Definition of a Topological Space

Read pages 35–42

Def. A *topology* on set X is collection τ of subsets of X satisfying:

- X and $\emptyset$ are in τ.
- Any union of sets in τ is in τ
- Any *finite* intersection sets in τ is in τ.

Note: The set X together with the collection τ of subsets is called a *topological space*. The subsets in the collection τ are called the *open sets* of the topological space.

Examples

- Any metric space (X, d) with τ the open sets in the metric space.
- Any set X with τ consisting of all subsets.
  This is called the *discrete* topology on X.
- Any set X with τ consisting only of X and $\emptyset$.
  This is called the *indiscrete* topology on X.

Other Examples:

Definitions

Let $A \subset X$, where X is a topological space.

A is *closed* in X if $X - A$ is open in X.

The *interior* of A in X, $\text{int}(A)$ or A°, is the union of all open sets of X that are contained in A.

The *exterior* of A in X, $\text{ext}(A)$, is the union of all open sets of X that are contained in $X \setminus A$.

The *frontier* or *boundary* of A in X, $\text{Bd}(A)$, is defined to be $\{x \in X | \forall U^{\text{open}}$ with $x \in U, U \cap A \neq \emptyset$ and $U \cap (X \setminus A) \neq \emptyset\}$

The *closure* of A in X, $\text{cl}(A)$ or $\overline{A}$, is defined to be $\text{int}(A) \cup \text{Bd}(A)$.

Examples:

Results about Closure and Interior

Let $A \subset X$, where X is a topological space.

- X is the disjoint union of $\text{int}(A)$, $\text{ext}(A)$, and $\text{Bd}(A)$
- $\overline{A}$ is closed in X
- if $A \subset B$, $\text{int}(A) \subset \text{int}(B)$ and $\text{ext}(B) \subset \text{ext}(A)$
- if $A \subset B$, $\overline{A} \subset \overline{B}$
- $\overline{A}$ is the intersection of all closed sets containing A.
- A is open if and only if $A = \text{int}(A)$
- A is closed if and only if $A = \overline{A}$
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$, $\quad \overline{A \cap B} \subset \overline{A} \cap \overline{B}$
- $\text{int}(A \cap B) = \text{int}A \cap \text{int}B$, $\quad \text{int}A \cup \text{int}B \subset \text{int}(A \cup B)$

Accumulation Points or Limit Points

Def. Let $A \subset X$, where X is a topological space. A point $p \in X$ is an *accumulation point* or *limit point* of A if every open set of X that contains p also contains a point of A *distinct* from p.

Def. Let $A \subset X$, where X is a topological space. The *derived set* of A, A', is the set of all accumulation points of A.

Theorem: $\overline{A} = A \cup A'$.

Note: This implies $A' \subset \overline{A}$.

Examples: