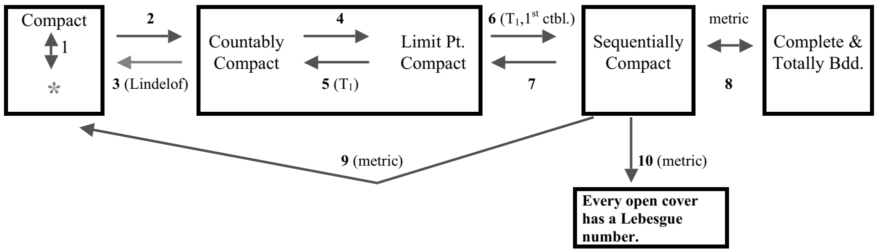
Diagram

Compactness Definitions and Implications



Compact: Every open cover has a finite subcover.

***:** Every collection of closed sets with the finite intersection property has nonempty intersection.

Finite Intersection Property: A collection of sets has this property if any finite intersection of the sets is nonempty.

Lindelof: Every open cover has a countable subcover.

Countably Compact: Every countable open cover has a finite subcover

Limit Point Compact: Every infinite set has a limit point.

Sequentially compact: Every sequence has a convergent subsequence.

Complete: Every Cauchy sequence converges.

Totally Bounded: For each ε>0 there is a finite covering by ε balls.

Lebesgue Number: ε>0 is a Lebesgue number of a cover if any set of diameter < ε is contained in some element of the cover.

Compactness Implications

1. X is compact if and only if every collection of closed subsets of X with the finite intersection property has nonempty intersection.

2. If X is compact then X is countably compact.

3. If X is countably compact and Lindelof, then X is compact.

4. If X is countably compact, then X is limit point compact.

5. If X is limit point compact and T_1, then X is countably compact.

Implications II

6. If X is limit point compact, T_1, and first countable, X is sequentially compact.

7. If X is sequentially compact, X is limit point compact.

8. A metric space (X, d) is sequentially compact if and only if it is complete and totally bounded.

10. If a metric space (X, d) is sequentially compact, then every open cover has an associated Lebesgue number.

Implications III

9. If a metric spaces (X, d) is sequentially compact, then it is compact.

Theorem: For a metric space (X, d), the following are equivalent: compactness, countable compactness, limit point compactness, sequential compactness, and the property of being complete and totally bounded.