13.4 – Triple Integrals

Read Lesson 21 in the Study Guide and Section 13.4 in the text.
- triple integrals over rectangular regions
- triple integrals over general regions

Suggested Homework:

Try 7, 11, 15-37 odd numbered, 41

Definition of Triple Integral

Def: The triple integral of $f(x, y, z)$ over a rectangular box $B = [a, b] \times [c, d] \times [r, s]$ is

$$\iiint_B f(x, y, z)dz =$$

$$\lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_{ijk}$$

provided the limit exists.

Existence and Properties

Notes:

- The sum in the definition is a triple Riemann sum.
- If f is continuous, the triple integral exists and does not depend on the choice of $(x^*_{ijk}, y^*_{ijk}, y^*_{ijk})$
- Same properties as double integrals

Evaluating

Fubini's Theorem:
If $f(x, y, z)$ is continuous on $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_R f(x, y, z)dV = \int_r^s \left(\int_c^d \left(\int_a^b f(x, y, z)dx \right) dy \right) dz$$

Note: Also equal to iterated integrals in the other 5 orders.

Examples

$$\iiint_R f(x,y,z)dV$$ where $f(x,y,z) = x\sin(x+y+z)$ and where

$$R = [0,1] \times [0,2] \times [0,3]$$

General Regions

If E is a *bounded region*, contained in a box B, and $f(x,y,z)$ is defined on E, Then

$$\iiint_E f(x,y,z)dV = \iiint_B F(x,y,z)dV$$

where $F(x,y,z) = f(x,y,z)$ if $(x,y,z) \in E$ and $= 0$ otherwise.

Special 3-dimensional regions

Def: A z-simple region is a region $\mathbf{E} =$

$$\{(x,y,z)|(x,y) \in D, u_1(x,y) \leq z \leq u_2(x,y)\}$$

Other simple regions are similarly defined.

Evaluating on Special Regions

If f is continuous on a z-simple region,

$$\iiint_E f(x,y,z)dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right) dA$$

Similarly for other simple regions.

Examples

- Evaluate $\iiint_E f(x,y,z)dV$ where $f(x,y,z) = xyz$ and where E is the solid region bounded by the coordinate planes and $2x + y + 3z = 6$

More Examples

- Set up iterated integrals for the volume of the region bounded by the planes $z = 0$, $x = 0$, $y = 0$, $y = 1 - x$ and the surface $z = 1 - x^2$