A New Class of Homology and Cohomology 3-Manifolds

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Abstract. We show that for any set of primes $\mathcal{P}$ there exists a space $M_{\mathcal{P}}$ which is a homology and cohomology 3-manifold with coefficients in $\mathbb{Z}_p$ for $p \in \mathcal{P}$ and is not a homology or cohomology 3-manifold with coefficients in $\mathbb{Z}_q$ for $q \notin \mathcal{P}$. In addition, $M_{\mathcal{P}}$ is not a homology or cohomology 3-manifold with coefficients in $\mathbb{Z}$ or $\mathbb{Q}$.

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1. Introduction

In 1908 Tietze [7] constructed famous 3-manifolds $L(p,q)$ called lens spaces. These spaces have many interesting properties. For example, lens spaces $L(5,1)$ and $L(5,2)$ have isomorphic fundamental groups and the same homology, but they do not have the same homotopy type (proved by Alexander [1] in 1919). It is well-known that for every prime q, the lens space $M^3 = L(q,1)$ has the following properties:

- M^3 is a 3-dimensional homology manifold with coefficients in $\mathbb{Z}_p$ (denoted as 3-$hm_{\mathbb{Z}_p}$) for every prime $p \neq q$;
- M^3 is not a 3-dimensional homology manifold with coefficients in $\mathbb{Z}_q$;
- M^3 is not a 3-dimensional homology manifold with coefficients in $\mathbb{Z}$.

We shall generalize this classical result as follows:

Theorem 1.1. *Given any set of primes $\mathcal{P}$ there exists a space $M_{\mathcal{P}}$ which is a homology and cohomology 3-manifold with coefficients in $\mathbb{Z}_p$ for $p \in \mathcal{P}$ and is not a homology or cohomology 3-manifold with coefficients in $\mathbb{Z}_q$ for $q \notin \mathcal{P}$. In addition, $M_{\mathcal{P}}$ is not a homology or cohomology 3-manifold with coefficients in $\mathbb{Z}$ or $\mathbb{Q}$.*

2. Preliminaries

First, we fix the terminology, notation, and remind the reader of some well-known facts. We let L be the ring of integers $\mathbb{Z}$ or a field.

Definition 2.1 (*cf. [2, Corollary 16.9]*). A space X is called an n-dimensional cohomology manifold over L (denoted $n - cm_L$) if:

(1) X is locally compact and has finite cohomological dimension over L;
(2) X is cohomologically locally connected over L (clc_L); and
(3) for all $x \in X$,

$$\check{H}^p(X, X\backslash\{x\}; L) \cong \begin{cases} L & \text{for } p = n \\ 0 & \text{for } p \neq n \end{cases}$$

where $\check{H}^*$ are Čech cohomology groups with coefficients in L.

Definition 2.2. A homology L-manifold of dimension n over L (denoted as $n - hm_L$) is a locally compact topological space X with finite cohomological dimension over L such that for any $x \in X$, the Borel–Moore homology groups $H_p(X, X\backslash\{x\}; L)$ are trivial unless $p = n$, in which case they are isomorphic to L. Homology manifolds will stand for homology $\mathbb{Z}$-manifolds.

Any n-dimensional cohomology manifold ($n - cm_L$) is an n-dimensional homology manifold ($n - hm_L$) by [2, Theorem 16.8]. Therefore we will construct only cohomology manifolds which will be automatically homology manifolds by this theorem.

For the construction and some simple properties of lens spaces see [4,6]. In particular, the homology groups of the lens space $M^3 = L(q, 1)$ are

$$H_n(M^3; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 3 \\ \mathbb{Z}_q & n = 1 \\ 0 & n = 2 \text{ or } n \geq 4 \end{cases}$$

By the Universal Coefficients Theorem we have for any abelian group G,

$$H_n(M^3; G) \cong H_n(M^3; \mathbb{Z}) \otimes G \oplus H_{n-1}(M^3; \mathbb{Z}) * G.$$

Therefore, if p and q are prime and $p \neq q$ then

$$H_n(M^3; \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p & \text{if } n = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

whereas

$$H_n(M^3; \mathbb{Z}_q) \cong \begin{cases} \mathbb{Z}_q & n = 0, 1, 3 \\ 0 & \text{otherwise} \end{cases}$$

Local conditions of Definitions 2.1 and 2.2 are satisfied since M^3 is a manifold therefore $M^3 = L(q, 1)$ is a 3-$hm_{\mathbb{Z}_p}$ and a 3-$cm_{\mathbb{Z}_p}$ but is neither a 3-$hm_{\mathbb{Z}_q}$ nor a 3-$cm_{\mathbb{Z}_q}$ if p and q are prime and $p \neq q$ (cf. [2]).

