Uncertainty Quantification Methods for Multiscale Modeling and Dimension Reduction

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Outline

1. **Polynomial Chaos for Multiscale Modeling**
   - Dispersive Maxwell System
   - Maxwell-Random Lorentz System
   - Viscoelastic Materials
   - Magnetohydrodynamics
2. **Karhunen-Loève for Dimension Reduction**
   - Reservoir Operations
   - Power Grid
   - Tsunami Loading

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Maxwell's Equations

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}, \text{ in } (0, T) \times \mathcal{D} \qquad \text{(Faraday)}$$

$$\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} - \nabla \times \mathbf{H} = \mathbf{0}, \text{ in } (0, T) \times \mathcal{D} \qquad \text{(Ampere)}$$

$$\nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{B} = 0, \text{ in } (0, T) \times \mathcal{D} \qquad \text{(Poisson/Gauss)}$$

$$\mathbf{E}(0, \mathbf{x}) = \mathbf{E_0}; \ \mathbf{H}(0, \mathbf{x}) = \mathbf{H_0}, \text{ in } \mathcal{D} \qquad \text{(Initial)}$$

$$\mathbf{E} \times \mathbf{n} = \mathbf{0}, \text{ on } (0, T) \times \partial\mathcal{D} \qquad \text{(Boundary)}$$

$\mathbf{E}$ = Electric field vector

$\mathbf{H}$ = Magnetic field vector

$\mathbf{J}$ = Current density

$\mathbf{D}$ = Electric flux density

$\mathbf{B}$ = Magnetic flux density

$\mathbf{n}$ = Unit outward normal to $\partial\mathcal{D}$

Constitutive Laws

$$\mathbf{D} = \epsilon \mathbf{E} + \mathbf{P}$$
$$\mathbf{B} = \mu \mathbf{H} + \mathbf{M}$$
$$\mathbf{J} = \sigma \mathbf{E} + \mathbf{J}_s$$

$\mathbf{P} =$	Polarization	$\epsilon =$	Electric permittivity
$\mathbf{M} =$	Magnetization	$\mu =$	Magnetic permeability
$\mathbf{J}_s =$	Source Current	$\sigma =$	Electric Conductivity

where $\epsilon = \epsilon_0 \epsilon_\infty$ and $\mu = \mu_0 \mu_r$.

Polarization

The polarization is defined as the average dipole moment in a material.

- For linear materials we can define $\mathbf{P}$ in terms of a convolution with $\mathbf{E}$

$$\mathbf{P}(t,\mathbf{x}) = g * \mathbf{E}(t,\mathbf{x}) = \int_0^t g(t-s,\mathbf{x};\mathbf{q})\mathbf{E}(s,\mathbf{x})ds,$$

  where g is the dielectric response function (DRF) and $\mathbf{q}$ is a vector which contains all of the necessary dielectric parameters for a model.

- In the frequency domain

$$\hat{\mathbf{D}} = \epsilon\hat{\mathbf{E}} + \hat{\mathbf{g}}\hat{\mathbf{E}} = \epsilon_0\epsilon(\omega)\hat{\mathbf{E}},$$

  where $\epsilon(\omega)$ is called the complex permittivity.

Saltwater Data

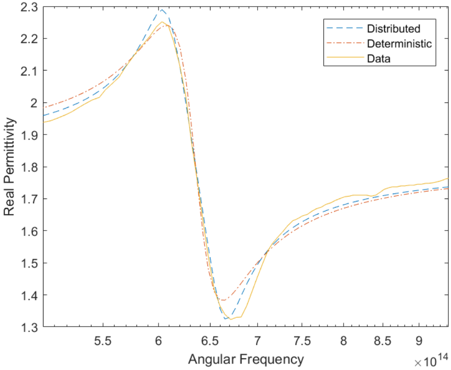

Figure: Fits for single-pole, saltwater data [Querry et al., 1972][1]

[1] J. Alvarez, A. Fisher, **N. L. Gibson**, "Approximating Dispersive Materials With Parameter Distributions in the Lorentz Model", Applied Mathematics, Modeling and Computational Science 2019 Proceedings, 11 pages. *To appear.*

Relaxation Polarization Models

$$\mathbf{P}(t,\mathbf{x}) = g * \mathbf{E}(t,\mathbf{x}) = \int_0^t g(t-s,\mathbf{x};\mathbf{q})\mathbf{E}(s,\mathbf{x})ds,$$

- Debye model [1913] $\mathbf{q} = [\epsilon_\infty, \epsilon_d, \tau]$

$$g(t,\mathbf{x}) = \epsilon_0\epsilon_d/\tau \;\; e^{-t/\tau}$$

$$\text{or} \quad \tau\dot{\mathbf{P}} + \mathbf{P} = \epsilon_0\epsilon_d\mathbf{E}$$

$$\text{or} \quad \epsilon(\omega) = \epsilon_\infty + \frac{\epsilon_d}{1+i\omega\tau}$$

  with $\epsilon_d := \epsilon_s - \epsilon_\infty$ and τ a relaxation time.

