Topological, Algebraic, and Analytical Aspects of Cantor Sets

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Main Topological Questions

For the purposes of this talk, all spaces will be –
subspaces of R^n for some n.

Two Main Questions in Topology:

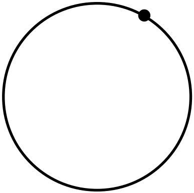
- Characterizing or Classifying Certain Spaces, Classes of Spaces, or Maps
- Determining when Embeddings of One Space in Another are Equivalent

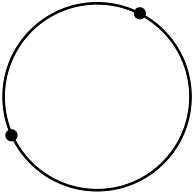
Characterization Example

The circle is the only space that has the following property:

No single point separates;

each pair of points separates.





Classification Examples

- Ever compact connected surface is either a 2-sphere, an n-holed torus, or the connected sum of n projective planes.

  These surfaces are distinguished by their orientability and Euler characteristic.

- Every map from S^1 to itself is homotopic to one of the maps f_i given by $f_i(z) = z^i$.

Equivalent Embeddings

As a specific example of the second question, given a subspace A of R^n, and two **embeddings**

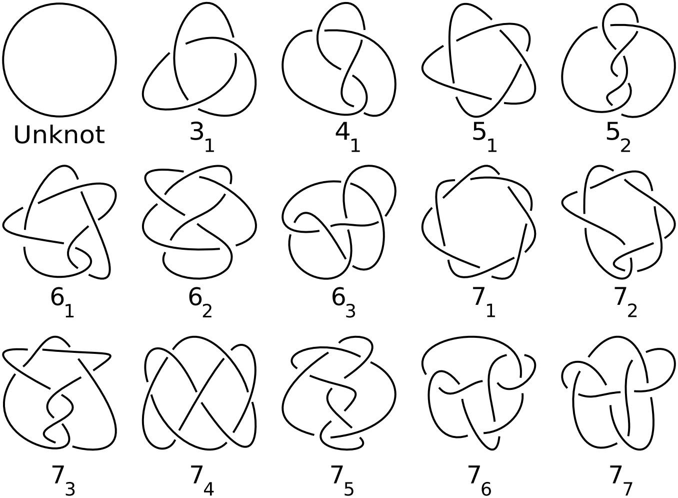
$$f : A \to R^n \text{ and } g : A \to R^n,$$

when should we view these embeddings as equivalent, or as topologically the same?

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ R^n & & R^n \end{tikzcd}$$

Knots

For example, consider knots in R^3 as embeddings of circles:



Definition of Equivalence

Def. Embeddings $f : A \to R^n$ and $g : A \to R^n$ are *equivalent* if there is a homeomorphism $h : R^n \to R^n$ such that $h \circ f = g$.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ R^n \arrow[rr, "h"] & & R^n \end{tikzcd}$$

Theorem: Any two embeddings of a circle in R^2 are equivalent.

This is known as the Schönflies Theorem, one consequence of which is the Jordan Curve Theorem.

The Cantor Set

The Standard Middle Thirds Cantor Set C in R^2 is defined as follows: $S_0 = [0,1], \quad S_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ Inductively, S_n has 2^n closed intervals of length $\frac{1}{3^n}$.

To get S_{n+1}, delete the open middle third of each closed interval in S_n.

Def. $C = \bigcap\limits_{i=0}^{\infty} S_i$.

Cantor Set in Base Three: Equivalently,

$$C = \left\{ \sum_{i=1}^{\infty} \frac{a_i}{3^i} \,\middle|\, a_i \in \{0,2\} \right\}$$

Stages in Construction

First 5 Stages of Cantor Set

Characterization

Topologically, the Cantor Set is characterized as follows:

Theorem:

A space X is homeomorphic to the Cantor set C *if and only if* X is

- totally disconnected
  (every component is a single point)
- compact
- every point is a limit point

Procedure for Producing Cantor Sets

The preceding characterization allows us to show that the following procedure always yields a space homeomorphic to the Cantor Set:

Let A_1 be a finite collection of pairwise disjoint compact subsets of R^n.

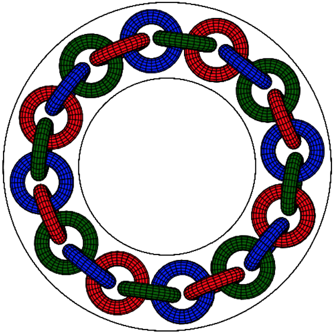
Assume that A_k is a finite collection of pairwise disjoint nonempty compact subsets of R^n so that each set in A_k is contained in some element of A_{k-1} and so that each element of A_{k-1} contains at least two elements of A_k.

Procedure, Continued

Theorem:

If the diameter of the sets in A_k goes to 0 as $k \to \infty$, then $X = \cap_{k=1}^{\infty} A_k$ is a Cantor set.

