Homogeneity Groups of Cantor sets in S^3

Dennis J. Garity
(joint work with Dušan Repovš)

Main Result

For every finitely generated abelian group G, we construct an unsplittable Cantor set C_G in S^3 with embedding homogeneity group isomorphic to G. (Pacific J. of Math., 2014)

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- Cantor sets C and D in S^3 are equivalent if there is a self homeomorphism of S^3 taking C to D.
- A Cantor set $C \subset S^3$ is rigidly embedded if the only self homeomorphism of C that extends to S^3 is the identity.

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e.g. unsplittable $\Longleftrightarrow$ irreducible

We phrase things in terms of Cantor sets in this talk.

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- At the other extreme are rigidly embedded Cantor sets, i.e. those Cantor sets for which only the identity homeomorphism extends.

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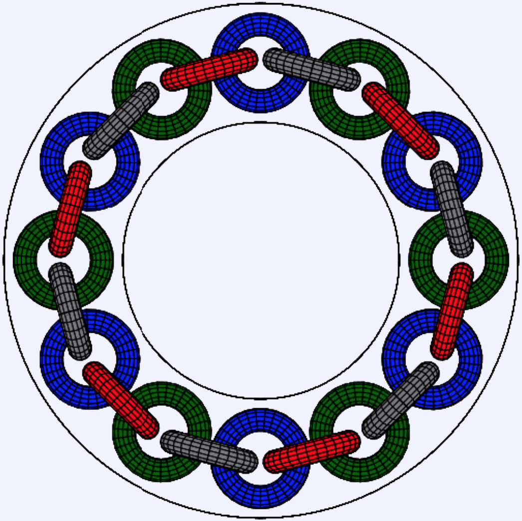
Antoine Cantor Sets



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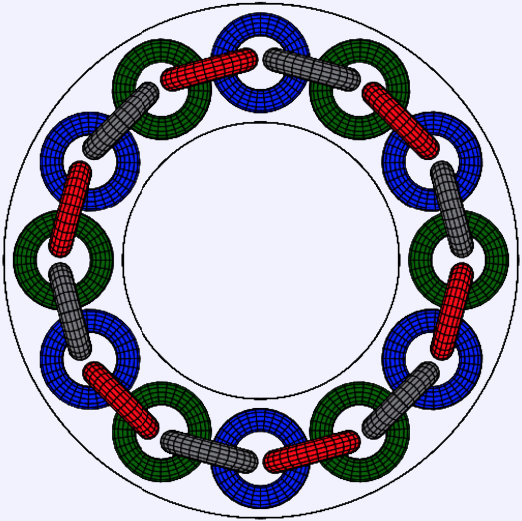

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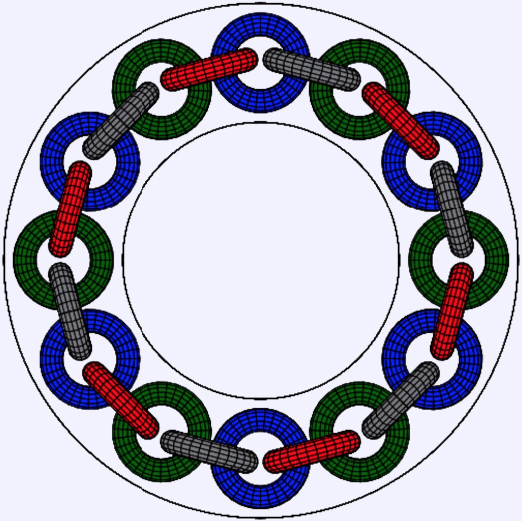

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$$C \equiv \bigcap_{i=0}^{\infty} S_i$$

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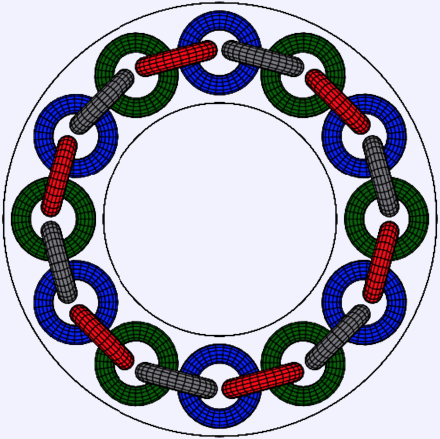
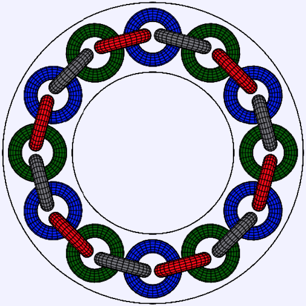
- [Shilepsky '74] This can be used to construct (uncountable many) inequivalent Antoine rigid Cantor sets.

Homogeneity group $\mathbb{Z}_p$



Antoine Chain With $\mathbb{Z}_6$ Group Action

$\mathbb{Z}_p$ Construction



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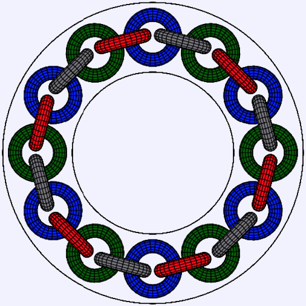
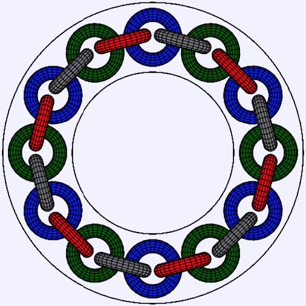


- S_0 an unknotted solid torus in S^3.
- $\{S_{(1,i)} \mid 1 \leq i \leq 4p\}$, an Antoine chain of length $4p$ in S_0, and

$$S_1 = \bigcup_{i=1}^{4p} S_{(1,i)}.$$

Construction, Continued



- $C_j, 1 \leq j \leq 4$, distinct rigid Antoine Cantor sets in $S_{(1,j)}$.

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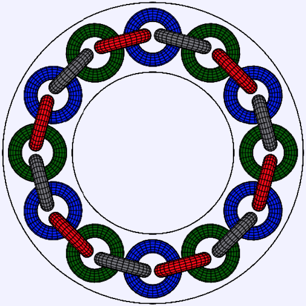

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- Let r be a homeomorphism of S^3, fixed on the complement of S_0, that takes $S_{(1,j)}$ to $S_{(1,j+4 \mod 4p)}$ for $1 \leq j \leq 4p$.

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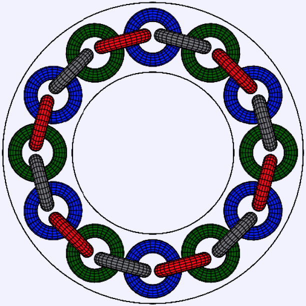

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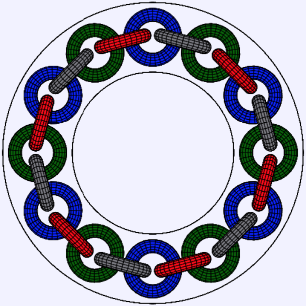
- Require that r^p is the identity on each $S_{(1,i)}$.

Construction Continued, II



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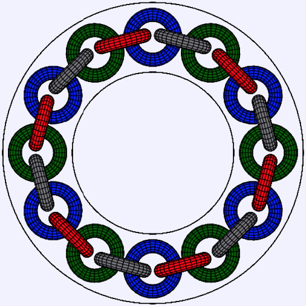

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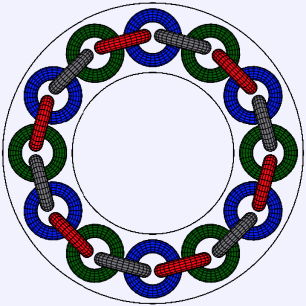
- $C_{\mathbb{Z}_p} = \bigcup_{i=1}^{4p} C_i$.

Sketch of Proof



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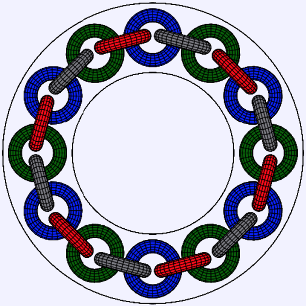

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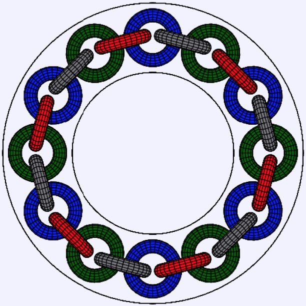

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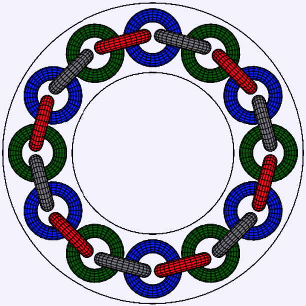

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- This inductively shows $h|_C = r^k$.

