The Topology of Gaussian and Eisenstein-Jacobi Interconnection Networks

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***Abstract*—Earlier authors have used quotient rings of Gaussian and Eisenstein-Jacobi integers to construct interconnection networks with good topological properties. In this article we present a unified study of these two types of networks. Our results include decomposing the edges into disjoint Hamiltonian cycles, a simplification of the calculation of the Eisenstein-Jacobi distance, a distribution of the distances between Eisenstein-Jacobi nodes, and shortest path routing algorithms. In particular, the known Gaussian routing algorithm is simplified.**

***Index Terms*—interconnection network, Gaussian integers, Eisenstein-Jacobi integers, routing in networks, diameter of a network**

I. Introduction

In parallel systems and multi-core systems, a number of processors are packed in a single VLSI chip and the interconnection topology of the system plays an important role in its performance. In the past (for example, in [1], [2], [8], [22]), many parallel systems were designed using the 2D and 3D toroidal topology. The topological properties of these toroidal networks can be studied relative to the Lee distance, a metric used in coding theory [7].

In [17], [18] the authors introduced two new classes of toroidal networks, called Gaussian and Eisenstein-Jacobi networks. (The term EJ network is often used for the latter, a convention we follow here.) These are two-dimensional networks whose nodes are labeled using a subset of the complex numbers, and in [17] it is proved that the class of regular 2D toroidal networks is a subclass of the Gaussian networks.

For any two positive integers a, b, the Gaussian networks are degree 4 symmetric networks with $a^2 + b^2$ nodes, and the EJ networks are symmetric degree 6 networks with $a^2 + b^2 + ab$ nodes. The diameter of Gaussian networks was calculated in [17, Theorems 10, 11], and it is much less than toroidal networks with same number of nodes. For example, when there is a Gaussian network with 200 nodes with diameter 10, while any two dimensional toroidal network with 200 nodes will have a diameter of at least 15.

There are many similarities between the Gaussian and EJ cases. We have tried to emphasize these connections when possible since the EJ networks are not well-understood compared to the significant work on Gaussian interconnection networks (for example, [17], [19], [20]). One reason for the scarcity of EJ results is that distance in EJ networks is more difficult to compute from the definition since the degree of each vertex is 6. Theorem 16 gives a simplification of this calculation that allows us to obtain properties of EJ networks similar to those already known for Gaussian networks [17], [20], including determining the upper bound on the EJ diameter (in Theorem 23(c)) and the distribution of EJ distances (in Theorem 27). Moreover, the more general point of view yields a simplification of the Gaussian shortest-path routing given in [17] that also applies to the EJ case.

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The paper is organized as follows. Section II reviews the necessary algebra as well as the interconnection topology of these networks, and the edge-disjoint Hamiltonian cycle decomposition for some of these networks (two cycles for Gaussian and three for EJ) is given in Section III. Section IV contains a study of the Gaussian and EJ distance functions, and our simplification of the calculation of the EJ distance as well as determining the upper bound on the diameter of EJ networks is given in Section V. The shortest-path routing algorithm is described in Section VI, and in Section VII we analyze the distance distribution of the EJ networks. The paper ends with some concluding remarks and a description of some open research problems in Section VIII.

II. The Interconnection Networks

In this section we describe the Gaussian and EJ networks in a way that capitalizes on their similarly. For this, we use σ to denote either i or $\rho := (1 + i\sqrt{3})/2$. For either choice of σ, the set $\mathbb{Z}[\sigma] := \{x + y\sigma \ : \ x, y \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$ since $i^2 = -1$ and $\rho^2 = -1 + \rho$. The rings $\mathbb{Z}[i], \mathbb{Z}[\rho]$ are respectively called the *Gaussian integers* and the *EJ integers*, and quotient rings of these $\mathbb{Z}[\sigma]$ will give the nodes of our networks.

The authors of earlier papers (for instance, [13], [18]) used $(-1 + i\sqrt{3})/2$ instead of our choice of ρ. Although the network for $a + b(-1 + i\sqrt{3})/2$ has $a^2 + b^2 - ab$ nodes and is different from ours for $a + b\rho$, the total set of possible networks that occur is the same regardless of choice. Our choice of $\rho = (1 + i\sqrt{3})/2$ allows us to demystify the calculation of EJ distance in Theorem 16.

In what follows the term *unit* of a ring will be used to refer to elements whose multiplicative inverses lie in the ring. For each choice of $\sigma = i, \rho$, the units in $\mathbb{Z}[\sigma]$ are the powers of σ. (Refer to [11, Section 14.4].)

For our purposes, it is often helpful to view complex numbers as points in $\mathbb{R}^2$, and for fixed nonzero $\alpha = a + b\sigma \in \mathbb{Z}[\sigma]$ define the half-open rhombus $\mathcal{P}_\alpha$ (with side-length $|\alpha|$) by

$$\mathcal{P}_\alpha = \{x\alpha + y\sigma\alpha \ : \ 0 \le x, y < 1\} \ . \tag{1}$$

In the Gaussian case, $\mathcal{P}_\alpha$ is a square of area $|\alpha|^2$, while in the EJ case, the area of $\mathcal{P}_\alpha$ equals $|\alpha|^2\cos(\pi/3) = |\alpha|^2\sqrt{3}/2$ since the acute angle is $\pi/3$ radians. The Gaussian rhombus for $a = 5+2i$ and the EJ rhombus for $\alpha = 3+2\rho$ are given in Figures 1 and 2, respectively. These have been defined in (1) in such a way that copies of $\mathcal{P}_\alpha$ tile the plane with no overlapping, and the translates used in the tiling are the Gaussian or EJ integer multiples of α.

For instance, Figure 1 illustrates four Gaussian tiles that are formed from the basic Gaussian rhombus for $\alpha = 5+2i$. Calling the basic rhombus the NE tile, the NW tile is the translate by $-\alpha$; the SW by $(-1-i)\alpha$, and the SE tile is translated by $-i\alpha$. Similarly, it can be checked that the tiles given in counterclockwise order from the basic EJ rhombus for $\alpha = 3+2\rho$ are the translates by $-\alpha, (-1-\rho)\alpha, -\rho\alpha$.

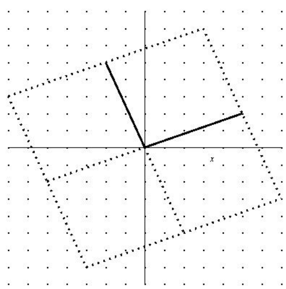

Fig. 1. The Gaussian rhombus $\mathcal{P}_\alpha$ for $\alpha = 5+2i$.

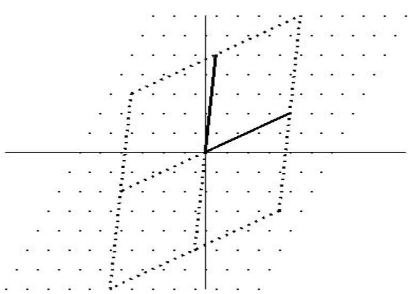

Fig. 2. The EJ rhombus $\mathcal{P}_\alpha$ for $\alpha = 3+2\rho$.

Definition 1. Let $\alpha \in \mathbb{Z}[\sigma]$ be nonzero. Then $w_1, w_2 \in \mathbb{Z}[\sigma]$ are *congruent modulo* α if there exists $\gamma \in \mathbb{Z}[\sigma]$ such that $w_2 - w_1 = \gamma\alpha$. Congruence is denoted by $w_2 \equiv w_1 \pmod{\alpha}$.

Returning to the tiling of $\mathbb{R}^2$ by translates of $\mathcal{P}_\alpha$, since the translates used are multiples of α, the tiling respects congruence modulo α in the sense that congruent points lie in the same position in their respective tiles. For example, for $\alpha = 5+2i$,

$$-2-5i = (1+2i) - (3+7i) = (1+2i) - (1+i)\alpha \ ,$$

and $-2-5i$ and $1+2i$ lie in the same relative position in their respective tiles (refer to Figure 1). Similarly, for $\alpha = 3+2\rho$,

$$3 - 3\rho + \rho\alpha = 3 + 2\rho^2 = 1 + 2\rho$$

since $\rho^2 = \rho - 1$, and so $1 + 2\rho = 3 - 3\rho \bmod \alpha$.