- Cole-Cole model [1941] (heuristic generalization) $\mathbf{q} = [\epsilon_\infty, \epsilon_d, \tau, \alpha]$

$$\epsilon(\omega) = \epsilon_\infty + \frac{\epsilon_d}{1+(i\omega\tau)^\alpha}$$

Polynomial Chaos for Random Debye

1. The Cole-Cole model corresponds to a fractional order ODE in the time-domain and is difficult to simulate.
2. Debye is efficient to simulate, but does not represent permittivity well.
3. We showed[2] that applying Polynomial Chaos to the Random Debye model preserves the efficiency of Debye with the fidelity of Cole-Cole.
4. Stability estimates for the continuous and discrete system were shown, and dispersion analyses were performed.
5. The inverse problem for the distribution of parameters was also addressed. [3]

[2] **N. L. Gibson**, "Polynomial Chaos for Dispersive Electromagnetics", Communications in Computational Physics, 18 (5), 1234–1263, 2015.

[3] M. Armentrout and **N. L. Gibson**, "Electromagnetic Relaxation Time Distribution Inverse Problems in the Time-Domain", Proceedings, WAVES 2011, 245–248, 2011.

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Lorentz Polarization Model

Lorentz model in Auxiliary Differential Equation (ADE) form:

$$\ddot{\mathbf{P}} + 2\nu\dot{\mathbf{P}} + \omega_0^2\mathbf{P} = \epsilon_0\omega_p^2\mathbf{E}$$

where ω_0 is the resonant frequency, ν is a damping coefficient, and ω_p is referred to as a plasma frequency.

Taking a Fourier transform of $\mathbf{D} = \epsilon\mathbf{E} + \mathbf{P}$, we get

$$\epsilon(\omega) = \epsilon_\infty + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i2\nu\omega}$$

with $\mathbf{q} = [\epsilon_\infty, \omega_0, \nu, \omega_p]$.

Distributions of Parameters

To account for the effect of distributions of parameters $\mathbf{q}$ in a polarization model, consider the following *polydispersive* DRF

$$h(t, \mathbf{x}; F) = \int_{\mathcal{Q}} g(t, \mathbf{x}; \mathbf{q}) dF(\mathbf{q}),$$

where $\mathcal{Q}$ is some admissible set and $F \in \mathfrak{P}(\mathcal{Q})$.
Then the polarization becomes:

$$\mathbf{P}(t, \mathbf{x}; F) = \int_0^t h(t-s, \mathbf{x}; F)\mathbf{E}(s, \mathbf{x}) ds.$$

Alternatively we can define the random polarization $\mathcal{P}(t, \mathbf{x}; \mathbf{q})$ to satisfy

$$\mathcal{P} = g(t, \mathbf{x}; \mathbf{q}) * \mathbf{E}$$

but with $\mathbf{q}$ random; the macroscopic polarization is then taken to be the expected value of the random polarization,

$$\mathbf{P}(t, \mathbf{x}; F) = \int_{\mathcal{Q}} \mathcal{P}(t, \mathbf{x}; \mathbf{q}) dF(\mathbf{q}).$$

Random Lorentz Polarization

We allow the parameter ω_0^2 be a random variable with probability distribution F on the interval (a, b). Then the random Lorentz model in ADE form is

$$\ddot{\mathcal{P}} + 2\nu\dot{\mathcal{P}} + \omega_0^2\mathcal{P} = \epsilon_0\omega_p^2 E$$

The macroscopic polarization is then taken to be the expected value of the random polarization,

$$\mathbf{P}(t, \mathbf{x}) = \int_a^b \mathcal{P}(t, \mathbf{x}; \omega_0^2)\, dF(\omega_0^2).$$

Saltwater Data

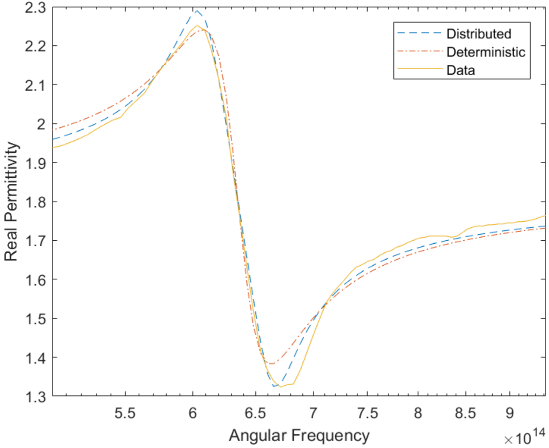

Figure: Real part of $\epsilon(\omega)$ fits for single-pole, saltwater data [1].

Saltwater Data

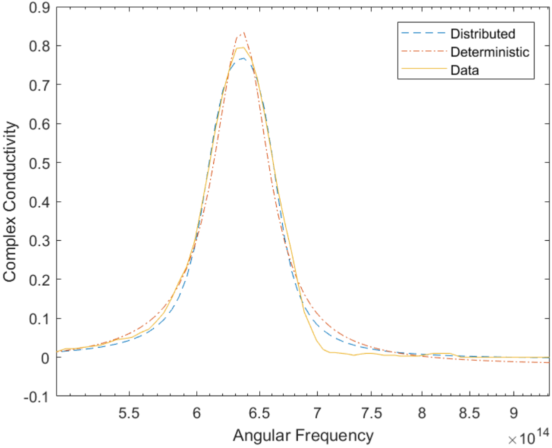

Figure: Imaginary part of $\epsilon(\omega)/\omega$, fits for single-pole, saltwater data [1].