3. Proof of Theorem 1.1

Let $\mathcal{P} = \{p_i\}_{i\in K}$, for $K = \mathbf{N}$ or $K = \{1, \ldots, k\}$, be a set of some prime numbers. If the set K is infinite then we define the numbers n_i as $n_i = p_1 \cdot p_2 \cdot p_3 \cdots p_i$. If the set K is finite and consists of exactly k elements, then define n_i as $n_i = p_1 \cdot p_2 \cdot p_3 \cdots p_k$ for all i.

Let X be a solenoid in the 3-dimensional sphere S^3, i.e., the inverse limit of solid tori corresponding to the following inverse system:

$$\mathbb{Z} \xleftarrow{n_1} \mathbb{Z} \xleftarrow{n_2} \mathbb{Z} \xleftarrow{n_3} \ldots$$

naturally embedded in S^3, see, e.g., [3, Chapter IIX, Exercise E.4].

Let us prove that the quotient space S^3/X is a cohomology 3-manifold $cm_{\mathbb{Z}_p}$. It is obvious that S^3/X is 3-dimensional, compact and metrizable. So the space S^3/X satisfies the condition (1) of Definition 2.1.

To prove that the space S^3/X satisfies the conditions (2) and (3) of Definition 2.1, let us calculate first the groups $\check{H}^n(S^3/X, \{x\}; G)$ with respect to the one-point subset $\{x\} = X/X$ for $G \cong \mathbb{Z}_p, p \in \mathcal{P}$; $G \cong \mathbb{Z}_q, q \notin \mathcal{P}$; $G \cong \mathbb{Z}$; $G \cong \mathbb{Q}$. Since S^3/X is connected and 3-dimensional it follows that

$$\check{H}^0(S^3/X, \{x\}; G) \cong 0 \quad \text{and} \quad \check{H}^n(S^3/X, \{x\}; G) \cong 0 \;\; \text{for } n > 3. \qquad (1)$$

Let U_i be the open ith solid torus neighborhood of X in S^3 (c.f. [3]). Then $\{U_i/X\}_{i\in\mathbb{N}}$ is a neighborhood base of x in S^3/X. By continuity of the Čech cohomology and by the Excision Axiom it follows that:

$$\check{H}^n(S^3/X, \{x\}; G) \cong \check{H}^n(\varprojlim S^3/\overline{U}_i, \overline{U}_i/\overline{U}_i; G)$$

$$\cong \varinjlim \check{H}^n(S^3/\overline{U}_i, \overline{U}_i/\overline{U}_i; G) \cong \varinjlim \check{H}^n(S^3, \overline{U}_i; G).$$

For $n = 1$ we have the exact sequence of the pair $(S^3, \overline{U_i})$:

$$\check{H}^0(S^3; G) \longrightarrow \check{H}^0(\overline{U_i}; G) \longrightarrow \check{H}^1(S^3, \overline{U}_i; G) \longrightarrow \check{H}^1(S^3, G) \longrightarrow \check{H}^1(\overline{U_i}; G).$$

Since the 1-dimensional cohomology of the 3-sphere is trivial for any group of coefficients G and $\overline{U_i}$ is connected space for every i, it follows that $\check{H}^1(S^3, \overline{U}_i; G) \cong 0$, therefore $\check{H}^n(S^3/X, \{x\}; G) \cong 0$ and, in particular,

$$\check{H}^1(S^3/X, \{x\}; \mathbb{Z}_p) \cong 0 \qquad (2)$$

and

$$\check{H}^1(S^3/X, \{x\}; \mathbb{Z}_q) \cong 0, \check{H}^1(S^3/X, \{x\}; \mathbb{Z}) \cong 0, \check{H}^1(S^3/X, \{x\}; \mathbb{Q}) \cong 0. \quad (3)$$

For $n = 2$ we have the following exact sequence of the pair $(S^3, \overline{U_i})$:

$$\check{H}^1(S^3; G) \longrightarrow \check{H}^1(\overline{U}_i; G) \longrightarrow \check{H}^2(S^3, \overline{U}_i; G) \longrightarrow \check{H}^2(S^3, G) \longrightarrow \check{H}^2(\overline{U_i}; G).$$

