Trace identities for solutions of the wave equation with initial data supported in a ball

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Abstract

Suppose u is the solution of the initial value problem

$$u_{tt} - \Delta_x u = 0, \qquad (x,t) \in R^n \times [0,\infty);$$
$$u(x,t{=}0) = f(x), \qquad u_t(x,t{=}0) = g(x), \qquad x \in R^n.$$

Suppose $n \geq 1$ is odd, f and g are supported in a ball B with boundary S, and one of f or g is zero. We derive identities relating the norm of f or g to the norm of the trace of u on $S \times [0,\infty)$. These identities are derived using integral geometric and multiplier methods.

1 Introduction

Let B_ρ represent the 0 centered open ball of radius ρ, $\overline{B_\rho}$ its closure, S_ρ its boundary, R^n_+ the half space $x_n \geq 0$ in R^n, and H its boundary (the hyperplane $x_n = 0$ in R^n). Below, all functions will be real valued.

Suppose u is the solution of the wave equation with initial data supported in $\overline{B_\rho}$. Our goal is the derivation of identities which relate the norm of the initial data to the norm of the trace of u on $S_\rho \times [0,\infty)$. Specifically, we prove the following two theorems.

Theorem 1 *Suppose $n \geq 1$ is an odd integer and $v(x,t)$ is the solution of the initial value problem*

$$\Box v \equiv v_{tt} - \Delta v = 0, \qquad x \in R^n, \quad t \in R \tag{1}$$

$$v(x,t{=}0) = f(x), \;\; v_t(x,t{=}0) = 0 \qquad x \in R^n. \tag{2}$$

If $f \in C_0^\infty(\overline{B_\rho})$ *then*

$$\begin{aligned}
\frac{\rho}{2}\int_{R^n} |f(x)|^2 \, dx &= \int_0^\infty \int_{|p|=\rho} t \, v(p,t)^2 \, dS_p \, dt \ , && (3)\\
\frac{\rho}{2}\int_{R^n} |\nabla f(x)|^2 \, dx &= \int_0^\infty \int_{|p|=\rho} t \, v_t(p,t)^2 \, dS_p \, dt \ , && (4)\\
\frac{\rho}{2}\int_{R^n} |\nabla f(x)|^2 \, dx &= \int_0^\infty \int_{|p|=\rho} t \, |\nabla v(p,t)|^2 \, dS_p \, dt \ . && (5)
\end{aligned}$$

Theorem 2 *Suppose* $n \geq 1$ *is an odd integer and* $w(x,t)$ *is the solution of the initial value problem*

$$w_{tt} - \Delta w = 0, \qquad x \in R^n, \quad t \in R \tag{6}$$

$$w(x,t{=}0) = 0, \;\; w_t(x,t{=}0) = g(x) \qquad x \in R^n. \tag{7}$$

If $g \in C_0^\infty(\overline{B_\rho})$ *then*

$$\begin{aligned}
\frac{\rho}{2}\int_{R^n} |g(x)|^2 \, dx &= \int_0^\infty \int_{|p|=\rho} t \, w_t(p,t)^2 \, dS_p \, dt \ , && (8)\\
\frac{\rho}{2}\int_{R^n} |g(x)|^2 \, dx &= \int_0^\infty \int_{|p|=\rho} t \, |\nabla w(p,t)|^2 \, dS_p \, dt \ . && (9)
\end{aligned}$$

In Thermoacoustic Tomography, one is interested in recovering a function supported inside a ball B, from the mean values of the function on spheres, of all possible radii, which are centered on the boundary of B. In [4] we studied this question and derived an inversion formula. It is well known (see [4]) that the mean values of a function f over families of spheres are related to the solution of the wave equation with initial data f. In [4] we derived the inversion formula from a trace identity similar to the ones listed in Theorems 1 and 2; in fact all but (9) follow fairly quickly from the identity derived in [4]. However deriving (9) takes quite a bit of work and its proof is the significant part of this article. The proof uses integral geometric and multiplier techniques and we do not know a proof of these identities using only multiplier techniques. The identities have been proved only when n is odd; we do not know whether they are valid when n is even.