Maxwell-Random Lorentz system

Combining with Maxwell's equations, we have

$$\mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E} \tag{1a}$$

$$\epsilon_0 \epsilon_\infty \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \frac{\partial \mathbf{P}}{\partial t} \tag{1b}$$

$$\ddot{\mathcal{P}} + 2\nu \dot{\mathcal{P}} + \omega_0^2 \mathcal{P} = \epsilon_0 \omega_p^2 \mathbf{E} \tag{1c}$$

with

$$\mathbf{P}(t, \mathbf{x}) = \int_a^b \mathcal{P}(t, \mathbf{x}; \omega_0^2) f(\omega_0^2) d\omega_0^2.$$

2D Maxwell-Random Lorentz Transverse Electric (TE) curl equations

$$\mu_0 \frac{\partial H}{\partial t} = -\mathrm{curl}\ \mathbf{E}, \tag{2a}$$
$$\epsilon_0 \epsilon_\infty \frac{\partial \mathbf{E}}{\partial t} = \mathbf{curl}\ H - \mathbf{J}, \tag{2b}$$
$$\frac{\partial \mathcal{P}}{\partial t} = \mathcal{J} \tag{2c}$$
$$\frac{\partial \mathcal{J}}{\partial t} = -2\nu \mathcal{J} - \omega_0^2 \mathcal{P} + \epsilon_0 \omega_p^2 \mathbf{E} \tag{2d}$$

where $\mathbf{E} = (E_x, E_y)^T$, $\mathbf{J} = (J_x, J_y)^T = \mathbb{E}[\mathcal{J}]$, $\mathcal{J} = (\mathcal{J}_x, \mathcal{J}_y)^T$, $\mathcal{P} = (\mathcal{P}_x, \mathcal{P}_y)^T$ and $H = H_z$.

Note $\mathrm{curl}\ \mathbf{U} = \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}$ and $\mathbf{curl}\ V = \left(\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial x}\right)^T$.

We introduce the random Hilbert space $V_F = (L^2(\Omega) \otimes L^2(\mathcal{D}))^2$ equipped with an inner product and norm as follows

$$(\mathbf{u}, \mathbf{v})_F = \mathbb{E}[(\mathbf{u}, \mathbf{v})_2],$$
$$\|\mathbf{u}\|_F^2 = \mathbb{E}[\|\mathbf{u}\|_2^2].$$

The weak formulation of the 2D Maxwell-Random Lorentz TE system is

$$\left(\mu_0 \frac{\partial H}{\partial t}, v\right)_2 = (-\mathrm{curl}\ \mathbf{E}, v)_2 \tag{3a}$$
$$\left(\epsilon_0 \epsilon_\infty \frac{\partial \mathbf{E}}{\partial t}, \mathbf{u}\right)_2 = (\mathbf{curl}\ H, \mathbf{u})_2 - (\mathbf{J}, \mathbf{u})_2 \tag{3b}$$
$$\left(\frac{\partial \mathcal{P}}{\partial t}, \mathbf{q}\right)_F = (\mathcal{J}, \mathbf{q})_F \tag{3c}$$
$$\left(\frac{\partial \mathcal{J}}{\partial t}, \mathbf{w}\right)_F = (-2\nu\mathcal{J}, \mathbf{w})_F + \left(-\omega_0^2 \mathcal{P}, \mathbf{w}\right)_F + \left(\epsilon_0 \omega_p^2 \mathbf{E}, \mathbf{w}\right)_F. \tag{3d}$$

for $v \in L^2(\mathcal{D})$, $\mathbf{u} \in H_0(\mathrm{curl}, \mathcal{D})^2$, and $\mathbf{q}, \mathbf{w} \in V_F$.

We have the following result[4]

Theorem (Stability of Maxwell-Random Lorentz)

Let $\mathcal{D} \subset \mathbb{R}^2$ and suppose that $\mathbf{E} \in C(0,T;H_0(\mathrm{curl},\mathcal{D})) \cap C^1(0,T;(L^2(\mathcal{D}))^2)$, $\mathcal{P},\mathcal{J} \in C^1(0,T;\left(L^2(\Omega)\otimes L^2(\mathcal{D})\right)^2)$, and $H(t) \in C^1(0,T;L^2(\mathcal{D}))$ are solutions of the weak formulation for the Maxwell-Random Lorentz system along with PEC boundary conditions. Then the system exhibits energy decay

$$\mathcal{E}(t) \leq \mathcal{E}(0) \quad \forall t \geq 0,$$

where the energy $\mathcal{E}(t)$ is defined as

$$\mathcal{E}(t)^2 = \left\|\sqrt{\mu_0}\, H(t)\right\|_2^2 + \left\|\sqrt{\epsilon_0\epsilon_\infty}\,\mathbf{E}(t)\right\|_2^2 + \left\|\sqrt{\frac{\omega_0^2}{\epsilon_0\omega_p^2}}\,\mathcal{P}(t)\right\|_F^2 + \left\|\frac{1}{\sqrt{\epsilon_0\omega_p^2}}\,\mathcal{J}(t)\right\|_F^2$$

where $\|u\|_F^2 = \mathbb{E}[\|u\|_2^2]$ and $\mathcal{J} := \frac{\partial \mathcal{P}}{\partial t}$.

