Section 3.4 - Uniform Continuity

Definition: Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$.

- f is uniformly continuous on E iff given $\varepsilon > 0$, $\exists \delta > 0$ so that

  $$\forall a \in E, \text{ if } x \in E \text{ and } |x-a| < \delta, \text{ then } |f(x)-f(a)| < \varepsilon$$

  Contrast with continuous on E:
  f is continuous on E iff
  $\forall a \in E$, given $\varepsilon > 0$, $\exists \delta > 0$ so that

  $$\text{if } x \in E \text{ and } |x-a| < \delta, \text{ then } |f(x)-f(a)| < \varepsilon$$

Relation to Cauchy condition

Lemma: If f is uniformly continuous on E, and $\{x_n\}$ is a Cauchy sequence in E, then $\{f(x_n)\}$ is also Cauchy.

Theorem: If $f : [a,b] \to \mathbb{R}$ is continuous, it is uniformly continuous.

Extending Functions

Theorem: A function $f : (a,b) \to \mathbb{R}$ is uniformly continuous if and only if it can be continuously extended to a function $g : [a,b] \to \mathbb{R}$.

That is , iff there is a continuous function $g : [a,b] \to \mathbb{R}$ so that $f(x) = g(x) \forall x \in (a,b)$.