Theorem 1 and Theorem 2 may be used to obtain a similar identity for solutions of the wave equation with arbitrary initial data which is supported in a ball. Suppose $u(x,t)$ is the solution of

$$u_{tt} - \Delta u = 0, \qquad x \in R^n, \quad t \in R \tag{10}$$

$$u(x,t{=}0) = f(x), \;\; u_t(x,t{=}0) = g(x), \qquad x \in R^n, \tag{11}$$

with f and g in $C_0^\infty(\overline{B_\rho})$. Then noting that $u(x,t) = v(x,t) + w(x,t)$; $v(x,t)$ is even in t and $w(x,t)$ is odd in t, for n odd, applying Theorems 1 and 2, one may show that

$$\frac{1}{2}\int_{R^n} (|\nabla f(x)|^2 + |g(x)|^2) \, dx = \frac{1}{2\rho}\int_{-\infty}^\infty \int_{|p|=\rho} |t| \; u_t(p,t)^2 \; dS_p \; dt. \tag{12}$$

(12) is also an isometry of the initial data space similar to a basic isometry in scattering theory, but "at a finite place" rather than asymptotic in space and time. Taking the left hand side of (12) to define the square of the H_E norm ($n \geq 3$), we can derive[1] the isometry in the following result of Friedlander, in [6], as a corollary of Theorems 1, 2, and Theorem 6 in [4].

Corollary 1 (Friedlander) *Suppose n is an odd integer, $n \geq 3$, and f and g are compactly supported smooth functions on R^n. If $u(x,t)$ is the solution of the initial value problem (10), (11), then $\lim_{r\to\infty}(r^{(n-1)/2}u_t(r\theta, r+\tau)) \equiv q(\theta,\tau)$ exists and $\|q\|_{L^2(S^{n-1}\times R)} = \|(f,g)\|_{H_E}$.*

From standard Sobolev space theory one knows that taking traces results in a loss of regularity of degree 1/2. From the standard well posedness theory for IVP for hyperbolic PDEs one knows that if $f \in H^1_{loc}(R^n)$ and $g \in L^2_{loc}(R^n)$ then v and w are in $H^1_{loc}(R^n \times [0,\infty)$ and hence have $H^{1/2}$ traces on $S_\rho \times [0,\infty)$. Our identities show that if f and g are supported in $\overline{B_\rho}$ then in fact the traces are in H^1 and that there is no loss of regularity. One minor application of our identities is that this improved trace regularity carries over to the case when we replace the wave operator by any first order perturbation of it and allow a non-zero forcing function. Trace regularity theorems are useful in control theory of hyperbolic PDE and also in studying inverse problems for hyperbolic PDEs.

Theorem 3 (Spherical Case) *Suppose $0 < \rho_1 < \rho$, $n \geq 1$ an odd integer, and*

- *$a_i(x,t), b(x,t)$ are bounded, measurable functions on $R^n \times [0,T]$ with a_i and b supported in $\overline{B_{\rho_1}} \times [0,T]$,*
- *$f \in H^1(R^n)$, $g \in L^2(R^n)$ with f, g supported in $\overline{B_\rho}$,*
- *$F \in L^2(R^n \times [0,T])$ with F supported in $\overline{B_{\rho_1}} \times [0,T]$.*

Suppose $\epsilon > 0$ and $u(x,t)$ is the solution of the IVP

$$\mathcal{L}u := \Box u + \sum_{i=1}^{n} a_i u_{x_i} + bu = F, \qquad R^n \times [0,T] \tag{13}$$

$$u(x,t{=}0) = f(x), \;\; u_t(x,t{=}0) = g(x), \;\; x \in R^n. \tag{14}$$

Then u has an H^1 trace, ∇u has L^2 trace, and u_t has L^2 trace on $S_\rho \times [\epsilon, T]$, for all $\epsilon > 0$, and

$$\int_\epsilon^T \int_{|p|=\rho} |u(p,t)|^2 + |\nabla u(p,t)|^2 + |u_t(p,t)|^2 \, dS_p \, dt \leq C \left(\|f\|_1^2 + \|g\|_0^2 + \int_0^T \|F(.,t)\|_0^2 \, dt \right) \tag{15}$$

with C independent of f, g, F.