Since by definition (1), $\mathcal{P}_\alpha$ has no boundary repetitions and $\mathbb{R}^2$ is tiled by the set of translates of $\mathcal{P}_\alpha$ by multiples of α, the elements of $\mathbb{Z}[\sigma]$ within the basic rhombus $\mathcal{P}_\alpha$ therefore form a complete system of representatives modulo α. We will denote the representative of w that lies in $\mathcal{P}_\alpha$ as $w \bmod \alpha$.

This choice of residue system is a standard one, and was already used in [10] for the Gaussian case. In [14], other Gaussian residue systems are given, including the "Utah" one used in [17], [19]. Huber's [13], [12] choice of representatives is different, since it is designed to minimize the complex modulus for both the EJ and Gaussian cases. For example, for $\alpha = 5+2i \in \mathbb{Z}[i]$ and $w = 3+2i$, Huber would divide w by α (as complex numbers)

$$\frac{3+2i}{5+2i} = \frac{(3+2i)(3-2i)}{|\alpha|^2} = \frac{19}{29} + \frac{4}{29}i$$

and then choose u, v to be the nearest integers to $19/29, 4/29$, respectively, yielding the representative $w - (u+vi)\alpha = -2$. (This is the division algorithm in [11, Section 12.8].)

As already noted, only one vertex of a fundamental rhombus is included in each tile (the bottom left corner) and it can be written in the form

$$u\alpha + v\sigma\alpha = (u+v\sigma)\alpha \text{ for some } u, v \in \mathbb{Z} \ ; \qquad (2)$$

that is, $w \in \mathbb{Z}[\sigma]$ is in the tile given by the vertex (2) if and only if $w - \gamma\alpha \in \mathcal{P}_\alpha$ for $\gamma = u + v\sigma \in \mathbb{Z}[\sigma]$.

Example 2. For $\alpha = 5+2i$, the tile containing $w = -2-2i$ can be located by solving $(-2,-2) = r(5,2) + s(-2,5)$ for r, s. Writing this as a matrix equation, we have

$$\begin{bmatrix} 5 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \ ,$$

which gives $r = -14/29, s = -6/29$. Since $-1 \le r < 0$ and $-1 \le s < 0$, w lies in the tile labeled by the vertex $(-1-i)\alpha$ and $w \bmod \alpha = w - (-1-i)\alpha = 1+5i$.

The procedure given in this example works in general since the linear independence of $\{\alpha, \alpha\sigma\}$ implies every $w \in \mathbb{Z}[\sigma]$ has unique $r, s \in \mathbb{R}$ such that $w = r\alpha + s\alpha\sigma$. Setting $c = a$ when $\sigma = i$ and $c = a + b$ when $\sigma = \rho$, we observe that the matrix $A := \begin{bmatrix} a & -b \\ b & c \end{bmatrix}$ is the change of basis matrix from $\{\alpha, \alpha\sigma\}$ to $\{1, \sigma\}$; that is, $w = r\alpha + s\alpha\sigma = w_1 + w_2\sigma$ are related by the matrix equation

$$A \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \ . \qquad (3)$$

For each choice of σ, $\det(A) = |\alpha|^2 \neq 0$ and so by Cramer's Rule there exist integers m, n such that $r = m/|\alpha|^2$ and $s = n/|\alpha|^2$.

Definition 3. Let nonzero $\alpha = a + b\sigma \in \mathbb{Z}[\sigma]$ be fixed. For $w \in \mathbb{Z}[\sigma]$, if m, n are integers such that $|\alpha|^2 w = m\alpha + n\alpha\sigma$ then m, n will be called the *coordinates* of w for α. We will denote this by $w = \langle m, n \rangle$.

Example 4. In Example 2, the coordinates of $w = -2-2i$ for $\alpha = 5+2i$ were found to be $\langle -14, -6 \rangle$. This can be verified by checking:

$$\begin{bmatrix} 5 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -14 \\ -6 \end{bmatrix} = \begin{bmatrix} -58 \\ -58 \end{bmatrix} = \det(A) \begin{bmatrix} -2 \\ -2 \end{bmatrix} \ ,$$

Next we find the coordinates of $-2-2\rho$ relative to $\alpha = 5+3\rho$. If $-2-2\rho = \langle m, n\rangle$ then $m = 49r, n = 49s$ where

$$\begin{bmatrix}5 & -3\\ 3 & 8\end{bmatrix}\begin{bmatrix}r\\ s\end{bmatrix} = \begin{bmatrix}-2\\ -2\end{bmatrix} \ ,$$

and by Cramer's Rule,

$$m = \det\begin{bmatrix}-2 & -3\\ -2 & 8\end{bmatrix} = -22 \quad ; \quad n = \det\begin{bmatrix}5 & -2\\ 3 & -2\end{bmatrix} = -4 \ ;$$

$-2-2\rho = \langle -22, -4\rangle$.

Note that $\alpha = \langle |\alpha|^2, 0\rangle$ and $\alpha\sigma = \langle 0, |\alpha|^2\rangle$. Also, by definition of $\mathcal{P}_\alpha$,

$$\langle m, n\rangle \in \mathcal{P}_\alpha \iff 0 \le m, n < |\alpha|^2 \ . \qquad (4)$$

Definition 5. For nonzero $\alpha = a + b\sigma \in \mathbb{Z}[\sigma]$, consider the associated (Gaussian or EJ) network whose nodes are the congruence classes $w \bmod \alpha$, and $v \bmod \alpha$ is adjacent to $w \bmod \alpha$ if and only if there exists j such that $v - w \equiv \sigma^j \pmod{\alpha}$. The nodes are labeled with their coordinates for α and the edge from $v \bmod \alpha$ to $w \bmod \alpha$ is labeled σ^j.

In the Gaussian case, the network is toroidal if $a = 0$ or $b = 0$. Moreover, for any nonzero $\alpha \in \mathbb{Z}[\sigma]$, the mapping $v \bmod \alpha \mapsto v\sigma \bmod \alpha\sigma$ is a bijection between the nodes of the networks for α and $\alpha\sigma$ which is a graph isomorphism. It what follows, we refer to the corners of the rhombus $\mathcal{P}_\alpha$ as its *vertices*. Notice that among the four vertices of $\mathcal{P}_\alpha$, only $(0,0)$ is a node of the network.

Returning to the definition of $\mathcal{P}_\alpha$ in (1), all points $x\alpha + y\sigma\alpha$ with $0 \le x < 1$ and $y = 1$ have been excluded from $\mathcal{P}_\alpha$. Since $x\alpha + \sigma\alpha \equiv x\alpha \pmod{\alpha}$, the interconnection is such that points on the excluded side $w = x\alpha + \sigma\alpha$ are wrapped onto points on the included side corresponding to α. Similarly, an element $\alpha + y\alpha\sigma$ on the side parallel to $\alpha\sigma$ is congruent to $y\alpha\sigma$.

For example, for $\alpha = 3+2\rho$ (as in Figure 2), the node $v = 1+6\rho$ is connected to the six nodes of the form $\beta_j := v + \rho^j$ for $j = 0, 1, \ldots, 5$, corresponding to the six EJ directions from v. We note that $\beta_1 \equiv 0 \bmod \alpha$, and β_j is itself a network node for $j = 3, 4$. The nodes corresponding to β_0, β_5 are found by wrapping around the right edge of the rhombus $\mathcal{P}_\alpha$; that is,

$$\beta_0 \bmod \alpha = \beta_0 - \alpha = (2+6\rho) - (3+2\rho) = -1+4\rho \ ;$$

$$\beta_5 \bmod \alpha = (1+6\rho-\rho^2) - (3+2\rho) = -1+3\rho \ ,$$

since $\rho^2 = \rho - 1$. Similarly, $\beta_2 \bmod \alpha$ is found by wrapping around the top side of $\mathcal{P}_\alpha$:

$$\beta_2 \bmod \alpha = \beta_2 - \rho\alpha = (1+6\rho+\rho^2) - (-2+5\rho) = 2+2\rho \ .$$

The simplicity of this wraparound is one of the reasons for our choice of these representatives.

The number of nodes in quotient rings is well-known (for instance [20, Theorems 4, 17]). In Theorem 7 we give a more geometric proof of this fact that is based on the following result.

