Embedding Theorem

Def. A space is *separable* if it has a countable dense subset. Recall that in a metric space, this is equivalent to having a countable basis.

Embedding Theorem: Every separable metric space is homeomorphic to a subspace of the Hilbert Cube $\equiv I^\omega \equiv [0,1]^\omega$

(In fact, finite dimensional separable metric spaces are homeomorphic to subspaces of R^n for some n. We won't prove this.)

Other Major Theorems

Urysohn Metrization Theorem: Every regular space with a countable basis is metrizable.

Cantor Image Theorem: Every separable metric space is the continuous image of a subspace of the Cantor set.

Proof of embedding theorem:

Step 1. Given separable (X, d) , replace d by $\overline{d}$.

Step 2. Let $\{U_1, U_2, \cdots\}$ be a countable basis for X. Define $f_i : X \to [0,1]$ by $f_i(x) = \overline{d}(x, X \setminus U_i)$. Each f_i is continuous.

Step 3. Define $f : X \to I^\omega$ by
$$f(x) = (f_1(x), f_2(x), \ldots)$$
f is continuous since each f_i is.

Step 4: $f : X \to Z \equiv f(X)$ is 1-1, onto and continuous.

Proof Continued

Step 5: Check that f^{-1} is continuous, or equivalently, that f takes open sets in X to open sets in Z.

Conclusion: These five steps show that $f : X \to Z \equiv f(X)$ is a homeomorphism.