Stochastic control in insurance*, Springer Science & Business Media, London, 2007.

[20] Erfaneh Sharifi, Hamid Bashiri, Arturo Leon, Yong Chen, and Nathan Gibson, *Valuation of flexibility for optimal reservoir operation*, Open Water Journal **4** (2017), no. 2, 5.

[21] Steven E. Shreve, John P. Lehoczky, and Donald P. Gaver, *Optimal consumption for general diffusions with absorbing and reflecting barriers*, SIAM Journal on Control and Optimization **22** (1984), no. 1, 55–75.

A Appendix

Proof of Lemma 3.1.

Proof. Consider X moves from m to $m+h$ for small $h>0$. Denote

$$\tau_h = \inf\left\{t > 0 | X_t = m+h, X_0 = m\right\}.$$

Then

$$V(r,m) = \sup_{\tau}\left\{E_{r,m}\left[\int_0^{\tau_h} dS_s - \phi dZ_s\right] + e^{r\tau_h} V(r_{\tau_h}, m+h)\right\}.$$

Subtracting $V(r,m)$ on both sides and dividing both sides by h,

$$\frac{V(r,m+h)-V(r,m)}{h} = -\frac{\sup_\tau\left\{\int_0^{\tau_h} dS_s - \phi dZ_s\right\}}{h}.$$

Notice that the supremum can be only achieved when the integral is equal to $e^r h$ as the consequence of the Law of Iterated Logarithm. Letting $h \to 0$,

$$\frac{\partial V}{\partial m}(r,m) = e^r \phi.$$

Since $V(r,m) = e^r F(m)$, then $F'(m) = \phi$.

On the other hand, suppose $x > y \geqslant m$. For $X_0 = x$, we can choose a non-optimal strategy for which we release amount $x-y$ of water immediately and then follow the optimal strategy when $X_0 = y$. We will have,

$$V(r,x) = \sup_{\tau}\left\{E_{r,x}\left[\int_0^{\infty} dS_s - \phi dZ_s\right]\right\} \geqslant x - y + V(r,y) > V(r,y).$$

Then V is increasing in x.

Lastly, we need to show that V is concave in x. Let $z = \alpha x + (1-\alpha)y$, where $x > z > y \geqslant m$ and $0 < \alpha < 1$. Then,

$$\begin{aligned} V(r,z) &= V(r,\alpha x + (1-\alpha)y) = \sup_{\tau} E_{r,z}\left[\int_0^{\infty} dS_s - \phi dZ_s\right] \\ &\geqslant \sup_{\tau} \alpha E_{r,x}\left[\int_0^{\infty} dS_s^1 - \phi dZ_s^1\right] + \sup_{\tau}(1-\alpha)E_{r,y}\left[\int_0^{\infty} dS_s^2 - \phi dZ_s^2\right] \\ &= \alpha V(r,x) + (1-\alpha)V(r,y) \end{aligned}$$

Therefore V is concave in x. □