[4] A. Fisher, J. Alvarez, **N. L. Gibson**, "Analysis of Methods for the Maxwell-Random Lorentz Model", Results in Applied Mathematics, vol. 8, 1–17, 2020.

Proof: (for 2D)

By choosing $v = H$, $\mathbf{u} = \mathbf{E}$, $\mathbf{q} = \mathcal{P}$ and $\mathbf{w} = \mathcal{J}$ in the weak form, and adding all equations into the time derivative of the definition of $\mathcal{E}^2$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d\mathcal{E}^2(t)}{dt} &= -\Big(\mathrm{curl}\ \mathbf{E}, H\Big)_2 + \Big(H, \mathrm{curl}\ \mathbf{E}\Big)_2 - \Big(\mathbf{J}, \mathbf{E}\Big)_2 + \left(\frac{\omega_0^2}{\epsilon_0\omega_p^2}\mathcal{J}, \mathcal{P}\right)_F \\ &\quad - \left(\frac{2\nu}{\epsilon_0\omega_p^2}\mathcal{J}, \mathcal{J}\right)_F - \left(\frac{\omega_0^2}{\epsilon_0\omega_p^2}\mathcal{P}, \mathcal{J}\right)_F + (\mathbf{E}, \mathcal{J})_F \\ &= -\left\|\sqrt{\frac{2\nu}{\epsilon_0\omega_p^2}}\mathcal{J}\right\|_F^2\end{aligned}$$

$$\boxed{\frac{d\mathcal{E}(t)}{dt} = \frac{-1}{\mathcal{E}(t)}\left\|\sqrt{\frac{2\nu}{\epsilon_0\omega_p^2}}\mathcal{J}\right\|_F^2 \leq 0.}$$

□

Polynomial Chaos

Let $\omega_0^2 = r\xi + m$ and $\xi \in [-1, 1]$. Suppressing the dimension of $\mathcal{P}$ and the spatial dependence, we have

$$\mathcal{P}(\xi, t) = \sum_{i=0}^{\infty} \alpha_i(t)\phi_i(\xi) \to \ddot{\mathcal{P}} + 2\nu\dot{\mathcal{P}} + (r\xi + m)\mathcal{P} = \epsilon_0\omega_p^2 E.$$

Utilizing the Triple Recursion Relation for orthogonal polynomials:

$$\xi\phi_n(\xi) = a_n\phi_{n+1}(\xi) + b_n\phi_n(\xi) + c_n\phi_{n-1}(\xi),$$

the differential equation becomes

$$\sum_{i=0}^{\infty} \left[\ddot{\alpha}_i(t) + 2\nu\dot{\alpha}_i(t) + m\alpha_i(t)\right]\phi_i + r\alpha_i(t)\left[a_i\phi_{i+1} + b_i\phi_i + c_i\phi_{i-1}\right] = \epsilon_0\omega_p^2 E\phi_0.$$

Galerkin Projection

We apply a Galerkin Projection onto the space of polynomials of degree at most p to get:

$$\ddot{\vec{\alpha}} + 2\nu\dot{\vec{\alpha}} + A\vec{\alpha} = \vec{f}$$

where $\vec{f} = \hat{e}_1\epsilon_0\omega_p^2 E$ and

$$A = rM + mI, \quad M = \begin{pmatrix} b_0 & c_1 & 0 & \cdots & 0 \\ a_0 & b_1 & c_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & a_{p-2} & b_{b-1} & c_p \\ 0 & \cdots & 0 & a_{p-1} & b_p \end{pmatrix}.$$

Or we can write as a first order system:

$$\dot{\vec{\alpha}} = \vec{\beta}$$
$$\dot{\vec{\beta}} = -A\vec{\alpha} - 2\nu\vec{\beta} + \vec{f}.$$

Theorem (Energy Decay for Maxwell-PC Lorentz-FDTD)

If the CFL condition is satisfied, then the Yee scheme for the 2D TE mode Maxwell-Polynomial Chaos-Lorentz system satisfies the discrete identity

$$\delta_t \mathcal{E}_h^{n+\frac{1}{2}} = \frac{-1}{\overline{\mathcal{E}}_h^{n+\frac{1}{2}}} \left\| \sqrt{\frac{2\nu}{\epsilon_0 \omega_p^2}} \overline{\vec{\beta}}^{n+\frac{1}{2}} \right\|_\alpha^2$$

for all n where

$$\mathcal{E}_h^n = \left(\mu_0 (H^{n+\frac{1}{2}}, H^{n-\frac{1}{2}})_H + \|\sqrt{\epsilon_0 \epsilon_\infty}\ \mathbf{E}^n\|_E^2 + \left\| \frac{A^{1/2}}{\sqrt{\epsilon_0 \omega_p^2}} \vec{\alpha}^n \right\|_\alpha^2 + \left\| \sqrt{\frac{1}{\epsilon_0 \omega_p^2}} \vec{\beta}^n \right\|_\alpha^2 \right)^{1/2}$$

defines a discrete energy.

Note that $\|\vec{\alpha}\|_\alpha^2 \approx \|\mathbb{E}[\mathcal{P}]\|_2^2 + \|\mathrm{StdDev}(\mathcal{P})\|_2^2 = \mathbb{E}[\|\mathcal{P}\|_2^2] = \|\mathcal{P}\|_F^2$ so that this is a natural extension of the Maxwell-Random Lorentz energy.[4]

[4] A. Fisher, J. Alvarez, **N. L. Gibson**, "Analysis of Methods for the Maxwell-Random Lorentz Model", Results in Applied Mathematics, vol. 8, 1–17, 2020.

Theorem

The discrete dispersion relation for the Maxwell-PC FDTD Lorentz scheme is given by

$$\frac{\omega_\Delta^2}{c^2}\epsilon_\Delta(\omega) = K_\Delta^2$$

where the discrete expected complex permittivity is given by

$$\epsilon_\Delta(\omega) := \epsilon_\infty + \omega_{p,\Delta}^2 \hat{e}_1^T \left(A_\Delta - \omega_\Delta^2 I - i2\nu_\Delta\omega_\Delta I\right)^{-1} \hat{e}_1$$

and the discrete wavenumber and quantity K_Δ are given by

$$k_\Delta := \sqrt{k_{x,\Delta}^2 + k_{y,\Delta}^2}, \quad K_\Delta := \sqrt{K_{x,\Delta}^2 + K_{y,\Delta}^2},$$

with

$$K_{x,\Delta} := \frac{2}{\Delta x}\sin\left(\frac{k_{x,\Delta}\Delta x}{2}\right), \quad K_{y,\Delta} := \frac{2}{\Delta y}\sin\left(\frac{k_{y,\Delta}\Delta y}{2}\right)\ldots$$

Theorem (Continued)

and the discrete PC matrix and discrete damping are given by

$$A_\Delta := \cos^2(\omega\Delta t/2)A, \quad \nu_\Delta := cos\left(\frac{\omega\Delta t}{2}\right)\nu.$$

Similarly,

$$\omega_\Delta := \frac{2}{\Delta t}\sin\left(\frac{\omega\Delta t}{2}\right), \quad \omega_{p,\Delta} := cos\left(\frac{\omega\Delta t}{2}\right)\omega_p.$$

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PC for Viscoelastic Volterra Kernels

The shear stress, σ, in a linear viscoelastic body is given as the following functional of shear strain, ε:

$$\sigma(t) = \mu(t)\varepsilon(0) + \int_0^t \mu(t-s)\dot{\varepsilon}(s)\,ds \tag{5}$$

where μ is a stress relaxation function, often of exponential form. However, many real materials can be observed to relax slower than exponentially. For this reason a *power law* dependence is often preferred wherein $\mu(t) = \mu_0 t^{-\alpha}$ for $\mu_0 > 0$ and $\alpha \in (0,1)$.
The disadvantage with this is that there is no local form of an ADE, the entire past history must be preserved when simulating. [5]

[5] **N. L. Gibson** and S. Shaw, "Polynomial Chaos for Viscoelastic Volterra Kernels", *in preparation*.

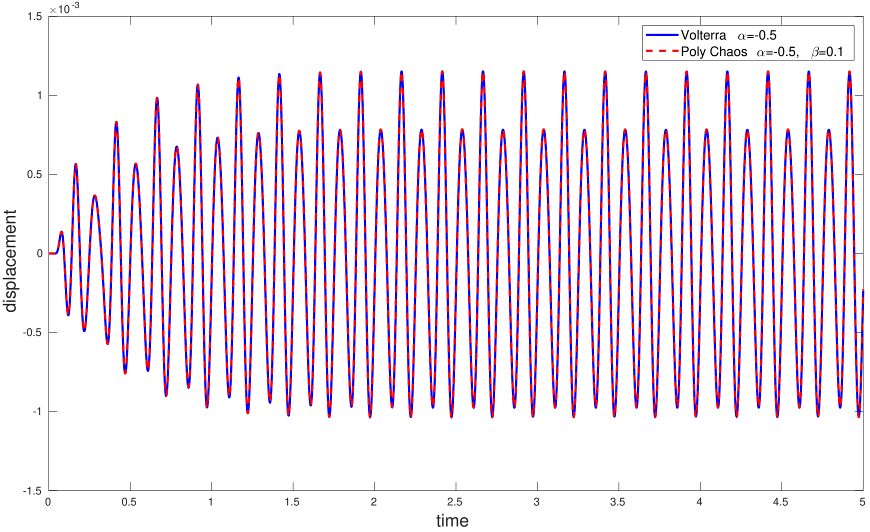

Figure: Comparison of simulations of power law Volterra kernel vs continuous spectrum with Polynomial Chaos.

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Magnetohydrodynamics (MHD)

$$\text{MHD} = \text{Maxwell} + \text{Navier-Stokes}$$

Collaborators

- Rigel Woodside, NETL Albany
- Duncan McGregor, ORISE Fellow, 9/2014–6/2016
- Evan Rajbhandari, ORISE Fellow, 7/2020–6/2022
- Vrushali Bokil (PI), NSF-DMS Computational Mathematics, 8/1/2020–7/31/2022

PC for Hall MHD

Ohm's Law becomes

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \beta_e(\mathbf{J} \times \mathbf{B})$$

where $\mathbf{u}$ is the velocity, and $\beta_e = \omega_e \tau_e$ is the Hall parameter.

$$\mathbf{J} = \overline{\sigma}(\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

where

$$\overline{\sigma} = \sigma \begin{bmatrix} \frac{1}{1+\omega_e^2\tau_e^2} & \frac{-\omega_e\tau_e}{1+\omega_e^2\tau_e^2} & 0 \\ \frac{\omega_e\tau_e}{1+\omega_e^2\tau_e^2} & \frac{1}{1+\omega_e^2\tau_e^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Compare to

$$\epsilon(\omega) = \epsilon_\infty + \epsilon_d \frac{1}{1+\omega^2\tau^2} + \epsilon_d \frac{\omega\tau}{1+\omega^2\tau^2}\mathrm{i}.$$

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Acknowledgments

- Co-PI, Bonneville Power Administration (BPA) Technology Innovation Program, "Towards reduction of uncertainty in the operation of reservoir systems", with Arturo Leon (PI, Civil Engineering) and Christopher Hoyle (Co-PI, Mechanical Engineering), 10/2012–9/2015, $665,593.
- Co-PI, BPA Technology Innovation Program, "Framework for Quantification of Risk and Valuation of Flexibility in the FCRPS", with Arturo Leon (PI, Civil Engineering), Christopher Hoyle (Co-PI, Mechanical Engineering), Claudio Fuentes (Co-PI, Statistics), Yong Chen (Co-PI, Applied Economics), 10/2015–5/2018, $1.2M.

Reservoir Operations

The broad context of the problem of interest is a PDE-constrained optimal control problem with uncertainty. In particular, one must

- maximize revenue (minimize cost to the consumer)
- minimize ecological violations
- meet electrical demand with hydro-power production
- at least 9 other constraints

Simulation of Unsteady Flows

- Most free surface flows are unsteady and nonuniform.
- Unsteady flows in river systems are most efficiently simulated using 1D models.

Saint-Venant equations: PDEs representing conservation of mass and momentum for a control volume:

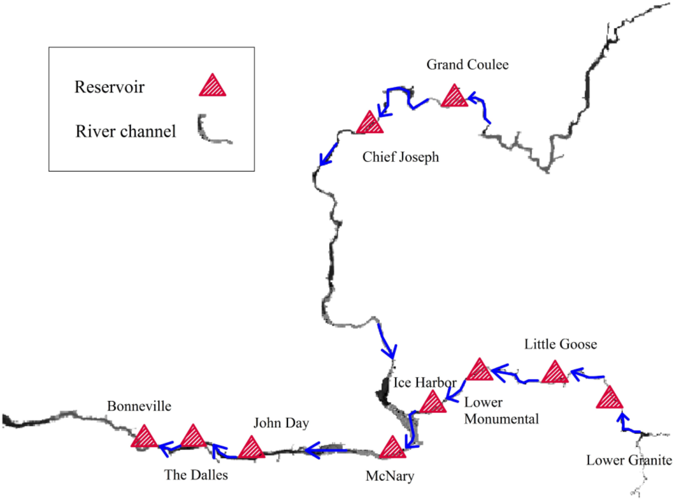
$$B\frac{\partial y}{\partial t} + \frac{\partial Q}{\partial x} = 0, \qquad (6)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\left(\frac{\partial y}{\partial x} + S_f - S_0\right) = 0, \qquad (7)$$

where x is a distance along the channel in the longitudinal direction, t is time, y is a water depth, Q is a flow discharge,
B is a width of the channel, g is an acceleration due to gravity,
A is a cross-sectional area of the flow, S_f is a friction slope, S_0 is a river bed slope.

Initial, boundary and interface conditions are required to close the system.

Big 10 Columbia River System



Unknowns: flow discharges Q_i for $i = 1, \ldots, 10$.

Dimension Reduction

- Simulations are required to be two weeks in duration.
- Solving for decision variables on each time step (daily or hourly) is computationally impractical.
- We wish to construct a reduced dimensional basis for the decisions.
- Fortunately there exist years worth of data.
- We solve only for the optimal coefficients of an expansion in this basis (Spectral Optimization Method[6]).
- Specifically, we construct a truncated Karhunen-Loeve (KL) expansion (or PCA) using the historical solutions

$$Q(t_j, \vec{\xi}) = \bar{Q}(t_j) + \sum_{k=1}^{M} \sqrt{\lambda_k} \psi_k(t_j) \xi_k$$

and optimize over $\xi_k, k = 1 \dots M$.