[1] We assume that the limit defining q exists in a manner which permits the interchange of the limit and the integral - this is a consequence of the results in ([6]).

Note that we do not make any claims about improved regularity in a neighborhood of t=0. This could be proved if f and g were supported away from S_ρ.

Using a trace identity of Bukhgeim and Kardakov in [3] for the solution of IVPs for the wave equation with initial data supported in a half plane, using standard arguments, one may also derive an optimal trace regularity for initial data supported in the half plane.

Theorem 4 (Hyperplane Case) *Suppose $\delta > 0$, $n \geq 1$ is an integer, and*

- *$a_i(x,t), b(x,t)$ are locally bounded, measurable functions on $R^n \times [0,T]$ with a_i and b supported in $\{x : |x_n| \geq \delta\} \times [0,T]$,*
- *$f \in H^1(R^n)$, $g \in L^2(R^n)$, and f and g supported in $|x_n| \geq \delta$,*
- *$F \in L^2(R^n \times [0,T])$ with F supported in $\{x : |x_n| \geq \delta\} \times [0,T]$.*

Suppose $u(x,t)$ is the solution of the IVP

$$\mathcal{L}u = F, \qquad \text{on } R^n \times [0,T], \tag{16}$$

$$u(x,t{=}0) = f(x), \;\; u_t(x,t{=}0) = g(x), \;\; x \in R^n. \tag{17}$$

Then has an H^1_{loc} trace, ∇u has L^2_{loc} trace, and u_t has L^2_{loc} trace on $H \times [0,T]$ and

$$\int_0^T \int_H \left(|u(p,t)|^2 + |u_t(p,t)|^2 + |\nabla u(p,t)|^2\right) dS_p\, dt \leq C \left(\|f\|_1^2 + \|g\|_0^2 + \int_0^T \|F(.,t)\|_0^2\, dt \right), \tag{18}$$

with C independent of f, g, F.

Note, here *we do* assert optimal regularity even near $t = 0$ but then we require that $f, g, F(.,t)$ be supported away from $x_n = 0$.

The main results of this article are the trace identities in Theorems 1 and 2 which are new. A simple application of this combined with some standard inequalities for solutions of hyperbolic PDE give us Theorems 3 and 4. Even though Theorems 3 and 4 are special cases of more general results of Bao and Symes in [1] and [2], we have chosen to include these theorems and their proofs because the proof is much simpler than the microlocal analysis based proofs in [1] and [2]; but then the Bao and Symes result is a more general result. They showed, for fairly general operators, with weak regularity assumptions, that singularities in the solution are generated by the propagation of the singularities, in the initial data or the forcing term (the RHS of the PDE), along the null bicharacteristics of the differential operator. They conjectured that the weaker regularity in the traces results from bicharacteristics which graze the cylindrical surface $S \times [0,T]$ and they showed that in the absence of such bicharacteristics the trace has optimal regularity, that is there is no loss of regularity in taking the trace. According to them it is hard to prove such trace results for time-like surfaces $S \times [0,T]$ because it is difficult to obtain energy estimates where one wishes to relate the energy on a time-like surface to the energy on a space-like surface (the initial data surface). They prove their result by modifying the wave operator so that there are no characteristic surfaces (presumably by taking microlocal cutoffs of the differential operator) and then modifying the energy type estimate.

Symes in [11], has a different proof of Theorem 4, and also constructed an example to show that the result is violated if the restrictions on the support are removed. Lasiecka and Triggiani in [7] also obtained results similar to Theorem 4 for the wave operator using Laplace Transforms and in [8] they gave a sharp characterization of the regularity of the traces if the support restrictions are dropped; Tataru in [12] has studied this sharp regularity problem (without the support conditions) in a fairly general situation.

2 Proof of Theorems 1, 2, and Corollary 1

2.1 Proof of Theorems 1 and 2

If $n=1$ then

$$v(x,t) = \frac{f(x+t)+f(x-t)}{2}, \qquad w(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \;.$$

If we interpret $\int_{|p|=\rho} h(p)\,dS_p$ to mean $h(\rho)+h(-\rho)$, then we may verify by a straight-forward calculation that Theorem 1 and Theorem 2 are valid when $n=1$.