The cohomology groups $\check{H}^1(S^3; G)$ and $\check{H}^2(S^3, G)$ are trivial, and the groups $\check{H}^1(\overline{U}_i; G)$ are isomorphic to G since U_i has the homotopy type of a circle. The homomorphisms $\check{H}^1(\overline{U}_i; G) \to \check{H}^1(\overline{U}_{i+1}; G)$ are multiplications by n_i that take the group G into itself. Therefore for the group of coefficients $G \cong \mathbb{Z}_p$ it follows that

$$\check{H}^2(S^3/X, \{x\}; \mathbb{Z}_p) \cong 0. \qquad (4)$$

However,

$$\check{H}^2(S^3/X,\{x\};\mathbb{Z}_q)\not\cong 0, \check{H}^2(S^3/X,\{x\};\mathbb{Z})\not\cong 0, \check{H}^2(S^3/X,\{x\};\mathbb{Q})\not\cong 0. \quad (5)$$

For $n=3$ consider the next cohomology exact sequence for the pair $(S^3,\overline{U_i})$:

$$\check{H}^2(\overline{U}_i;G)\longrightarrow \check{H}^3(S^3,\overline{U}_i;G)\longrightarrow \check{H}^3(S^3,G)\longrightarrow \check{H}^3(\overline{U}_i;G).$$

Since $\overline{U}_i\simeq S^1$, it follows that:

$$\check{H}^3(S^3/X,x;G)\cong G. \quad (6)$$

Let us calculate the groups $\check{H}^n(S^3/X-\{x\};\mathbb{Z}_p)$. The space $S^3/X-\{x\}$ is the union $\bigcup_{i=1}^{\infty}(S^3-U_i)$ of an increasing sequence of "complementary" solid tori.

For $n=1$ we have the following exact sequence of Milnor–Harlap [5, Theorem 1]:

$$0\to \varprojlim{}^{(1)}\check{H}^0(S^3-U_i;\mathbb{Z}_p)\to \check{H}^1(S^3-X;\mathbb{Z}_p)\to \varprojlim \check{H}^1(S^3-U_i;\mathbb{Z}_p)\to 0,$$

where $\varprojlim^{(1)}$ is the first derived functor of the functor of the inverse limit. Since $p\in\mathcal{P}$ it follows that the inverse limit $\varprojlim\check{H}^1(S^3-U_i;\mathbb{Z}_p)$ is trivial. The group $\varprojlim^{(1)}\check{H}^0(S^3-U_i;\mathbb{Z}_p)$ is trivial since the corresponding inverse sequence satisfies the Mittag-Leffler (ML) condition, so we have

$$\check{H}^1(S^3-X;\mathbb{Z}_p)\cong 0. \quad (7)$$

Analogously, it is easy to see that

$$\check{H}^1(S^3-X;\mathbb{Z}_q)\not\cong 0 \text{ for } q\notin\mathcal{P},\quad \check{H}^1(S^3-X;\mathbb{Z})\cong 0,\ \check{H}^1(S^3-X;\mathbb{Q})\not\cong 0. \quad (8)$$

Let $n=2$, then we have the Milnor–Harlap exact sequence for the presentation $S^3/X\backslash\{x\}=\bigcup_{i=1}^{\infty}(S^3-U_i)$:

$$0\to \varprojlim{}^{(1)}\check{H}^1(S^3-U_i;\mathbb{Z}_p)\to \check{H}^2(S^3-X;\mathbb{Z}_p)\to \varprojlim \check{H}^2(S^3-U_i;\mathbb{Z}_p)\to 0.$$

The groups $\varprojlim^{(1)}\check{H}^1(S^3-U_i;\mathbb{Z}_p)$ are trivial since the groups $\check{H}^1(S^3-U_i;\mathbb{Z}_p)$ are isomorphic to the finite group $\mathbb{Z}_p$ and the corresponding inverse sequence satisfies the ML condition. The groups $\check{H}^2(S^3-U_i;\mathbb{Z}_p)$ are also trivial since the "complementary" solid tori have the homotopy type of the circle. Therefore

$$\check{H}^2(S^3-X;\mathbb{Z}_p)\cong 0. \quad (9)$$

For $n=3$ we have the exact sequence of Milnor–Harlap for the same presentation of $S^3/X\backslash\{x\}$ as before:

$$0\to \varprojlim{}^{(1)}\check{H}^2(S^3-U_i;\mathbb{Z}_p)\to \check{H}^3(S^3-X;\mathbb{Z}_p)\to \varprojlim \check{H}^3(S^3-U_i;\mathbb{Z}_p)\to 0.$$

The groups $\check{H}^3(S^3-U_i;\mathbb{Z}_p)$ and $\check{H}^2(S^3-U_i;\mathbb{Z}_p)$ are trivial since the spaces S^3-U_i have the homotopy type of a circle. Therefore:

$$\check{H}^3(S^3-X;\mathbb{Z}_p)\cong 0. \quad (10)$$

Next, let us calculate the groups $\check{H}^n(S^3/X,S^3/X-X/X;G)$ for certain groups G.

