Compactness of Products

Read pages 60 – 67.

Tube Lemma: If Y is compact and U is an open set containing $\{x\} \times Y$ in $X \times Y$, there is an open set V containing x in X with $V \times Y \subset U$.

Theorem: If X and Y are compact, so is $X \times Y$ with the product topology.

Theorem: If $\{X_i \mid i = 1, 2, \cdots\}$ is a countable collection of compact spaces, then $X = \prod_{i=1}^{\infty} X_i$ with the product topology is also compact.

Examples:

Compactness in R^n

Theorem: **(Heine Borel Theorem)**
A closed interval in R is compact.

Theorem: A subspace of R^n is compact if and only if it is closed and bounded.

Examples:

Types of Compactness

Def.

- X is *countably compact* if every countable open cover of X has a finite subcover.
- X has the *Bolzano-Weierstrass Property* if every infinite subset of X has a limit point in X.
- X is *sequentially compact* if every sequence in X has a convergent subsequence.

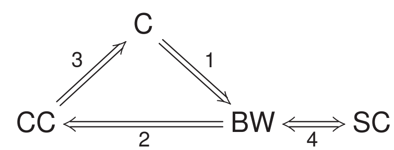
Examples:

Compactness in Metric Spaces

Theorem: For metric spaces, the following are equivalent:

- Compactness (C)
- Countable Compactness (CC)
- Bolzano Weierstrass Property (BW)
- Sequential Compactness (SC)

Proof:



Sequences in Metric Spaces

Def:

- A sequence (s_n)in X *converges* to p if for each open set U in X containing p, there is an $N > 0$ so that if $n \geq N$ then $s_n \in U$.
- A sequence (s_n) in a metric space (X, d) is *Cauchy* if for each $\varepsilon > 0$ there is an $N > 0$ so that if $n, m \geq N$ then $d(s_n, s_m) < \varepsilon$
- A metric space (X, d) is *totally bounded* if for each $\varepsilon > 0$ there is a finite cover of X by ε balls.
- A metric space (X, d) is *complete* if every Cauchy sequence in X converges.

Characterization in Metric Spaces

Theorem: If (X, d) is a compact metric space, then X is complete and totally bounded.

Note: The converse is also true.

Consequences of Compactness

Theorem: If f is a continuous function from a compact space X *onto* a space Y, then Y is compact.

Corollary: If f is a continuous *one-to-one* function from a compact space X *onto* a Hausdorff space Y, then f is a homeomorphism.

Corollary: If f is a continuous *real valued* function on a compact space X, then f achieves a maximum and minimum value.