One pattern that leads to a non-standard Cantor Set in R^3:



Properties of the Cantor Set

- Every compact metric space is the continuous image of a subspace of C
- C contains a copy of every 0 dimensional space
- $C \cong \prod_{i=1}^{\infty} \{0, 1\}$
- $\prod_{i=1}^{\infty} C \cong C$
- The measure of the removed intervals from $[0, 1]$ to obtain C is 1. $\left(\sum_{i=0}^{\infty} \frac{2^i}{3^{i+1}} = 1 \right)$
- C is uncountable
- C is homogenous (in fact there is a self homeomorphism taking any countable dense subset to any other such subset)

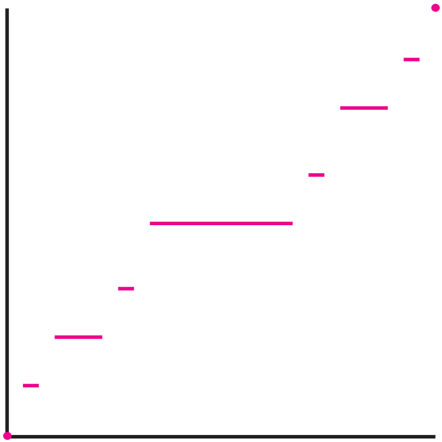
More Properties

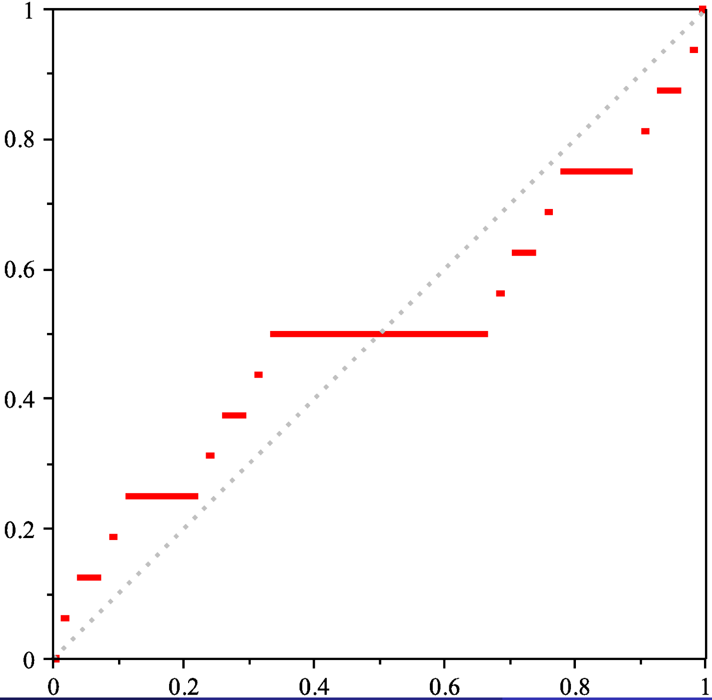
- There is a continuous nondecreasing function f from C **onto** $I = [0, 1]$.
- There is a continuous function f^∞ from $C = \prod_{i=1}^{\infty} C$ **onto** $\prod_{i=1}^{\infty} I = I^\omega$.

$$f\left(\sum_{i=1}^{\infty} \frac{n_i}{3^i}\right) = \sum_{i=1}^{\infty} \frac{n_i}{2 \cdot 2^i}$$

More Properties (Analysis)

There is a continuous nondecreasing function from I onto I that is constant on $I - C$. So there is a continuous function from I to I, with derivative 0 almost everywhere, that is not constant.




1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

Yet More Properties

- Any two copies of C in R^2 are equivalent
- There are uncountably many inequivalent copies of C in R^3.

Higher Dimensional Analogs

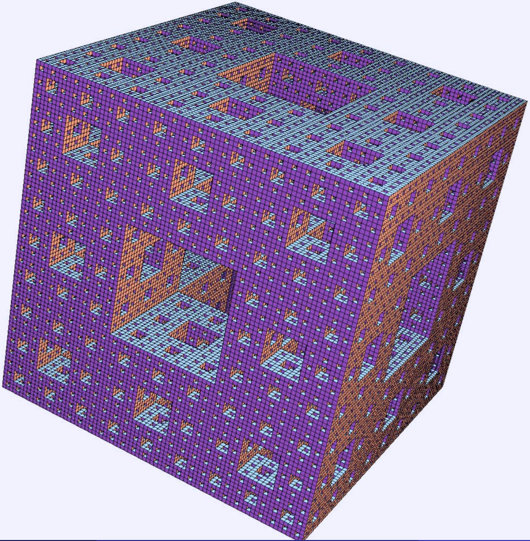
Note: For each positive integer n, there is an n-dimensional analog of the Cantor Set, μ_n in R^{2n+1} characterized by:

- compact
- n-dimensional
- $n-1$ connected (C^{n-1})
- locally $n-1$ connected (LC^{n-1})
- Disjoint n cells property