Since the space S^3/X is connected and $\dim S^3/X = 3$ it follows that these groups are trivial groups for $n = 0, n > 3$.

Since the space $S^3 - X$ is connected and $\check{H}^1(S^3/X;\mathbb{Z}_p) \cong 0$ by (2), it follows by the exact cohomology sequence of the pair $(S^3/X, S^3/X - X/X)$ or the pair $S^3/X, S^3\backslash X$ $(S^3/X\backslash X/X = S^3\backslash X)$ that

$$\check{H}^1(S^3/X, S^3 - X;\mathbb{Z}_p) \cong 0. \tag{11}$$

By the exact sequence:

$$\check{H}^1(S^3 - X;\mathbb{Z}_p) \xrightarrow{\delta} \check{H}^2(S^3/X, S^3 - X;\mathbb{Z}_p) \longrightarrow \check{H}^2(S^3/X;\mathbb{Z}_p) \longrightarrow \check{H}^2(S^3 - X;\mathbb{Z}_p)$$

and since the groups $\check{H}^1(S^3 - X;\mathbb{Z}_p)$ and $\check{H}^2(S^3/X;\mathbb{Z}_p)$ are trivial by (7) and (4) it follows that

$$\check{H}^2(S^3/X, S^3 - X;\mathbb{Z}_p) \cong 0. \tag{12}$$

For any group of coefficients the corresponding homomorphism δ is a monomorphism by (3). Since the groups $\check{H}^1(S^3 - X;\mathbb{Z}_q)$ for $q \notin \mathcal{P}$, $\check{H}^1(S^3 - X;Q)$ are nontrivial by (8), and the groups $\check{H}^1(S^3/X;\mathbb{Z}_q), \check{H}^1(S^3/X;Q)$ are trivial if follows that

$$\check{H}^2(S^3/X, S^3 - X;\mathbb{Z}_q) \ncong 0, \check{H}^2(S^3/X, S^3 - X;\mathbb{Q}) \ncong 0. \tag{13}$$

Consider the following exact sequence of the pair $(S^3/X, S^3 - X;\mathbb{Z}_p)$:

$$\check{H}^2(S^3 - X;\mathbb{Z}_p) \xrightarrow{\delta} \check{H}^3(S^3/X, S^3 - X;\mathbb{Z}_p) \longrightarrow \check{H}^3(S^3/X;\mathbb{Z}_p) \longrightarrow \check{H}^3(S^3 - X;\mathbb{Z}_p).$$

The groups $\check{H}^2(S^3 - X;\mathbb{Z}_p)$ and $\check{H}^3(S^3 - X;\mathbb{Z}_p)$ are trivial by (9) and (10) respectively. Next, observe that $\check{H}^3(S^3/X;\mathbb{Z}_p) \cong \mathbb{Z}_p$ therefore

$$\check{H}^3(S^3/X, S^3 - X;\mathbb{Z}_p) \cong \mathbb{Z}_p. \tag{14}$$

Let us show that S^3/X is a clc_{Z_p} space at all points. Obviously, the space S^3/X is a clc_{Z_p} space for all points except at the point $x = X/X$ since $S^3\backslash X$ is an open manifold. As mentioned before, the sets $\{U_i/X\}_{i\in\mathbb{N}}$ form a neighborhood base of the point x. Consider the groups $\check{H}^n(U_i/X, X/X;\mathbb{Z}_p)$. By the Excision Axiom it follows that $\check{H}^n(U_i/X, X/X;\mathbb{Z}_p) \cong \check{H}^n(U_i, X;\mathbb{Z}_p)$.

From the following commutative diagram with exact rows

$$\begin{array}{ccccc} 0 \cong \check{H}^0(X;\mathbb{Z}_p) & \longrightarrow \check{H}^1(U_i, X;\mathbb{Z}_p) & \longrightarrow \check{H}^1(U_i;\mathbb{Z}_p) & \cong \mathbb{Z}_p \\ \downarrow & \downarrow \pi^i & & \downarrow \times n_i \\ 0 \cong \check{H}^0(X;\mathbb{Z}_p) & \longrightarrow \check{H}^1(U_{i+1}, X;\mathbb{Z}_p) & \longrightarrow \check{H}^1(U_{i+1};\mathbb{Z}_p) & \cong \mathbb{Z}_p \end{array}$$

it follows that for a large enough i, the homomorphism

$$\check{H}^1(U_i, X;\mathbb{Z}_p) \xrightarrow{\pi^i} \check{H}^1(U_{i+1}, X;\mathbb{Z}_p)$$

is trivial. Therefore

$$S^3/X \text{ is a } 1 - clc_{\mathbb{Z}_p}\text{space}. \tag{15}$$

By the analogous diagram for the group of coefficients $\mathbb{Z}$ it is easy to see that the homomorphism $\check{H}^1(U_i, X; \mathbb{Z}) \xrightarrow{\pi^i} \check{H}^1(U_{i+1}, X; \mathbb{Z})$ is a monomorphism of the group $\mathbb{Z}$. Therefore

$$S^3/X \text{ is not } 1 - clc_{\mathbb{Z}}. \tag{16}$$

By the exact sequence

$$\check{H}^1(X; \mathbb{Z}_p) \longrightarrow \check{H}^2(U_i, X; \mathbb{Z}_p) \longrightarrow \check{H}^2(U_i, \mathbb{Z}_p)$$

since $\check{H}^2(U_i, \mathbb{Z}_p) \cong 0$ and the Čech cohomology group $\check{H}^1(X; \mathbb{Z}_p)$ is obviously isomorphic to the direct limit of the sequence

$$\mathbb{Z}_p \overset{\times n_1}{\to} \mathbb{Z}_p \overset{\times n_2}{\to} \mathbb{Z}_p \overset{\times n_3}{\to} \cdots$$

it follows that $\check{H}^2(U_i, X; \mathbb{Z}_p) \cong 0$. By the Excision Axiom it follows that $\check{H}^2(U_i/X, X/X; \mathbb{Z}_p) \cong 0$ and

$$S^3/X \text{ is a } 2 - clc_{\mathbb{Z}_p} \text{space.} \tag{17}$$

By the following exact sequence of the pair (U_i, X)

$$\check{H}^2(X; \mathbb{Z}_p) \longrightarrow \check{H}^3(U_i, X; \mathbb{Z}_p) \longrightarrow \check{H}^3(U_i, \mathbb{Z}_p)$$

and since the space X is 1-dimensional and U_i is homotopy equivalent to the circle it follows that $\check{H}^3(U_i, X; \mathbb{Z}_p) \cong 0$ therefore $\check{H}^3(U_i/X, X/X; \mathbb{Z}_p) \cong 0$ and

$$S^3/X \text{ is a } 3 - clc_{\mathbb{Z}_p} \text{space.} \tag{18}$$

By the local connectedness of the space S^3/X, by (15), (17), (18) and since $\dim S^3/X = 3$ it follows that S^3/X is a $clc_{\mathbb{Z}_p}$ space and S^3/X satisfies the condition (2) of Definition 2.1.

By (11), (12), and (14) it follows that S^3/X satisfies the condition (3) of Definition 2.1, therefore S^3/X is a $cm_{\mathbb{Z}_p}$ and a $hm_{\mathbb{Z}_p}$ 3-manifold.

However, the space S^3/X is neither a $3 - cm_{\mathbb{Z}_q}$ nor a $3 - cm_{\mathbb{Q}}$ since $\check{H}^2(S^3/X, S^3 - X; \mathbb{Z}_q) \ncong 0$ and $\check{H}^2(S^3/X, S^3 - X; \mathbb{Q}) \ncong 0$ by (16), and is not a $3 - cm_{\mathbb{Z}}$ since it is not a $1 - clc_{\mathbb{Z}}$. This completes the proof. □

4. Epilogue

The spaces which we have constructed are not ANR's, so there is an interesting question:

Question 4.1. Let $\mathcal{P}$ be any set of prime numbers. Does there exist a 3-dimensional ANR X with the following properties:

(1) for every prime $p \in \mathcal{P}$, X is a 3-hm_p
(2) for every prime $q \notin \mathcal{P}$, X is not a 3-hm_q?

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