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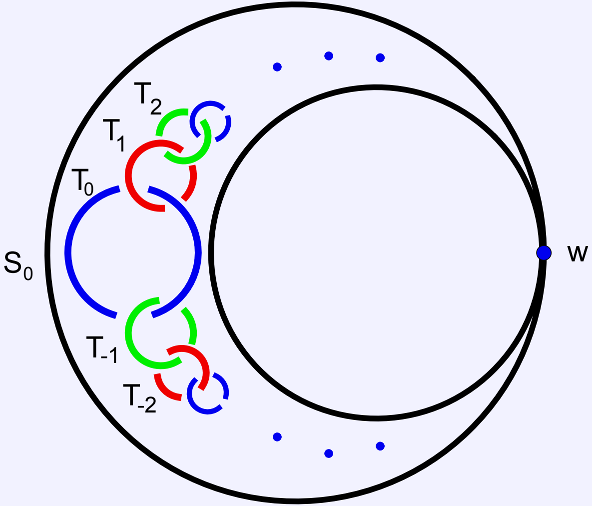
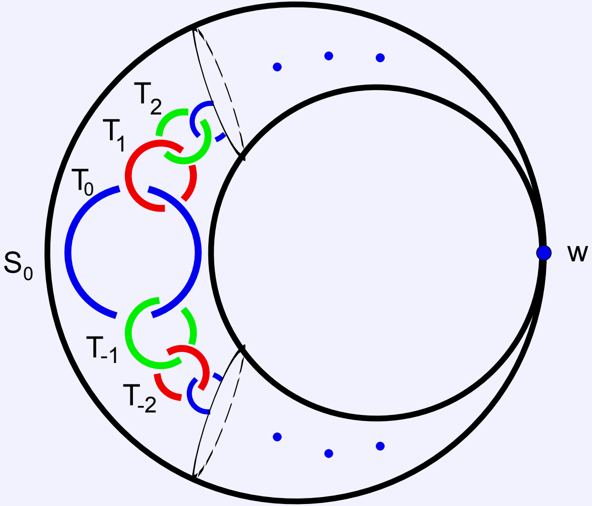
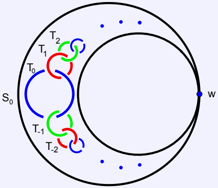


Genus 2 at w



$\mathbb{Z}$ construction



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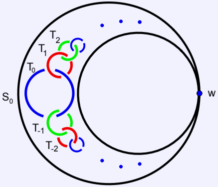

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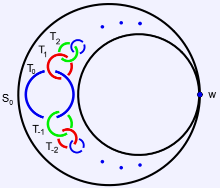

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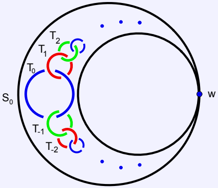

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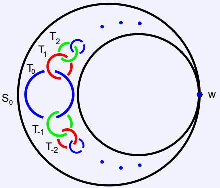

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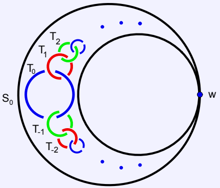

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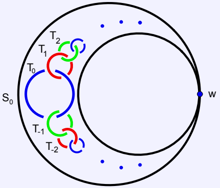

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- $C = \cup C_i \cup \{w\}$

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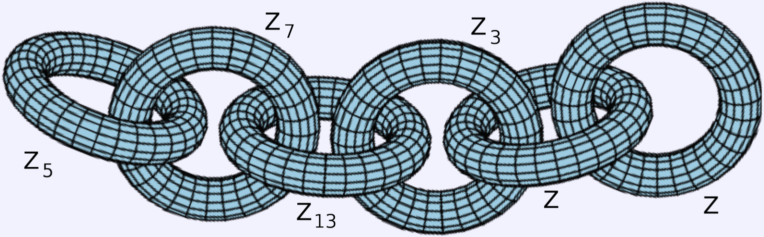
- Unsplittable

Finitely Generated Abelian G



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Step 1: There is a general position homeomorphism h_1, fixed on C, so that $h_1(\partial(M_1)\cup\partial(M_2))$ is in general position with $\partial(N_1)\cup\partial(N_2)$. The curves of intersection of $h_1(\partial(M_1)\cup\partial(M_2))\cap(\partial(N_1)\cup\partial(N_2))$ can be eliminated by a homeomorphism h_2 also fixed on C.