Theorem 6 (Pick's Theorem)**.** For any convex polygon, let I be the number of integer points strictly interior to the polygon and B be the number of integer points on the boundary. If every vertex of the polygon has integer coordinates, then the area of the polygon equals $I + B/2 - 1$.

Proof: Refer to [21]. ∎

Theorem 7. For any nonzero $\alpha \in \mathbb{Z}[\sigma]$ there are $|\alpha|^2$ congruence classes modulo α. In particular, the Gaussian network corresponding to $a+bi$ has a^2+b^2 nodes while the EJ network for $a+b\rho$ has a^2+b^2+ab nodes.

Proof: We use Pick's Theorem on the closure of $\mathcal{P}_\alpha$, which has B boundary points and I interior points.

For $\sigma = i$, the points on the boundary of the closed polygon that are in $\mathcal{P}_\alpha$ are: one vertex and the integer points interior to two sides, and each side has the same number of integer points. Therefore, the total number of congruence classes is: $I + (B-4)/2 + 1 = I + B/2 - 1$, which by Pick's Theorem equals the area of the square with side $|\alpha|$. Therefore, the Gaussian network for $a+bi$ has $|\alpha|^2 = a^2+b^2$ nodes.

For $\sigma = \rho$, the number of congruence classes is the number of EJ integers $c + d\rho$ within $\mathcal{P}_\alpha$. In order to calculate this, consider the matrix B of the linear transformation taking $(1,0) \mapsto (1,0)$ and $(1/2, \sqrt{3}/2) \mapsto (0,1)$, with $\det(B) = 2/\sqrt{3}$. This matrix has the property that the EJ integers $c + d\rho \in \mathcal{P}_\alpha$ are exactly the integer points in $B\mathcal{P}_\alpha$ where $\text{area}(B\mathcal{P}_\alpha) = \det(B)\cdot\text{area}(\mathcal{P}_\alpha)$. Since $\text{area}(\mathcal{P}_\alpha) = |\alpha|^2\sqrt{3}/2$, then $\text{area}(B\mathcal{P}_\alpha) = |\alpha|^2$, and by Pick's Theorem the number of congruence classes modulo α again equals $|\alpha|^2$. Since

$$a + b\rho = a + b\Big(\frac{1+i\sqrt{3}}{2}\Big) = \Big(a + \frac{b}{2}\Big) + ib\frac{\sqrt{3}}{2}$$

then

$$|\alpha|^2 = \Big(a + \frac{b}{2}\Big)^2 + \frac{3b^2}{4} = a^2 + ab + b^2 \ .$$

∎

Example 8. The Gaussian network for $\alpha = 1+i$ has two nodes and so one nonzero node. Since $\mathcal{P}_\alpha$ is a square of side $\sqrt{2}$ whose diagonal has endpoints 0 and $2i$, the nonzero node is i. For $\alpha = 1+\rho$ there are three vertices in the corresponding EJ network. Since α is at an angle of $\pi/6$, the diagonal of $\mathcal{P}_\alpha$ is at an angle of $\pi/3$, and each of $\rho, 2\rho$ lies in $\mathcal{P}_\alpha$. Therefore, the three distinct congruence classes are: $0, \rho, 2\rho$.

III. The Hamiltonian decomposition

Many efficient communication algorithms in parallel networks have been designed using edge disjoint Hamiltonian cycles. For example, the all-to-all broadcasting algorithm, where each node broadcasts a message to all other nodes [4], [6], [9], [15], [16]. In a single I/O port model, a node can receive at most one message per unit time and must receive messages from all other nodes. The problem can be solved optimally by first generating a Hamiltonian cycle in the network. If the network were decomposed into edge disjoint Hamiltonian cycles, the algorithm can be extended to a multiple I/O ports model in which a node can send or receive from all n of its neighbors in unit time. The time taken by this algorithm is $O(Nmt_s/n)$ (where N is the number of nodes, m is the message size, and t_s is the transfer time per word) and so it is optimal.

In this section we show the edges of our interconnection networks can be decomposed into edge disjoint Hamiltonian cycles. For the Gaussian case, the two Hamiltonian cycles were identified in [3, pp. 113-135], and this is also discussed in [17, Section 4]. One of the open problems showcased in [5] was to "construct networks with a good diameter which can be decomposed into Hamiltonian cycles."

Example 9. For $\alpha = 3+\rho$, consider the path beginning at 0 and successively adding $\rho \bmod \alpha$. This is depicted in Figure 3. From the picture we see that ρ lies in $\mathcal{P}_\alpha$, and this can be verified by calculating the coordinates of ρ for α. For this, we first solve

$$\begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ,$$

and by Cramer's Rule

$$r = \frac{\det \begin{bmatrix} 0 & -1 \\ 1 & 4 \end{bmatrix}}{13} = \frac{1}{13} \quad \text{and} \quad s = \frac{\det \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}}{13} = \frac{3}{13} ;$$

$\rho = \langle 1,3 \rangle$. Therefore, the sequence of path vertices begins with $\langle 0,0\rangle, \langle 1,3\rangle, \langle 2,6\rangle, \langle 3,9\rangle, \langle 4,12\rangle$. The next vertex $5\rho = \langle 5,15\rangle$ is outside $\mathcal{P}_\alpha$ since $15 > \det(A)$. From its coordinates we know it is in the tile with vertex $\alpha\rho = \langle 0,13\rangle$, and

$$5\rho - \alpha\rho = \langle 5,15\rangle - \langle 0,13\rangle = \langle 5,2\rangle \in \mathcal{P}_\alpha .$$

The path continues from $\langle 5,2\rangle$ with $\langle 6,5\rangle, \langle 7,8\rangle, \langle 8,11\rangle, \langle 9,1\rangle, \langle 10,4\rangle, \langle 11,7\rangle, \langle 12,10\rangle, \langle 0,0\rangle$. It can be checked that this is a Hamiltonian cycle.

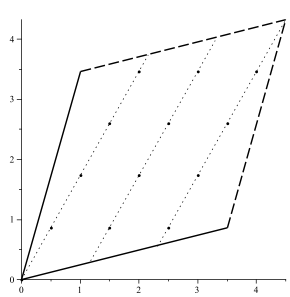

Fig. 3. The northeast Hamiltonian cycle for $\alpha = 3+\rho$.

The next theorem shows this example is not unusual: For any unit σ^j the path obtained by successively adding $\sigma^j \bmod \alpha$ is always a Hamiltonian cycle.

Theorem 10. If nonzero $\alpha = a + b\sigma \in \mathbb{Z}[\sigma]$ is not a unit and $\gcd(a,b) = 1$ then the edges of the network modulo α can be partitioned into disjoint Hamiltonian cycles.

Proof: Since the network is regular of degree 4 or 6, it suffices to describe two disjoint Hamiltonian cycles in the Gaussian case and three disjoint Hamiltonian cycles in the EJ case.

For any fixed unit ϵ, begin at the node 0 and traverse the path modeled by successively adding $\epsilon \bmod \alpha$. Since there are finitely many nodes, the path must eventually cycle and so there exists a node w and an integer $k > 0$ such that $w + k\epsilon \equiv w \pmod{\alpha}$. But then $k \cdot 1 \equiv 0 \pmod{\alpha}$; that is, the path is itself a cycle whose length is the additive order of 1. The cycle for $-\epsilon$ is the same as the cycle for ϵ; for units ϵ_1, ϵ_2 with $\epsilon_1 \neq \pm\epsilon_2$ the cycles cannot have a common edge since at every vertex the edge labeled ϵ_1 is different from the edge labeled $\pm\epsilon_2$. Therefore, it suffices to prove the cycles are Hamiltonian, by showing the order of 1 equals $|\alpha|^2$, which is the number of nodes in the network.

