Cross Products of Vectors

Read Lesson 4 in the study guide (11.4) in the text.

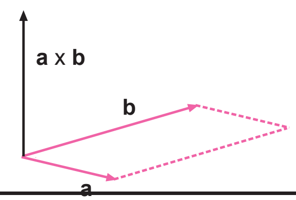
Finish Project One by Friday

Try 13–21 odd numbered, and 25, 27, 31, 35, 37, 39

Motivation

Motivation: Define a vector product so that:
$\mathbf{i}\times\mathbf{j}=\mathbf{k}\qquad \mathbf{j}\times\mathbf{k}=\mathbf{i}\qquad \mathbf{k}\times\mathbf{i}=\mathbf{j}$

Def: The *cross product* of vectors **a** and **b**, $\mathbf{a}\times\mathbf{b}$, is the vector perpendicular to both **a** and **b**, in a direction determined by the right hand rule, with magnitude $|\mathbf{a}||\mathbf{b}|\sin\theta$, where θ is the angle between **a** and **b**



Algebraic Definition

Alternate Definition: If $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$, $\mathbf{a}\times\mathbf{b}$, is

$$\langle a_2b_3-a_3b_2,\quad a_3b_1-a_1b_3,\quad a_1b_2-a_2b_1\rangle =$$

$$(a_2b_3-a_3b_2)\,\mathbf{i}-(a_1b_3-a_3b_1)\,\mathbf{j}+(a_1b_2-a_2b_1)\,\mathbf{k}$$

$$=\begin{vmatrix} a_2 & a_3\\ b_2 & b_3\end{vmatrix}\mathbf{i}-\begin{vmatrix} a_1 & a_3\\ b_1 & b_3\end{vmatrix}\mathbf{j}+\begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{k}$$

where $\begin{vmatrix} a & b\\ c & d\end{vmatrix}\equiv ad-bc$ **Note:** Result is a vector!

Computing using Determinants

Def.

$$\begin{vmatrix} a & b & c\\ d & e & f\\ g & h & i\end{vmatrix}\equiv a\begin{vmatrix} e & f\\ h & i\end{vmatrix}-b\begin{vmatrix} d & f\\ g & i\end{vmatrix}+c\begin{vmatrix} d & e\\ g & h\end{vmatrix}$$

So: $\langle a_1,a_2,a_3\rangle\times\langle b_1,b_2,b_3\rangle=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

Examples

Equivalence of Definitions

Theorem: The two definitions of cross product are the same.

Proof:

Corollary: $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $|\mathbf{a} \times \mathbf{b}| = 0$.

Commutativity, Associativity

Note:

$$\begin{array}{ll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{i} = -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

Note:

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} \neq \mathbf{i} \times (\mathbf{i} \times \mathbf{k})$$

Cross Product Properties

1. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ (Scalar triple product)
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$

Check:

Scalar Triple Product

Note: $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is the volume of the solid parallelipiped determined by the three vectors.