Section 12.5–The Chain Rule

This lesson covers the material in Section 12.5 on the chain rule.

Read Lesson 13 in the Study Guide and Section 12.5 in the text.

Continue working on online homework.

Try: 7-21 odd numbered, 27, 33

The Chain Rule, I

The Chain Rule, Case 1:

If $f(x, y)$ is a function of two variables x and y, and if $x = x(t)$ and $y = y(t)$ are functions of a third variable t, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

Why this works -
Write f as a function of t and differentiate.

Diagram

$$\begin{tikzcd} & f \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ t & & t \end{tikzcd}$$

Chain Rule, II:

If f is a function of x and y, and x and y are functions of u and t, then

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\begin{tikzcd} & & f \arrow[dl] \arrow[dr] & & \\ & x \arrow[dl] \arrow[d] & & y \arrow[d] \arrow[dr] & \\ u & t & & u & t \end{tikzcd}$$

Why this works:

Chain Rule, General Version:

Example $f(x,y) = x^2\sin(y)$

$x(u,t) = u^2t^3 \qquad y(u,t) = ut - u^3$

General Version If f is a function of $x_1, x_2, \cdots x_n$ and each of $x_1, x_2, \cdots, x_n$ is a function of $u_1, u_2, \cdots, u_m$, then

$$\frac{\partial f}{\partial u_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial u_i}$$

Examples:

Implicit Differentiation:

An equation of the form $F(x,y) = 0$ **implicitly** defines y as a function of x near a point (a,b)

if

the partial derivatives of F are continuous on a disc containing (a,b) and if $\frac{\partial F}{\partial y} \neq 0$ on this disc.

In this case -

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Surfaces:

$F(x,y,z) = 0$ **implicitly** defines z as a function of x and y near a point (a,b,c)

if

the partial derivatives of F are continuous on an open ball containing (a,b,c) and if $\frac{\partial F}{\partial z} \neq 0$ there.

In this case -

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Examples:

$x^2 + y^2 - 4 = 0$

$F(x,y,z) = x^2 + 3y^2 - 2z^2 = 4$