Complete Regularity

Read Section 38

Def. A regular space X is *completely regular* if for each point p of X and for each closed set A of X not containing p, there is a map $f : X \to [0,1]$ such that $f(p) = \{1\}$ and $f(A) = \{0\}$

Note: preserved by products and subspaces.

Theorem: A space X is completely regular iff it is homeomorphic to a subspace of $[0,1]^J$ for some J.

Stone-Čech Compactification

Def. A *compactification* of a space X is a compact Hausdorff space Y containing X such that $\overline{X} = Y$. Two compactifications Y_1 and Y_2 of X are equivalent if there is a homeomorphism $h : Y_1 \to Y_2$ such that $h(x) = x$ for each x in X.

Thm: Let X be completely regular. Then there exists a compactification $\beta(X)$ of X, called the Stone-Čech Compactification, having the property that every bounded map $f : X \to R$ extends uniquely to a continuous map of $\beta(X)$ into R.

Uniqueness of $\beta(X)$

Lemma: If $A \subset X$ and $f : A \to Y^{\text{Hausdorff}}$ is a map, f has at most one extension to a function from $\overline{A} \to Y$.

Thm: Let X be completely regular. Given any map $f : X \to C^{\text{Compact Hausdorff}}$, f extends uniquely to a map $g : \beta(X) \to C$.

Thm: Any two compactifications Y_1 and Y_2 as above are equivalent.

Note: $\beta(X)$ is maximal in the sense that any other compactification is a quotient space of it.