Details, II

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Assume first: $T \subset \mathrm{IntS}$.

If the geometric index of T in S is 0, then all components of $h_2 \circ h_1(M_1)$ are in the interior of S. This is a contradiction since there are points of C not in S. So the geometric index of T in S is greater than or equal to 1.

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This geometric index must then be 2 and the geometric index of T in S must be 1.

Details IV

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The net result is that it is possible to construct a homeomorphism h_3' taking the components of $h_2 \circ h_1(M_1)$ to the components of N_1. One now proceeds inductively, matching up further stages in the constructions, obtaining the desired homeomorphism h as a limit.

Ideas on Non Abelian Case

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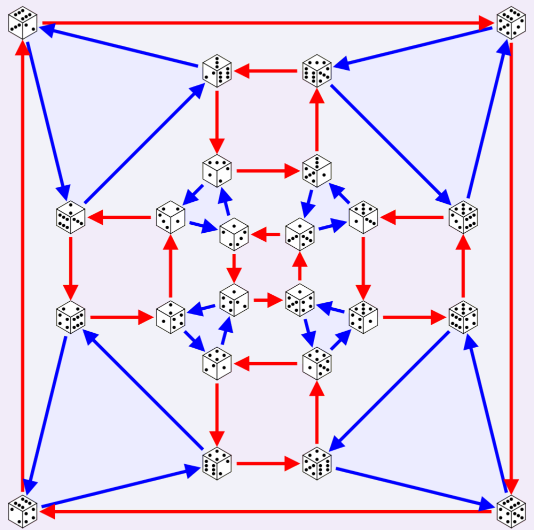
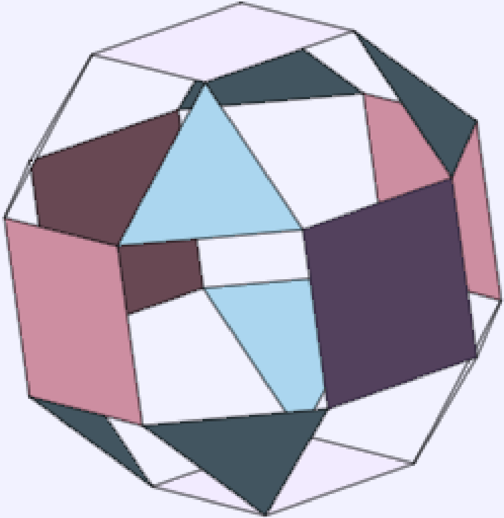
- Other finitely generated groups?

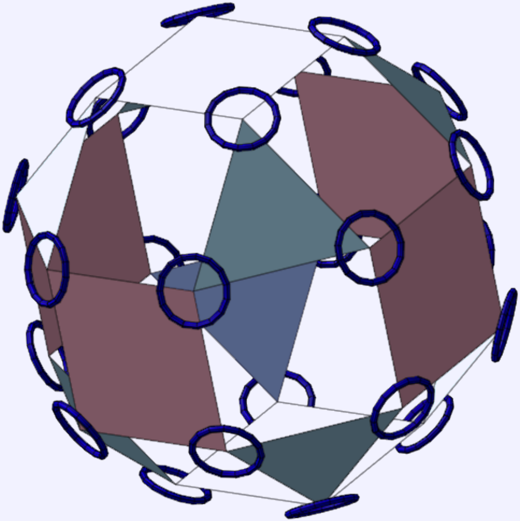
Generalization - A Cayley Graph



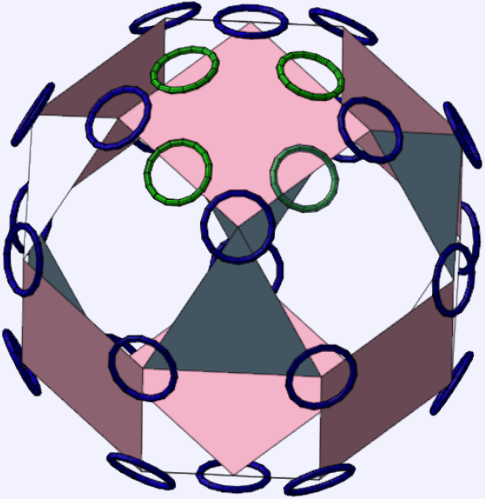
Generalization - A Cayley Graph- II



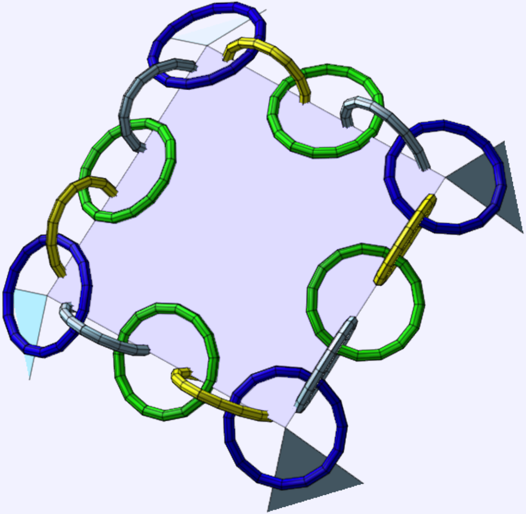
Generalization - Vertices



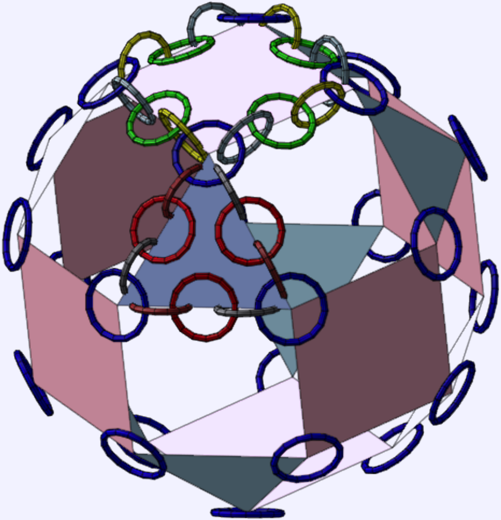
Generalization - Edges I



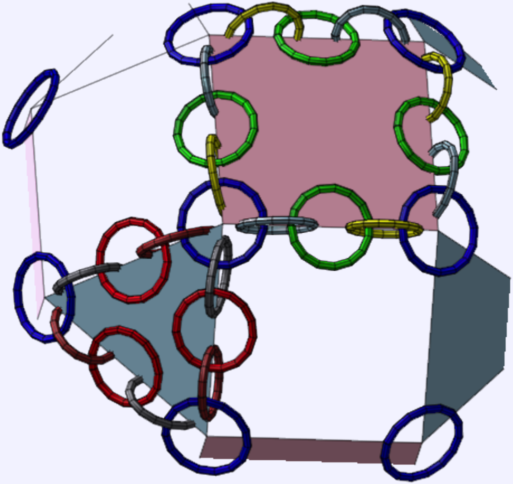
Generalization - Edges II



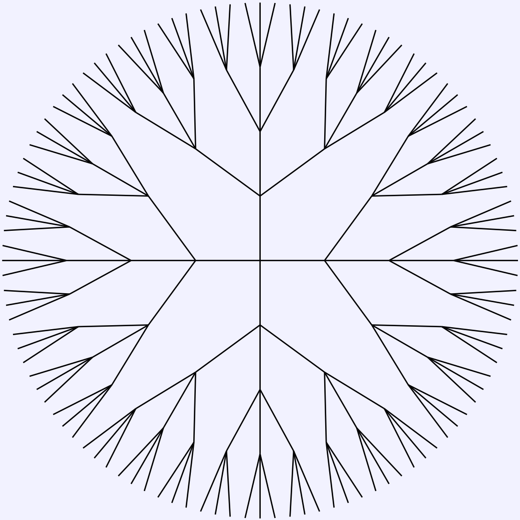
Generalization - Edges III



Closeup at Vertiex



Generalization - F2- Stage 4



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