Motivation

Read §46. Most of these results will be presented in summary form.

Given a homotopy between maps f and $g : X \to Y$, we would like to know if there is a "path" of continuous functions from X to Y. That is, in $C(X,Y)$, is there a path joining f to g?

Topology of Pointwise Convergence

Def: For $x \in X$ and $U^{open} \subset Y$ $S(x,U) \equiv \{f \in Y^X | f(x) \in U\}$.The topology of *pointwise convergence* on Y^X is the topology having for a basis finite intersections of sets of this form.

Note: this is just the product topology.

Thm: f_n converges to f in Y^X with the top. of pointwise convergence iff for each x in X, $f_n(x)$ converges to $f(x)$ in Y.

Topology of Compact Convergence

Def. If (Y,d) is metric and X is a space, given $f \in Y^X$ and $C^{compact} \subset X$, $B_C(f,\varepsilon)$ consists of all g for which

$$\sup\{d(f(x),g(x)) | x \in C\} < \varepsilon$$

These sets form a basis for a topology on Y^X called the *topology of compact convergence*.

Thm: f_n converges to f in Y^X with the top. of compact convergence iff for each compact C in X, $f_n|_C$ converges uniformly to $f|_C$.

Comparison of Topologies

Thm: Let X be a space and (Y,d) be metric. For Y^X,

$$\text{(uniform)} \supset \text{(cpct. conv)} \supset \text{(ptwise conv.)}$$

If X is compact, the first two coincide.

If X is discrete, the last two coincide.

Compactly Generated Spaces

Def. X is compactly generated if a set U is open in X iff $U \cap C$ is open in C for each compact C in X.

Lemma: If X is locally compact or first countable, then X is compactly generated.

Lemma: If X is compactly generated, then $f : X \to Y$ is continuous iff for each compact C in X, $f|_C$ is continuous.

Thm: If X is compactly generated and (Y, d) is metric, then $C(X, Y)$ is closed in Y^X with the topology of compact conv.

Compact Open Topology

Def. Let X and Y be spaces. If $C^{\text{cpct}} \subset X$ and $U^{\text{open}} \subset Y$, let

$$S(C, U) = \{f \in C(X, Y) | f(C) \subset U\}$$

Note: Finite intersections of these sets form a basis for a topology on Y^X called the *compact open topology*.

Thm: If X is a space and (Y, d) is metric, then on $C(X, Y)$, the compact open topology and the topology of compact convergence coincide.

Evaluation Map e

Cor: If (Y, d) is metric, the compact convergence topology on $C(X, Y)$ does not depend on the specific metric d. So if X is cpct, the uniform topology on $C(X, Y)$ does not depend on the metric d.

Thm: Let X be locally cpct Hausdorff and let $C(X, Y)$ have the cpct open topology. Then $e : X \times C(X, Y) \to Y$ given by $e(x, f) = f(x)$ is continuous.

Homotopic maps and paths in $C(X, Y)$

Def: Given a continuous $f : X \times Z \to Y$, there is a $F : Z \to C(X, Y)$ given by $(F(z))(x) = f(x, z)$. Conversely, given F, this defines f. F is induced by f.

Thm: Let $C(X, Y)$ have the compact open topology. If $f : X \times Z \to Y$ is continuous, so is $F : Z \to C(X, Y)$. The converse holds if X is locally compact Hausdorff.

Thm: Let X be locally compact Hausdorff, Y be a space and let $C(X, Y)$ have the compact open topology. $H : X \times I \to Y$ is continuous if and only if the induced $F : I \to C(X, Y)$ is.