If k is any integer such that $k \equiv 0 \pmod{\alpha}$, there exists $x + y\sigma \in \mathbb{Z}[\sigma]$ such that

$$k = (a+b\sigma)(x+y\sigma) = ax + (bx+ay)\sigma + by\sigma^2 . \quad (5)$$

Since k is an integer, its imaginary part is zero. For the Gaussian case, the imaginary part of k is $bx+ay$ and so (x,y) is an integer solution of $bx+ay=0$. Since $\gcd(a,b)=1$, this means there exists an integer c such that $(x,y) = c(a,-b)$. So,

$$k = (a+bi)(x+yi) = c(a+bi)(1-bi) = c(a^2+b^2) ;$$

$k \geq a^2+b^2 = |\alpha|^2$, and the cycle is Hamiltonian. For the EJ case, $\sigma^2 = -1+\sigma$ and (5) give

$$k = (ax-by) + (bx+ay+by)\sigma$$

and $bx + (a+b)y = 0$ with $\gcd(a, a+b) = \gcd(a,b) = 1$. Therefore, there exists an integer $c \geq 1$ such that $(x,y) = c(a+b,-b)$;

$$k = c(a+b\rho)(a+b-b\rho) = c(a^2+ab+b^2) \geq a^2+ab+b^2 = |\alpha|^2 ,$$

proving the EJ cycle is Hamiltonian. ■

Example 11. The Gaussian network for $\alpha = 3+2i$ has 13 nodes. Here we calculate its "east" cycle, modeled by adding $1 \bmod \alpha$. From Property (4), the negative second coordinate in $1 = \langle 3,-2\rangle$ indicates $1 \notin \mathcal{P}_\alpha$. Also, $1 \equiv 1+\alpha i \bmod \alpha$ where $1+\alpha i = \langle 3,11\rangle$, and moreover the successor of any $\langle x,y\rangle$ is $\langle x,y\rangle + \langle 3,-2\rangle \bmod \alpha$. The cycle is: $\langle 0,0\rangle, \langle 3,11\rangle, \langle 6,9\rangle, \langle 9,7\rangle, \langle 12,5\rangle, \langle 2,3\rangle, \langle 5,1\rangle, \langle 8,12\rangle, \langle 11,10\rangle, \langle 1,8\rangle, \langle 4,6\rangle, \langle 7,4\rangle, \langle 10,2\rangle, \langle 0,0\rangle$.

The east Hamiltonian cycle in the EJ network for $\alpha = 3+\rho$ can be found using $1 = \langle 4,-1\rangle$. The cycle is: $\langle 0,0\rangle, \langle 4,12\rangle, \langle 8,11\rangle, \langle 12,10\rangle, \langle 3,9\rangle, \langle 7,8\rangle, \langle 11,7\rangle, \langle 2,6\rangle, \langle 6,5\rangle, \langle 10,4\rangle, \langle 1,3\rangle, \langle 5,2\rangle, \langle 9,1\rangle, \langle 0,0\rangle$.

IV. Taxicab norms

In order to understand the calculation of distance in these networks, it is helpful to define two norms on the complex numbers. For $w \in \mathbb{C}$, its *i-taxicab norm* is

$$\|w\|_i = |x| + |y| \quad \text{where } w = x + iy , \quad (6)$$

and its *ρ-taxicab norm* is

$$\|w\|_\rho := \min\{|x|+|y|+|z| \ : \ w = x + y\rho + z\rho^2\} . \quad (7)$$

The i-taxicab norm is often called the *Manhattan norm*, or simply the *taxicab norm.*

As we have already said, an understanding of these metrics as defined on all of $\mathbb{C}$ is necessary for computing distance in the interconnection networks. In particular, we are interested

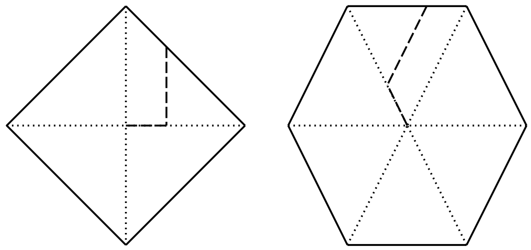

Fig. 4. Examples of circles in the i-taxicab and ρ-taxicab metric.

in recognizing "circles" for the taxicab metrics, that is, the set of all points taxicab-equidistant (say at a radius of r) from a fixed central point. The left picture in Figure 4 is the graph of a circle in the i-taxicab metric. To see this is true, first note that the two points at a horizontal/vertical Euclidean distance of $+r$ are also at a distance of r in the i-taxicab metric and so are on the circle. Since the slope of the line between these points equals 1, all points on the line segment between them are also at a distance of r from the center. This argument works for each quadrant, and the circle is as pictured. In the case of the ρ-taxicab metric, all six points on the rays through $\pm 1, \pm\rho, \pm\rho^2$ at a Euclidean distance of r are on the circle. Since the line segments are either horizontal or have slope $\pm\sqrt{3}$, we obtain a ρ-taxicab circle as pictured in the right of Figure 4. For instance, every point w on the top horizontal segment can be written as $w = x\rho + y\rho^2$ for $0 \le x, y < r$ with $x + y = r$ and so w can be obtained by proceeding a distance of y along the ray ρ^2 and then a distance of x along the ray ρ.

Considering the lattice of elements of $\mathbb{Z}[\sigma]$ as nodes in an infinite graph where the edges connect $v \in \mathbb{Z}[\sigma]$ with all $v + \sigma^j$, the value of $\|v - w\|_\sigma$ is the length of the shortest path between v and w. Accordingly, in the network for α, the distance between $v, w \in \mathcal{P}_\alpha$ equals the value of the norm of $v - w$ defined as follows.

Definition 12. For any node v in a Gaussian network, the *Gaussian norm* is

$$\|v\|_G = \min\{\|v - \gamma\alpha\|_i : \gamma \in \mathbb{Z}[i]\}\ . \tag{8}$$

The *EJ norm* of a node v in an EJ network is

$$\|v\|_E = \min\{\|v - \gamma\alpha\|_\rho : \gamma \in \mathbb{Z}[\rho]\}\ . \tag{9}$$

When the type of network is not specified, we will denote the norm by $\|v\|_Q$.

As commented earlier, these norms are different from Huber's [13], [12] who chose representatives by minimizing the complex modulus. Both [10], [20] contain a proof that the norms in (8), (9) induce a distance function on the associated network, and that the function induced by Huber's norm does not satisfy the triangle inequality.

In this section we develop other properties of the taxicab norms in preparation for our simplification of the norms (see Theorem 23) and our determination of the EJ diameter in Section VI.

Calculating the ρ-taxicab norm is complicated by the fact that each of $\{1, \rho\}$, $\{1, \rho^2\}$, $\{\rho, \rho^2\}$ is an $\mathbb{R}$-basis for $\mathbb{C}$, and so the representation of $w \in \mathbb{C}$ as $w = x + y\rho + z\rho^2$ is not unique. For example, using $\rho^2 = -1 + \rho$,

$$-1 + 3\rho = 2\rho + \rho^2 = 1 + \rho + 2\rho^2 \text{ etc.}$$

Although Gaussian networks have been extensively studied, the EJ networks have received considerably less attention, possibly because calculating the EJ norm seems difficult. Theorem 16 in combination with Algorithm 17 can be used to calculate the ρ-taxicab norm. For this calculation it is important to use or definition of ρ rather than the one used in earlier papers.

We first observe that multiplying any $w \in \mathbb{C}$ by i rotates w about the origin by $\pi/2$, and so moves it to the next counter-clockwise quadrant. Analogously, multiplying any element of $\mathbb{Z}[\rho]$ by the unit ρ^j rotates the point counterclockwise about the origin by an angle of $j\pi/3$ radians. This motivates the following definition.

Definition 13. For $\sigma = i, \rho$ and $w \in \mathbb{C}$, w is in the *j-th sector for σ* if the angle w makes with the positive x-axis is between the angles for σ^{j-1} and σ^j.

In the Gaussian case, the signs of x, y indicate the sector of $x + iy$ relative to $\sigma = i$. Since there are six sectors for $\sigma = \rho$, the sectors relative to ρ cannot be identified by four sign choices. Rather, we have the following alternate characterization:

Theorem 14. Let $\sigma = i, \rho$ and nonzero $w \in \mathbb{C}$. Then w is in the j-th sector for σ if and only if there exist $x, y \ge 0$ such that $w = x\sigma^{j-1} + y\sigma^j$.

Proof: The result holds for $\sigma = i$. To prove the result for $\sigma = \rho$, it suffices to prove it for w in the first sector, since multiplying w by ρ^k rotates w from the first sector to the k-th sector. If $w = x + y\rho$ with $x, y \ge 0$ then w corresponds to the endpoint of the vector obtained from adding a vector on the ray 1 to a vector on the ray ρ. By the parallelogram law w must lie between these two rays and so by definition is in the first sector for ρ. On the other hand, if $x + y\rho$ is in the first sector, the argument can be reversed to give $x, y \ge 0$. ■

Example 15. We see that $w = 1 + i$ is in the first sector for $\sigma = i$, and that $-1 + 3\rho = 2\rho + \rho^2$ is in the second sector for ρ.

Theorem 16. Let $\sigma = i, \rho$, and $w \in \mathbb{C}$. If w is in the j-th sector for σ then $\|w\|_\sigma = x + y$ where $w = x\sigma^{j-1} + y\sigma^j$ and $x, y \ge 0$.

Proof: The result holds for $\sigma = i$. Considering w in the first sector for ρ, we write $w = w_1 + w_2\rho$ where $w_1, w_2 \ge 0$. Let $x, y, z \in \mathbb{R}$ be any other choice with $w = x + y\rho + z\rho^2$. Since $\rho^2 = -1 + \rho$, $w = (x - z) + (y + z)\rho$ and by linear independence, $x - z = w_1$ and $y + z = w_2$. Therefore, $x + y = w_1 + w_2$ and so

$$|x| + |y| + |z| \ge x + y + |z| = w_1 + w_2 + |z| \ge w_1 + w_2\ .$$

This proves $\|w\|_\sigma = w_1 + w_2$. ■

If w is a node in an EJ network and given in the form $w = w_1 + w_2\rho$, how do we find the linear combination $w = x\rho^{j-1} + y\rho^j$ with $x, y \geq 0$? The following algorithm answers this question.

Algorithm 17. INPUT: $w = w_1 + w_2\rho \in \mathbb{Z}[\rho]$.
OUTPUT: j and $x, y \geq 0$ such that $w = x\rho^{j-1} + y\rho^j$.

1) If w_1, w_2 have different signs, go to Step 2.
   $x = |w_1|, y = |w_2|$
   If $w_1 > 0$ then $j = 1$ else $j = 4$.
2) If $|w_1| < |w_2|$ go to Step 3.
   $x = |w_2|, y = |w_1| - |w_2|$
   If $w_2 > 0$ then $j = 3$ else $j = 6$.
3) $x = |w_2| - |w_1|, y = |w_1|$
   If $w_2 > 0$ then $j = 2$ else $j = 5$.

Proof: Since all choices of x, y are non-negative, it suffices to prove $w = x\rho^{j-1} + y\rho^j$ for the given values of j.

First consider the case when $w_2 \geq 0$. If $w_1 \geq 0$ then $w = |w_1| + |w_2|\rho$, and $j = 1$. When $w_1 < 0$, then

$$w = -|w_1| + |w_2|\rho \ . \tag{10}$$

We consider the two cases given by the sign of $|w_1| - |w_2|$. If this is positive, then (10) can be rewritten as

$$w = -|w_1| + |w_2|(1 + \rho^2) = |w_2|\rho^2 + (|w_1| - |w_2|)\rho^3 \ .$$

On the other hand, if $|w_1| - |w_2| < 0$ then from (10)

$$w = |w_1|(-\rho + \rho^2) + |w_2|\rho = (|w_2| - |w_1|)\rho + |w_1|\rho^2 \ ,$$

in the second sector.

When $w_2 < 0$ then

$$\rho^3 w = -|w_1| + |w_2|\rho$$

satisfies the previous case. Using $x, y \geq 0$ and j such that $\rho^3 w = x\rho^{j-1} + y\rho^j$ and $\rho^6 = 1$, we obtain $w = x\rho^{k-1} + y\rho^k$ for $k = j + 3$, as required. ■

Example 18. For example, to find $\|-1 + 4\rho\|_\rho$, we use Algorithm 17 with $w_1 = -1, w_2 = 4$. The third alternative applies, giving $x = |w_2| - |w_1| = 3$ and $y = |w_1| = 1$ and $\|-1 + 4\rho\|_\rho = 1 + 3 = 4$.

The properties of taxicab norms in the following result will be used in the next section to determine the diameter of EJ networks.

Proposition 19. Let $\alpha = a + b\sigma \in \mathbb{Z}[\sigma]$ with $|a| \neq |b|$ and $ab \neq 0$.

(a) For all nonzero $\gamma \in \mathbb{Z}[\sigma]$, $\|\gamma\alpha\|_\sigma \geq \|\alpha\|_\sigma$.
(b) $\|\gamma\alpha\|_\sigma = \|\alpha\|_\sigma \iff \gamma$ is a unit.
(c) Let $w \in \mathbb{Z}[\sigma]$ with $\|w\|_\sigma \leq \max\{|a|, |b|\}$. If $\gamma \in \mathbb{Z}[\sigma]$ such that $\|w - \gamma\alpha\|_\sigma < \|w\|_\sigma$ then γ is a unit.

Proof: Refer to Appendix A. ■

The following is a useful application of Proposition 19.

Theorem 20. Let $\alpha = a + b\sigma \in \mathbb{Z}[\sigma]$ with $|a| \neq |b|$. and $w \in \mathbb{Z}[\sigma]$. If $\gamma_0 \in \mathbb{Z}[\sigma]$ such that $\|w - \gamma_0\alpha\|_\sigma \leq \|\alpha\|_\sigma/2$ then $\|w\|_Q = \|w - \gamma_0\alpha\|_\sigma$.

Proof: By Proposition 19(a), for any $\gamma \neq \gamma_0$ in $\mathbb{Z}[\sigma]$, $\|(\gamma - \gamma_0)\alpha\|_\sigma \geq \|\alpha\|_\sigma$, and so

$$\|\alpha\|_\sigma \leq \|\gamma\alpha - \gamma_0\alpha\|_\sigma \leq \|\gamma\alpha - w\|_\sigma + \|w - \gamma_0\alpha\|_\sigma \ ,$$

by the triangle inequality. Therefore,

$$\|\gamma\alpha - w\|_\sigma \geq \|\alpha\|_\sigma - \|w - \gamma_0\alpha\|_\sigma \geq \|\alpha\|_\sigma/2$$

by hypothesis and so $\|\gamma\alpha - w\|_\sigma \geq \|\gamma_0\alpha - w\|_\sigma$. ■

Example 21. For $w := 2 + 5\rho$, the value of $\|w\|_E$ in the EJ network for $\alpha := 4 + 3\rho$ can be easily calculated, since for $\gamma_0 = 1$,

$$w - \gamma_0\alpha = (2 + 5\rho) - (4 + 3\rho) = -2 + 2\rho = 2\rho^2$$

gives

$$\|w - \gamma_0\alpha\|_\rho = 2 < 3.5 = \|\alpha\|_\rho/2$$

and so $\|w\|_E = 2$.

Earlier we commented that the networks for α and $\sigma\alpha$ are isomorphic under the mapping $v \mapsto \sigma v$. This allows us to reduce consideration to α in the first sector. Also, the transformation $x + y\sigma \mapsto y + x\sigma$ is a bijection from nodes in the network of $a + b\sigma$ to nodes of the network of $b + a\sigma$ in which edges labeled ± 1 correspond to edges labeled $\pm\sigma$ (and conversely), and in the EJ case, any edge of the form $\pm\rho^2$ corresponds to $\mp\rho^2$. The network relative to $a + b\sigma$ is therefore isomorphic to that relative to $b + a\sigma$, and without loss of generality we may assume $\alpha = a + b\sigma$ with $a \geq b \geq 0$.

V. Calculating distance in the networks

In this section we show that the distance from any node w to the node 0 can be obtained by finding the minimum distance from w to the vertices $0, \alpha, \sigma\alpha, (\sigma + 1)\alpha$; that is, we prove that in each of (8), (9) the minimum is attained at a vertex $\gamma\alpha$ of $\mathcal{P}_\alpha$. For Gaussian networks, this was proved in Proposition 1 of [10] for prime $|\alpha|^2$. In contrast, [17] has 13 candidates that must be tested. (Refer to Lemma 15 and Algorithm 2 there.)

Since the set of nodes in the networks under consideration is closed under subtraction, the length of the shortest path between v, w equals $\|(v - w) \bmod \alpha\|_Q$, and is calculated by comparing the distance from $v - w$ to each of the four vertices of the rhombus $\mathcal{P}_\alpha$.

We next describe a useful canonical decomposition of $\mathcal{P}_\alpha$. First, at each vertex $\gamma\alpha$ form the block consisting of all $w \in \mathcal{P}_\alpha$ such that $\|w - \gamma\alpha\|_\sigma \leq \|\alpha\|_\sigma/2$. In the Gaussian case (refer to the first picture in Figure 5), the boundary of each such block is part of the i-circle of radius $\|\alpha\|_\sigma/2$ about the vertex $\gamma\alpha$. (Refer back to the left picture in Figure 4.) Each of these is a square and the remaining points in $\mathcal{P}_\alpha$ lie in a "central" square whose diagonals are parallel to the coordinate axes. In the EJ case (refer to the second picture in Figure 5), each block is part of a hexagon about a vertex (as in the right picture in Figure 4), and the remaining points lie in one of two triangles. The pictures given in Figure 5 are typical under the assumption that $a \geq b \geq 0$. For our purposes, the central square in the Gaussian decomposition is further divided into the four congruent triangles formed by its

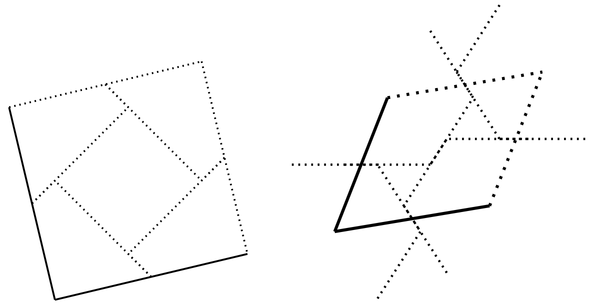

Fig. 5. Examples of Gaussian and EJ decompositions of $\mathcal{P}_\alpha$.

diagonals, and among these triangles the one closest to 0 will be referred to as the *special triangle*. Setting $T := (a+b)/2$ and $S := (a-b)/2$, the vertices of the lower triangle in the EJ decomposition are

$$\alpha - T \ , \ T\rho \ , \ \alpha\rho - T\rho^2 \ ,$$

and these are

$$\alpha - T = S + b\rho \ , \ T\rho \ , \ \alpha\rho - T\rho^2 = S + (a-S)\rho \ .$$

It can be checked that it is Euclidean-equilateral with side length S and centroid

$$S + b\rho + \frac{2S}{3}(\rho - \frac{1}{2}) = \frac{a-b}{3} + \frac{a+2b}{3}\rho \ . \qquad (11)$$

Therefore, the lines from the centroid to its vertices decompose it into three congruent triangles, and the left one will be called the *EJ special triangle*. Examples of special triangles are given in Figure 6.

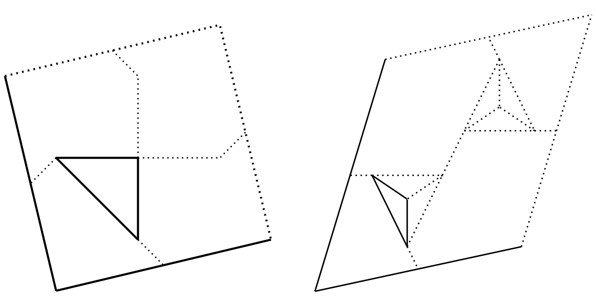

Fig. 6. Examples of Gaussian and EJ special triangles in $\mathcal{P}_\alpha$.

Lemma 22. Let $\alpha = a + b\sigma$ with $a \geq b \geq 0$ and $w = w_1 + w_2\sigma \in \mathcal{P}_\alpha$. Then

(a) w is σ-closer to 0 than any other vertex of $\mathcal{P}_\alpha$ if and only if either $\|w\|_\sigma \leq \|\alpha\|_\sigma/2$ or w is in the special triangle.

(b) If $w = w_1 + w_2\sigma$ is in the special triangle, then $\|w\|_\sigma = w_1 + w_2 \leq a$.

(c) In the special triangle, the σ-furthest point from 0 is the center of the Gaussian rhombus and the centroid of the EJ lower triangle.

Proof: Set $T := (a+b)/2$ and $S := (a-b)/2$. By Theorem 20, $\|w\|_\sigma \leq T$ implies 0 is the closest vertex, and so to prove part (a) it suffices to show that the points in the special triangle are the closest points to 0 with $\|w\|_\sigma \leq T$.

By construction, the vertices of the Gaussian special triangle are: $Ti, S+Ti, \alpha-T = S+bi$ (written in clockwise order) and its sides are parallel to the coordinate axes by which i-distance is measured. Points on the upper side are i-equidistant from 0 and αi, and the common distance increases from T to a as the side is traversed from left to right. Therefore, w inside the special triangle are i-closer to 0 than they are to αi and $\|w\|_i \leq a$. Similarly, points on the right side of the special triangle are i-closer to 0 than to α. The i-furthest point from 0 is $S + Ti$ (the center of $\mathcal{P}_\alpha$) with $\|S + Ti\|_i = S + T = a$, and every point in the special triangle lies in the first sector.

Inspecting the EJ special triangle, its top-left vertex $w = T\rho$ is ρ-equidistant from 0 and $\alpha\rho$. Since the angle of its top-right side is $-\pi/6$ radians, every point on the side is also ρ-equidistant from 0 and $\alpha\rho$, with the common distance increasing from T to $(2a+b)/3$. Points below this side are closer to 0 than to $\alpha\rho$. A similar argument shows the points in the special triangle are closer to 0 than to α. Among all points in the triangle, the vertex $S + (a-S)\rho$ is ρ-furthest from 0. ■

Using the Utah representation the authors of [17] reduce the calculation of $\|\beta\|_G$ to finding the minimum over thirteen values of $\delta \equiv \beta \pmod{\alpha}$. We have simplified this further by showing it is enough to check only the four values $\beta - j\alpha$ for $j = 0, 1, i, 1+i$. For example, the Gaussian norm of the node $\beta = 1 + 3i$ in the Gaussian graph for $\alpha = 5 + 2i$ (refer to Figure 1) is found by comparing $\|\beta - 0\|_i = 4$, $\|\beta - \alpha\|_i = \|-4+i\|_i = 5$, $\|\beta - i\alpha\|_i = \|3 - 2i\|_i = 5$, and $\|\beta - (1+i)\alpha\|_i = \|-2-4i\|_i = 6$. This gives $\|1+3i\|_G = 4$.

Theorem 23. Let $\alpha = a + b\sigma \in \mathbb{Z}[\sigma]$ be nonzero with $a \geq b \geq 0$.

(a) If $w \in \mathcal{P}_\alpha$ then

$$\|w\|_Q = \min\{\|w - \gamma\alpha\|_\sigma : \gamma = 0, 1, \sigma, 1+\sigma\} \ .$$

(b) The diameter of the Gaussian network is at most a which is attained if and only if $a \equiv b \pmod 2$.

(c) The diameter of the EJ network is at most $(2a+b)/3$ which is attained if and only if $a \equiv b \pmod 3$.

The diameter of Gaussian networks has already been found in [17], [20].

Proof: Once part (a) is proved, the other parts would follow from Lemma 22(c) since the center of the Gaussian rhombus is a Gaussian integer if and only if $a \equiv b \pmod 2$ and the centroid of the lower EJ triangle is an EJ integer if and only if $a \equiv b \pmod 3$.

