Acceleration and Curvature

This section answers the following questions about a path $\mathbf{r}(t)$ traced out over time:

- How does the acceleration increases or decreases the speed and change the direction of the path?
- What is the curvature of the path?

Read Lesson 8 in the study guide

Read Section 11.9 in the text

Try 11, 13, 15, 17, 23, 25, 31, 33, 39

Velocity and Acceleration

Position: $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$

Velocity: $\mathbf{v}(t) = \mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$

Acceleration:
$\mathbf{a}(t) = \mathbf{r}''(t) = \langle f''(t), g''(t), h''(t)\rangle$

Speed:
$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$

Unit Tangent Vector: $\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$

Arc Length

Arc Length: $s(t) = \displaystyle\int_{t_0}^{t} |\mathbf{v}(u)|\, du$

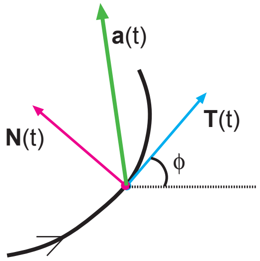
Note: $\dfrac{ds}{dt} = |\mathbf{v}(t)| = \text{speed}.$

Curvature:

Def. The *curvature* κ to $\mathbf{r}(t)$ at a point is $\left|\dfrac{d\mathbf{T}}{ds}\right|$.

Note: $\kappa(t) = \dfrac{\left|\frac{d\mathbf{T}}{dt}\right|}{\frac{ds}{dt}} = \dfrac{\left|\frac{d\mathbf{T}}{dt}\right|}{|\mathbf{v}(t)|} = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$

Curvature as change in angle:



Note: In the two dimensional case, the curvature is also equal to $\left|\dfrac{d\phi}{ds}\right|$

Other formulas for Curvature in R^2

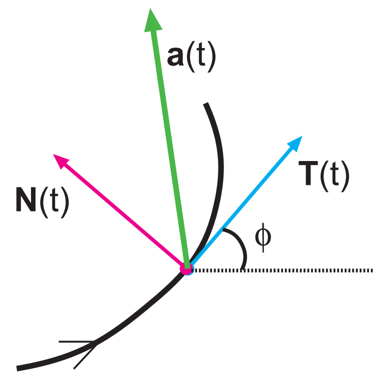
Theorem: If $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then

$$\kappa = \frac{|x'y'' - y'x''|}{|(x')^2 + (y')^2|^{(3/2)}}$$

So if the curve is given by $y = f(x)$, so that $\mathbf{r}(t) = \langle t, f(t) \rangle$, then

$$\kappa = \frac{|f''|}{|1 + (f')^2|^{(3/2)}}$$

Principal Unit Normal Vector



Def: The principal unit normal vector $\mathbf{N}(t)$ is defined to be

$$\mathbf{N}(t) = \frac{\frac{d\mathbf{T}}{ds}}{\left|\frac{d\mathbf{T}}{ds}\right|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$$

Note: $\mathbf{N}$ has length 1, and is perpendicular to $\mathbf{T}$

Relation to Acceleration:

Theorem: $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$

$$= \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \kappa\,\mathbf{N}$$

$$= \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|}\mathbf{T} + \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|}\mathbf{N}$$

Note: From the above, $\kappa = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$

Osculating Circle:

The osculating circle to $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$ is the circle through $\mathbf{r}(t_0)$ tangent to the curve with curvature κ.

Center: $\mathbf{r}(t_0) + \dfrac{1}{\kappa}\mathbf{N}(t_0)$ **Radius:** $\dfrac{1}{\kappa}$

Given $\mathbf{r}(t)$

To Find	Compute
$\mathbf{v}$	$\mathbf{r}'$
speed	$\lvert\mathbf{r}'\rvert$
$\mathbf{a}$	$\mathbf{r}''$
$\mathbf{T}$	$\dfrac{\mathbf{r}'}{\lvert\mathbf{r}'\rvert}$
$a_{\mathbf{T}}$	$\dfrac{\mathbf{r}'\cdot\mathbf{r}''}{\lvert\mathbf{r}'\rvert} = \frac{d^2s}{dt^2}$
$a_{\mathbf{N}}\mathbf{N}$	$\mathbf{a} - a_{\mathbf{T}}\mathbf{T}$

To Find	Compute
$a_{\mathbf{N}} = \left(\frac{ds}{dt}\right)^2 \kappa$	$\lvert a_{\mathbf{N}}\mathbf{N}\rvert = \dfrac{\lvert\mathbf{r}'\times\mathbf{r}''\rvert}{\lvert\mathbf{r}'\rvert}$
$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$	$a_{\mathbf{N}}\mathbf{N}/a_{\mathbf{N}}$
$\kappa = \dfrac{\lvert\mathbf{r}'\times\mathbf{r}''\rvert}{\lvert\mathbf{r}'\rvert}$	$a_{\mathbf{N}}/\lvert\mathbf{r}'\rvert^2$
$\kappa = \left\lvert\frac{d\mathbf{T}}{ds}\right\rvert = \left\lvert\frac{d\phi}{ds}\right\rvert$	$\frac{\lvert x'y''-y'x''\rvert}{\lvert(x')^2+(y')^2\rvert^{(3/2)}}$ $\frac{\lvert f''\rvert}{\lvert 1+(f')^2\rvert^{(3/2)}}$

Binormal Vector B

Def. The binormal vector $\mathbf{B}(t)$ is defined to be

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

Note: The vector $\mathbf{B}(t)$ has length 1, is perpendicular to $\mathbf{N}(t)$ and $\mathbf{T}(t)$. These three vectors can be used to provide a three dimensional coordinate system at a point on a curve.

Example: