§ 12.6–Directional Derivatives and the Gradient

This lesson covers the material in Section 12.6

Read Lesson 13 in the Study Guide and Section 12.6 in the text.

Continue working on online homework.

Try: 9-19 odd numbered, 23, 27, 33, 35, 37, 45-51 odd numbered

Directional Derivatives

Def: The directional derivative of f at $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(\mathbf{p}) = \lim_{h\to 0} \frac{f(\mathbf{p}+h\mathbf{u}) - f(\mathbf{p})}{h}$$

if this limit exists.

Theorem: Let f be differentiable at $\mathbf{p}$. Then f has directional derivatives in the direction of any unit vector $\mathbf{u} = \langle u_1, u_2\rangle$ and

$$D_{\mathbf{u}}f(\mathbf{p}) = \mathbf{u}\cdot\nabla f(\mathbf{p})$$

Gradient Vector

Def. The gradient of f at $\mathbf{p}$ is the vector

$$\nabla f(\mathbf{p}) = \langle f_x(\mathbf{p}), f_y(\mathbf{p})\rangle$$

Note f is differentiable at $\mathbf{p}$ if and only if

$$f(\mathbf{p}+\mathbf{h}) = f(\mathbf{p}) + \nabla f(\mathbf{p})\cdot\mathbf{h} + \epsilon(\mathbf{h})\cdot\mathbf{h}$$

where $\epsilon(\mathbf{h}) \to 0$ as $\mathbf{h}\to 0$.

Properties of Gradient Vectors

- $\nabla(f(\mathbf{p}) + g(\mathbf{p})) = \nabla f(\mathbf{p}) + \nabla g(\mathbf{p})$
- $\nabla(\alpha f(\mathbf{p})) = \alpha\nabla f(\mathbf{p})$
- $\nabla(f(\mathbf{p})g(\mathbf{p})) = f(\mathbf{p})\nabla g(\mathbf{p}) + g(\mathbf{p})\nabla f(\mathbf{p})$

Examples:

- $f(x,y) = \sin(xy)$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $\mathbf{p} = (\frac{\pi}{8}, 1)$
- $f(x,y) = x^3y + xy^2$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$, $\mathbf{p} = (1,2)$

Note: If $w = f(x,y,z)$, we can define the directional derivative of f at $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ as

$$D_{\mathbf{u}}f(\mathbf{p}) = \lim_{h\to 0} \frac{f(\mathbf{p}+h\mathbf{u}) - f(\mathbf{p})}{h}$$

if this limit exists, just as for functions of two variables.

Gradient in Three Dimensions

Def. The gradient of $f(x,y,z)$ at $\mathbf{p}$ is

$$\nabla f(\mathbf{p}) = \langle f_x(\mathbf{p}), f_y(\mathbf{p}), f_z(\mathbf{p})\rangle$$

Just as before,

Theorem: Let f be differentiable at $\mathbf{p}$. Then f has directional derivatives in the direction of any unit vector $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and

$$D_{\mathbf{u}}f(\mathbf{p}) = \mathbf{u}\cdot\nabla f(\mathbf{p})$$

Maximum Rate of Change:

Theorem: If f is a differentiable function of 2 or 3 variables the maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{p})$ is $|\nabla f(\mathbf{p})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{p})$

Reason:

Example:

Significance of gradient vector:

2-Dimensional Case: $f(x,y)$

- $\mathrm{D}_{\mathbf{u}}f(x,y) = \mathbf{u}\cdot\nabla f(x,y)$
- *Maximum value* of $\mathrm{D}_{\mathbf{u}}f(x,y)$ is $|\nabla f(x,y)|$ and occurs when $\mathbf{u}$ is in the direction of $\nabla f(x,y)$. So the gradient vector points in the direction of maximum increase of the function.
- For a level curve $f(x,y) = c$, the equation of the tangent line to the curve at (a,b) is $\nabla f(a,b)\cdot\langle x-a, y-b\rangle = 0$
- The gradient vector is perpendicular to the level curves.

3-Dimensional Case: $F(x, y, z)$

- $D_{\mathbf{u}}F(x, y, z) = \mathbf{u} \cdot \nabla \mathbf{F}(x, y, z)$
- *Maximum value* of $D_{\mathbf{u}}F(x, y, z)$ is $|\nabla F(x, y, z)|$ and occurs when $\mathbf{u}$ is in the direction of $\nabla F(x, y, z)$. So the gradient vector points in the direction of maximum increase of the function.
- For a level surface $F(x, y, z) = k$, the equation of the tangent plane to the surface at (a, b, c) is $\nabla f(a, b, c) \cdot \langle x - a, y - b, z - c\rangle = 0$.
- The gradient vector is perpendicular to the level surfaces.

Special Case:

$z = f(x, y)$ or equivalently
$F(x, y, z) = f(x, y) - z = 0$

In this case, $\nabla F = \langle f_x, f_y, -1\rangle$ and the equation of the tangent plane to the surface at $(a, b, f(a, b))$ is

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$$

or

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

as before.

Summary

THEOREM 12.11 Directions of Change

Let f be differentiable at (a, b).

1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of increase in this direction is $|\nabla f(a, b)|$.

2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of decrease in this direction is $-|\nabla f(a, b)|$.

3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.