By Theorem 20 and symmetry it suffices to consider w lying within the special triangle for $\mathcal{P}_\alpha$, and by Lemma 22, such $w = w_1 + w_2\sigma$ satisfies $w_1 + w_2 = \|w\|_\sigma \leq a$. If $\gamma \in \mathbb{Z}[\sigma]$ such that $\|w\|_Q = \|w - \gamma\alpha\|_\sigma$ then by Proposition 19(c) either $\gamma = 0$ or γ must be a unit. It remains to exclude $\gamma = \pm\rho^2$ in the EJ case as well as $\gamma = -1, -\sigma$ in both cases. We will use

$$a \geq \|w\|_\sigma = w_1 + w_2 \geq \frac{a+b}{2} \geq b \ . \qquad (12)$$

First of all, since $w + \alpha\sigma$ is in the first sector, its norm is at least $w_1 + a \geq \|w\|_\sigma$. Likewise, $\|w + \alpha\sigma\|_\sigma \geq w_2 + a$ follows immediately unless $\sigma = \rho$ and $w_2 - b < 0$. In that case we can use the fact that $\alpha\rho$ is in the second sector to get $w + \alpha\rho = (w_1 - b) + (w_2 + a)\rho$ is in the second sector, and

Algorithm 17 again gives $\|w+\alpha\sigma\|_\sigma \geq w_2 + a$. It remains to consider $\gamma = \pm\rho^2$ in the EJ case. Since

$$w + \rho^2\alpha = (w_1 + w_2 - b)\rho + (a + b - w_1)\rho^2$$

is a linear combination with positive coefficients,

$$\|w + \rho^2\alpha\|_\rho = (w_1 + w_2 - b) + (a + b - w_1) = a + w_2 \geq a \ .$$

Also, $w - \rho^2\alpha$ can be written as

$$(w_1 + b + a) + (w_2 - a)\rho = (a - w_2)(-\rho^2) + (w_1 + w_2 + b)$$

and so is in either the first or sixth sector depending on the sign of $a - w_2$. In either case $\|w - \rho^2\alpha\|_\rho \geq \|w\|_\rho$. ■

VI. Shortest-Path Routing

The properties developed in the last section allow us to find the shortest-path routing between any two nodes. First we give an example that describes the basic idea.

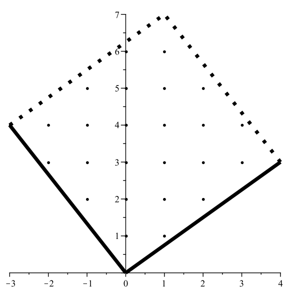

Fig. 7. The Gaussian network for $\alpha = 4 + 3i$.

Example 24. Referring to Figure 7, let $v := \langle 11, 23\rangle$, $w := \langle 20, 10\rangle$. Since $\det(A) = |4 + 3i|^2 = 25$, these are the points

$$v = \frac{11}{25}\alpha + \frac{23}{25}i\alpha \quad \text{and} \quad w = \frac{20}{25}\alpha + \frac{10}{25}i\alpha \ .$$

Each of v, w is a node in the network. To find the shortest path from v to w, we calculate the shortest path from 0 to $u := w - v \bmod \alpha$ and then translate by v. Here

$$u = \langle 20, 10\rangle - \langle 11, 23\rangle \equiv \langle 9, -13 + 25\rangle = \langle 9, 12\rangle \bmod \alpha$$

is the corresponding node label for u.

The next step is to calculate $\|u\|_G$ by writing u in standard complex coordinates and then calculating the distance from u to each vertex of the fundamental square. For this,

$$\frac{1}{25}\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

gives $u = 3i$. Since the Gaussian distance from 0 to u is $\|3i\|_i = 3 < 3.5 = \|\alpha\|/2$, by Theorem 20 0 is the closest vertex and $\|u\|_G = 3$. This norm calculation contains additional information, since $u = 3i$ says the path of length 3 that begins at v and follows the three edges corresponding to i ends at w: since $i = \langle 3, 4\rangle$ the path is

$$v = \langle 11, 23\rangle, \langle 14, 2\rangle, \langle 17, 6\rangle, \langle 20, 10\rangle = w \ .$$

Algorithm 25 (Shortest paths)**.** Let $\alpha = a + b\sigma$.
INPUT: Two nodes v, w in the network for α.
OUTPUT: A shortest path from v to w.

1) Find $u := w - v \bmod \alpha$ and transfer to standard complex coordinates.
2) Calculate $\|u - \gamma\alpha\|_\sigma$ for $\gamma = 0, 1, 1 + \sigma, \sigma$, and let γ_0 give the minimal value. In this process, if $\|u - \gamma\alpha\|_\sigma \leq \|\alpha\|/2$ we have identified $\gamma_0 = \gamma$.
3) Find $j, x, y \geq 0$ such that
$$u - \gamma_0\alpha \bmod \alpha = x\sigma^{j-1} + y\sigma^j$$
where Algorithm 17 is used when $\sigma = \rho$.
4) A shortest path begins at v and uses x copies of σ^{j-1} and y copies of σ^j.

Since $u \in \mathcal{P}_\alpha$, by Theorem 23(a) at most four values need to be checked in Step 2. This contrasts with the 13 values required in the earlier shortest path algorithm for Gaussian networks in [17, Algorithm 2].

Example 26. For $\alpha := 4 + 3\rho$, we find a shortest path from $v := \langle 18, 24\rangle$ to $w := \langle 10, 1\rangle$. Since $\det(A) = |4+3\rho|^2 = 37$, each is a node in the network. We set $u := w - v \bmod \alpha = \langle 29, 14\rangle$, where

$$\frac{1}{37}\begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 29 \\ 14 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2 + 5\rho \ .$$

In Example 21 we already calculated $\|2 + 5\rho\|_E = 2$ from $u - \alpha = 2\rho^2$. Since $1 = \langle 7, -3\rangle$ and $\rho = \langle 3, 4\rangle$ then $\rho^2 = -1 + \rho = \langle -4, 7\rangle$ and the path is

$$v = \langle 18, 24\rangle, \langle 14, 31\rangle, \langle 10, 1\rangle = w \ .$$

VII. The Distribution of norms in EJ networks

Since the Gaussian norm distribution has already been given in [17, Theorems 10, 11] and [20, Theorems 6, 7], we only consider the EJ case but we note that the argument given here also works for the Gaussian case.

Theorem 27. Let $\alpha = a + b\rho$ be nonzero with $a \geq b \geq 0$ and $T := (a + b)/2$ and $M := (2a + b)/3$. For any positive integer t, let $W(t)$ be the number of nodes w in the EJ network for α with $\|w\|_E = t$. Then

$$W(t) = \begin{cases} 1 & \text{if } t = 0 \\ 6t & \text{if } 1 \leq t < T \\ 18(M - t) & \text{if } T < t < M \\ 2 & \text{if } a \equiv b \pmod 3 \text{ and } t = M \\ 0 & \text{if } t > M \end{cases} .$$

The value of $W(T)$ depends on whether T, M are integers; that is, it depends on the value of $a - b \pmod 6$. The value of $W(T)$ can be found by subtracting the sum of the weights already listed from the total number of nodes $a^2 + b^2 + ab$.

Proof: By Theorem 23(c), the diameter is at most M. Also, M is attained if and only if $a \equiv b \pmod 3$ in which case $W(M) = 2$, one for the centroid of each triangle in the EJ decomposition. This gives the conclusion for all $t \geq M$.

If the region of ρ-closeness about each vertex of $\mathcal{P}_\alpha$ (from Theorem 23(a)) is translated to 0, the resulting set is the ρ-circle $\|w\|_\rho = T$ with center 0 and radius T plus six additional triangles appended to its sides. This is illustrated in Figure 8, where the dotted lines indicate the translated regions of ρ-closeness.

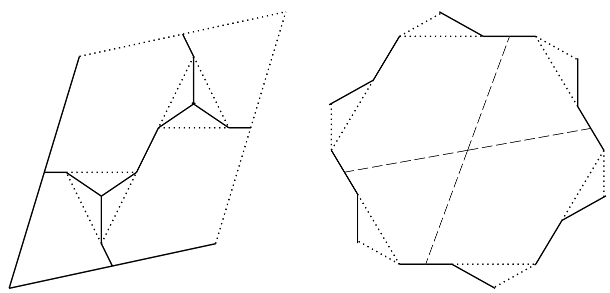

Fig. 8. A basic EJ decomposition and its translation to 0.

The first step of our analysis is to count the number of EJ integers w that satisfy $0 < \|w\|_\rho < T$ by noting that exactly one-sixth of them lie in the first sector (where the ray at an angle of $\pi/3$ radians is excluded). This is the triangle with vertices $0, T, T\rho$ as illustrated in Figure 9. Each complex

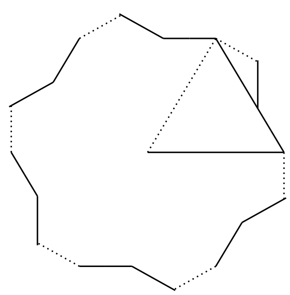

Fig. 9. The norm-count within the hexagon.

number w within this triangle lies on a line segment from t to $t\rho$, and this can be written as $w = t - j\rho^2$ for $0 \leq j < t$. When t is an integer,

$$w = t - j\rho^2 \in \mathbb{Z}[\rho] \iff j \in \mathbb{Z},$$

and so there are exactly t EJ integers on this line segment. From this we obtain $W(t) = 6t$, as required.

We next calculate $W(t)$ for $T < t < M$ by returning to the special triangle, and to prevent over-counting only its vertical side is included. Denoting the lower vertex of the special triangle by $V = \alpha - T$ and its upper left vertex by $U = T\rho$, we will call a horizontal translate of UV that contains elements of $\mathbb{Z}[\rho]$ an *Eisenstein line*. When $a \equiv b \pmod 2$, UV is itself Eisenstein and every Eisenstein line is a horizontal translate of UV by some $s \in \mathbb{Z}$. If $a \not\equiv b \pmod 2$ the Eisenstein lines are the translates by $s + 1/2$ for $s \in \mathbb{Z}$. In either case, every $w \in \mathbb{Z}[\rho]$ in the special triangle is in the first sector and $\|w\|_E = T + t$ for t such that w lies on the horizontal translate of UV by t.

Consider a line that is a translate of UV by a horizontal distance of s for any s with $T < T + s < M$. (Refer to Figure 10.) Since the special triangle is isosceles and the angles at U, V measure $\pi/6$ radians, this line intersects the triangle in the line segment from $v := V + s(\rho^2 + \rho)$ to $u = U + s(1 - \rho^2)$ where

$$u - v = (U - V) + s(1 - \rho - 2\rho^2) = (S - 3s)\rho^2 \ . \quad (13)$$

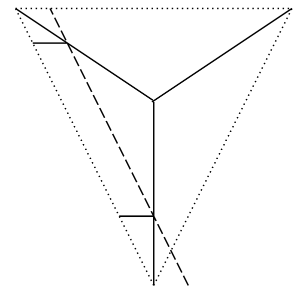

Fig. 10. The norm-count within the special triangle.

Therefore, since the upper side is not included in our count, $W(t) = 6(S - 3s)$ for integer $t = T + s < M$, where

$$S - 3s = S - 3(t - T) = 2a + b - 3t = 3(M - t) \ .$$

This completes the proof. ∎

Example 28. For the EJ network in Figure 2, we have $\alpha = 3 + 2\rho$ with $9 + 6 + 4 = 19$ nodes in the network, and $T = 5/2$, $M = 8/3$. By Theorem 27 the nodes are distributed according to:

$$W(0) = 1 \ , \ W(1) = 6 \ , \ W(2) = 12 \ ,$$

accounting for all 19 nodes. On the other hand, the EJ network for $\alpha = 3 + \rho$ (as pictured in Figure 3) has $9 + 3 + 1 = 13$ nodes and $T = 2$ is an integer. Since $M = 7/3 < 3$, from Theorem 27, $W(0) = 1, W(1) = 6$ and so $W(2) = 13 - 7 = 6$.

VIII. Conclusion

In recent years promising new networks were developed using quotient rings of Gaussian and EJ integers. Here we have given a unified approach to the study of the topology of these networks. This allowed us to develop new properties of the networks that helped to simplify routing algorithms, to find the EJ distance distribution, and to enumerate edge-disjoint Hamiltonian cycles in some networks. Some of the open problems include fault-tolerant routing, and multicasting with edge and node failures.

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Appendix

Proof of Proposition 19: Parts (a) and (b) can be proved at the same time by showing if $\gamma \in \mathbb{Z}[\sigma]$ is such that $0 < \|\gamma\alpha\|_\sigma \leq \|\alpha\|_\sigma$ then γ is a unit. Since $\|w\|_\sigma = \|\sigma^j w\|_\sigma$ for all $w \in \mathbb{Z}[\sigma]$, we may multiply α, γ by units as necessary and so may assume both α, γ are in the first sector. Therefore, $a, b > 0$, $\|\alpha\|_\sigma = a + b$, and $\gamma = u + v\sigma$ for $u, v \geq 0$.

If $u = 0$ then $\gamma\alpha = v\sigma\alpha = va\sigma + vb\sigma^2$ with $va, vb \geq 0$ and so $\|\gamma\alpha\|_\sigma = va + vb$;

$$a + b = \|\alpha\|_\sigma \geq va + vb \geq a + b \ ,$$

forcing equality throughout. Then $v = 1$, and γ must equal the unit σ. Similarly, $v = 0$ implies $\gamma = 1$. Both u, v may therefore be assumed to be positive.

Considering the Gaussian case,

$$\gamma\alpha = (u + vi)(a + bi) = (ua - vb) + (av + bu)i$$

gives

$$\|\gamma\alpha\|_i = |ua - vb| + (av + bu) \geq a + b = \|\alpha\|_\sigma \ , \quad (14)$$

and the hypothesis $\|\gamma\alpha\|_\sigma \leq \|\alpha\|_\sigma$ again forces equality throughout. Therefore, $ua - vb = 0$ and $u = v = 1$, giving $a = b$, which has been excluded.

In the EJ case, since γ, α are both in the first sector, $\gamma\alpha$ is in either the first or the second sector, and in order to calculate $\|\gamma\alpha\|_\rho$ the size of the coefficients of the linear combinations relative to $1, \rho$ and ρ, ρ^2 must be compared, where

$$\begin{aligned}\gamma\alpha &= (u + v\rho)(a + b\rho) = au + (av + bu)\rho + bv\rho^2 \\ &= \begin{cases}(au - bv) + (av + bu + bv)\rho \\ (-au + bv)\rho^2 + (av + bu + au)\rho\end{cases} .\end{aligned}$$

Therefore,

$$\|\gamma\alpha\|_\rho \geq (av + bu) + \min\{bv, au\} \ . \quad (15)$$

This cannot happen, since $u, v, a, b \geq 1$ would imply the contradiction

$$a + b \geq \|\gamma\alpha\|_\rho > av + bu \geq a + b \ .$$

This completes the proof of parts (a) and (b).

For part (c), by way of contradiction we suppose there exists a non-unit $\gamma \in \mathbb{Z}[\sigma]$ such that

$$\|w - \gamma\alpha\|_\sigma < \|w\|_\sigma \leq \max\{|a|, |b|\} \ .$$

Then

$$\|\gamma\alpha\|_\sigma = \|\gamma\alpha - w + w\|_\sigma \leq \|w - \gamma\alpha\|_\sigma + \|w\|_\sigma$$

by the triangle inequality, and so by assumption

$$\|\gamma\alpha\|_\sigma < \|w\|_\sigma + \|w\|_\sigma \leq 2\max\{|a|, |b|\} \ .$$

Since γ is a non-unit, by part (a)

$$\|\alpha\|_\sigma < \|\gamma\alpha\|_\sigma < 2a \ , \quad (16)$$

and the proof is completed by showing this cannot happen.

If γ is such that (16) holds, as above we may multiply each of α, γ by units as necessary to get both in the first sector, and so the appropriate one of (14), (15) holds. In the Gaussian case,

$$\|\gamma\alpha\|_i \geq av + bu \ .$$

If $\max\{|a|, |b|\} = a$, this combined with (16) thus implies $v = 1$ and (14) becomes

$$\begin{aligned}\|\gamma\alpha\|_i &= |ua - b| + (a + bu) = (ua - b) + (a + bu) \\ &= (u + 1)a + (u - 1)b \geq 2a \ ,\end{aligned}$$

contrary to (16). If $\max\{|a|, |b|\} = b$ the same argument gives $u = 1$ and a similar contradiction.

In the EJ case from (15), we likewise obtain $\|\gamma\alpha\|_\rho \geq av + bu$. Therefore, when $\max\{|a|, |b|\} = a$ then $v = 1$ and (15) becomes

$$\|\gamma\alpha\|_\rho \geq (au - b) + (a + bu) + b = a(u + 1) + bu > 2a \ ,$$

a contradiction. Since a similar contradiction is obtained from $\max\{|a|, |b|\} = b$, this completes the proof. ■

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