[6] D. Chen, A. S. Leon, **N. L. Gibson**, P. Hosseini, "Dimension reduction of decision variables for multi-reservoir operation: A spectral optimization model", Water Resources Research, 52 (1), 36–51, 2016.

Grand Coulee outflows

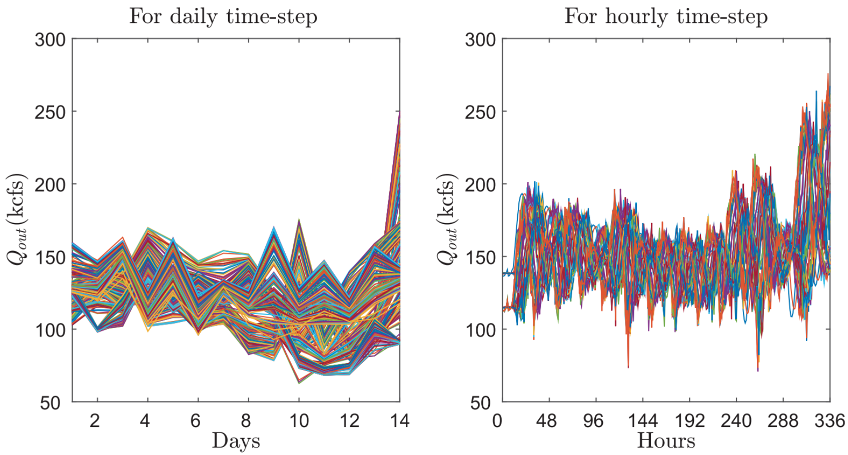


Figure: Synthetic data for outflows at Grand Coulee.

Grand Coulee KL Expansion

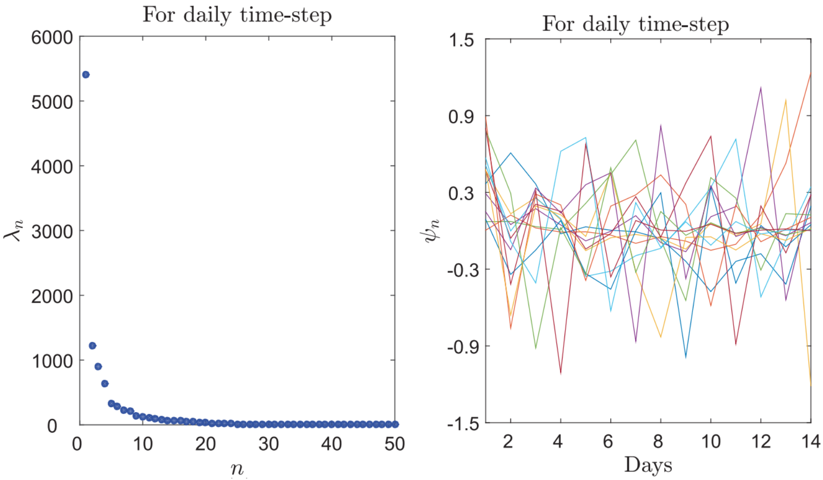


Figure: Eigenvalues and eigenvectors of the covariance matrix for Grand Coulee.

Optimization Results

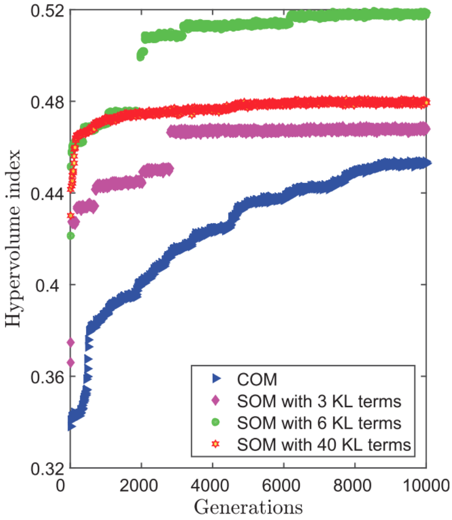


Figure: Goodness of multi-objective optimization results with various reduced dimension decision spaces.

Outline

1. **Polynomial Chaos for Multiscale Modeling**
   - Dispersive Maxwell System
   - Maxwell-Random Lorentz System
   - Viscoelastic Materials
   - Magnetohydrodynamics

2. **Karhunen-Loève for Dimension Reduction**
   - Reservoir Operations
   - Power Grid
   - Tsunami Loading

Probabilistic Load Flow (PLF) problem

- A load flow analysis tests the reliability of the power grid under various supply and demand scenarios.
- Probabilistic Load Flow allows for each supply source and/or demand location to be a random variable.
- These are typically solved via Monte-Carlo methods or so-called convolution methods.