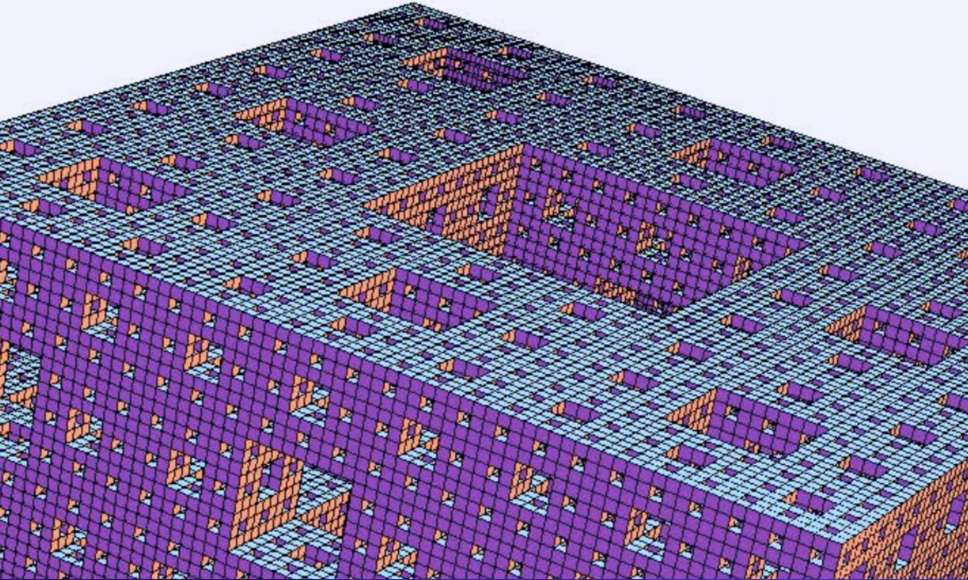
μ_n has analogous properties to C, in particular, it contains a copy of every n-dimensional space.

The One Dimensional Universal Space μ_1

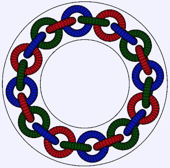
μ_1 is the Menger cube or Menger sponge.



Closeup View



Inequivalent Cantor Sets



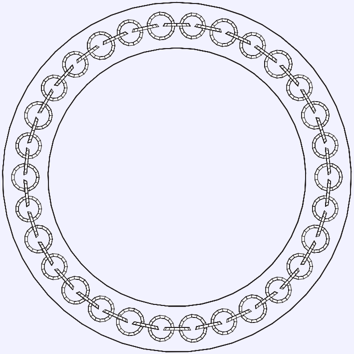
Sher (1968) showed that any two constructions as above in R^3 that yield equivalent Cantor Sets must have the same number of tori at each stage.

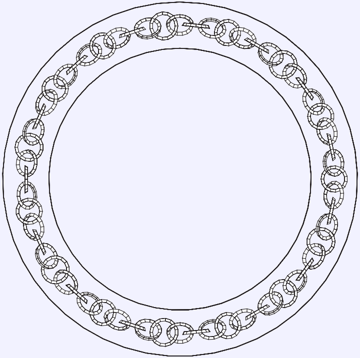
How to get inequivalent such Cantor sets?

Vary the number of tori at each stage.

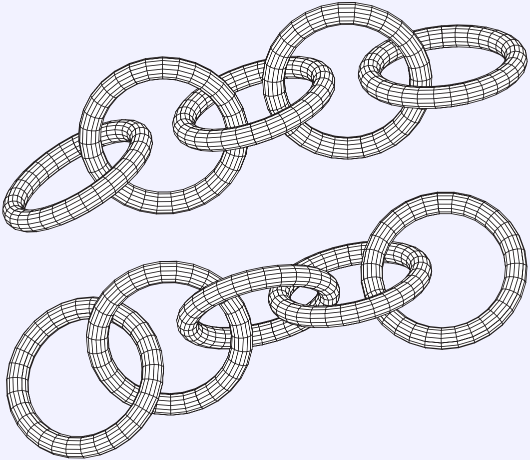
What if numbers are kept the same?

Links with Twists





Close Up View



Other types of Embeddings

There are nonstandard Cantor sets C in R^3:

- that have simply connected complement,
- that are rigidly embedded (the only self homeomorphism of C that extends to a homeomorphism of R^3 is the identity), and
- that have both of the above properties

Relation to Algebra

The *homogeneity group* of $C \subset R^3$ is group of homeomorphisms of C that extend to homeomorphisms of R^3.

The standardly Cantor, at one extreme, is *strongly homeogeneously embedded*. That is, the homogeneity group is the full group of self-homeomorphisms of the Cantor set, an extremely rich group (there is such a homeomorphism taking any countable dense set to any other).

At the other extreme are *rigidly embedded* Cantor sets, i.e. those Cantor sets for which only the identity homeomorphism extends.

Result, Conjecture

Theorem: (G, Repovš-2013) For every finitely generated Abelian group G, there is a Cantor set C_G in R^3 with homogeneity group G.

Conjecture: For every finitely generated group G, there is a Cantor set C_G in R^3 with homogeneity group G.

Dimension - Topological and Hausdorff

Every Cantor set has topological dimension 0.

The standard Cantor set in R has Hausdorff dimension

$$\frac{\ln(2)}{\ln(3)} \sim 0.6309\ldots$$

Iterated Function Systems

The Cantor set is the invariant set of the iterated function system:

$$f_1(x) = \frac{x}{3} \qquad f_2(x) = \frac{x}{3} + \frac{2}{3}$$