Probabilistic Load Flow (PLF) problem (continued)

- Our approach[7]: treat all of the solar power sources as spatial points within one underlying stochastic process.
- KL expansion results in 194 dependent random variables reduced to 12 uncorrelated random variables.
- Enables Anisotropic Sparse Grid Interpolation (Stochastic Collocation)

[7] B. Johnson, **N. L. Gibson** and E. Cotilla-Sanchez, "A Coupled Karhunen-Loève and Anisotropic Sparse Grid Interpolation Method for the Probabilistic Load Flow Problem", Electric Power Systems Research, 24 pages. *Accepted, to appear.*

Anisotropic Sparse Grid

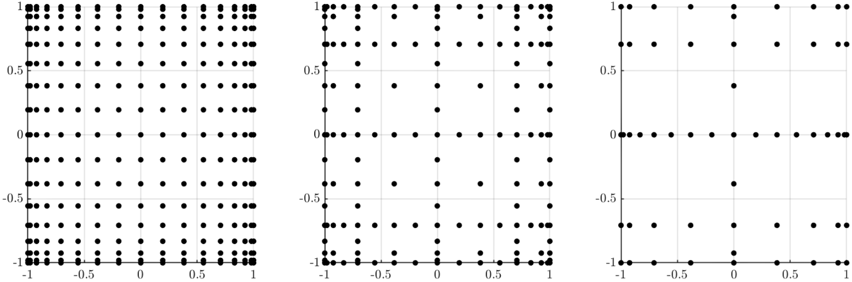


Figure: Demonstration of full tensor grid, sparse grid, and anisotropic sparse grid.

PLF Results

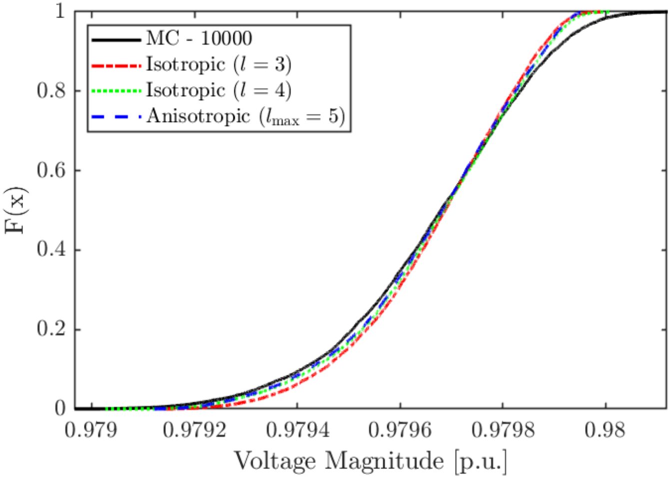


Figure: Voltage magnitude CDF at Bus 93 in the IEEE 118-Bus system.

Outline

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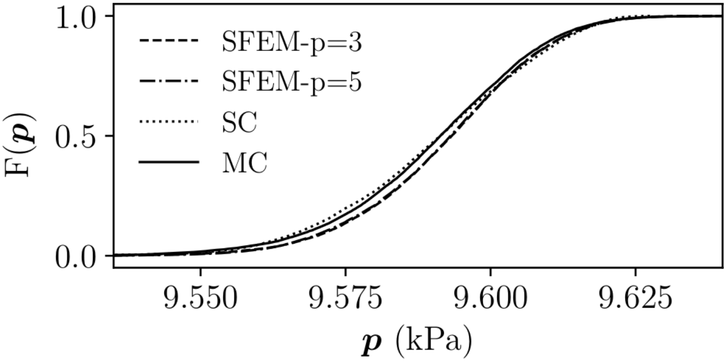
Tsunami Loading

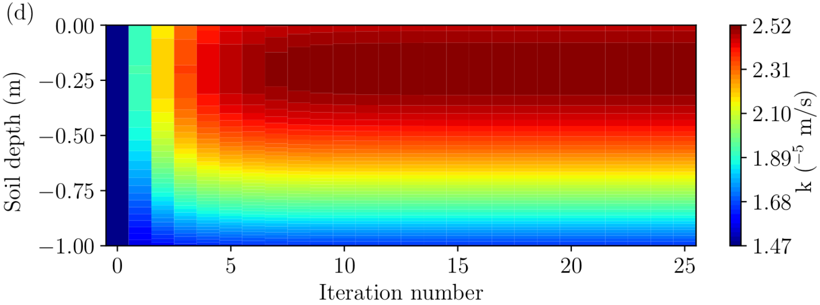
In collaboration with Ben Mason and Yingqing Qiu (Civil Engineering)

- We have applied KL expansions to an uncertain hydraulic conductivity for a numerical soil model under wave loading.
- We compared Stochastic Finite Element Method to Stochastic Collocation and Monte Carlo.



Tsunami Loading Inverse Problem

- Determine the coefficients of a KL expansion for an unknown hydraulic conductivity given measurements of pressure.
- Data obtained from Hinsdale wave lab experiment.
- 20 dimensions reduced to 6; 50 Newton iterations reduced to 15.



Summary

- KL expansions used for dimension reduction
  - Forward problems in power flow and wave loading
  - Optimal control for reservior modeling
  - Parameter estimation for wave loading
- PC expansions used for multiscale modeling
  - Random Lorentz polarization
    - Well-posedness (Continuous and Discrete)
    - Dispersion analysis
  - Viscoelastic relaxation
  - Toward Hall MHD