For odd $n \geq 3$, the theorems will be proved in parallel and the proofs are based on an identity stated in Theorem 6 of [4]. There we showed that if w_i, $i=1,2$ are solutions of the IVP (6), (7), with $g=g_i$, where g_i are smooth functions supported in $\overline{B_\rho}$ then

$$\frac{1}{2}\int_{R^n} g_1(x)\,g_2(x)\,dx = \frac{-1}{\rho}\int_0^\infty \int_{|p|=\rho} t\,w_1(p,t)\,w_{2tt}(p,t)\,dS_p\,dt \;. \tag{19}$$

Taking $g_1=g_2=g$ in this equation, integrating by parts, and noting that w is zero on $S_\rho(0)$ for $t=0$ and for t large (note n is odd), we obtain

$$\begin{aligned}
\frac{1}{2}\int_{R^n} |g(x)|^2\,dx &= \frac{-1}{\rho}\int_0^\infty \int_{|p|=\rho} t\,w(p,t)\,w_{tt}(p,t)\,dS_p\,dt \\
&= \frac{1}{\rho}\int_0^\infty \int_{|p|=\rho} t\,|w_t(p,t)|^2\,dS_p\,dt \;+\; \frac{1}{\rho}\int_0^\infty \int_{|p|=\rho} w(p,t)\,w_t(p,t)\,dS_p\,dt \\
&= \frac{1}{\rho}\int_0^\infty \int_{|p|=\rho} t\,|w_t(p,t)|^2\,dS_p\,dt, \;+\; \frac{1}{2\rho}\int_0^\infty \int_{|p|=\rho} \frac{d}{dt}\left(w(p,t)^2\right)dS_p\,dt \\
&= \frac{1}{\rho}\int_0^\infty \int_{|p|=\rho} t\,|w_t(p,t)|^2\,dS_p\,dt,
\end{aligned}$$

proving (8).

We note that w_t is a solution of (1), (2) with f replaced by g, hence (3) follows from (8). If we define $v_1 = \partial_j v$ then v_1 also satisfies the wave equation (1) except its initial conditions are

$$v_1(.,t{=}0) = \partial_j v(.,t{=}0) = \partial_j f, \qquad v_{1t}(.,t{=}0) = (\partial_j \partial_t v)(.,t{=}0) = 0 \ .$$

Hence, (3) applied to v_1 implies

$$\frac{\rho}{2}\int_{R^n} |\partial_j f(x)|^2\,dx = \int_0^\infty \int_{|p|=\rho} t\,|v_1(p,t)|^2\,dS_p\,dt = \int_0^\infty \int_{|p|=\rho} t\,|\partial_j v(p,t)|^2\,dS_p\,dt, \ .$$

Adding these for all $j = 1, 2, \cdots, n$ we obtain (5).

Now we prove (4). Take $w_1(.,t) = \int_0^t v(.,s)\,ds$ and $w_2 = v_t$. Then w_1 and w_2 are solutions of (6), except their ICs are $w_1(.,t{=}0) = 0$, $w_{1t}(.,t{=}0) = v(.,t{=}0) = f$; $w_2(.,t{=}0) = v_t(.,t{=}0) = 0$, and $w_{2t}(.,t{=}0) = v_{tt}(.,t{=}0) = \Delta v(.,t{=}0) = \Delta f$. Hence, from the bilinear form of (8), we have

$$\begin{aligned}\frac{1}{2}\int_{R^n} f(x)\,(\Delta f)(x)\,dx &= \frac{1}{\rho}\int_0^\infty \int_{|p|=\rho} t\,w_{1t}(p,t)\,w_{2t}(p,t)\,dS_p\,dt \\ &= \frac{1}{\rho}\int_0^\infty \int_{|p|=\rho} t\,v(p,t)\,v_{tt}(p,t)\,dS_p\,dt \ .\end{aligned}$$

Integrating by parts both sides of the equation and noting that f has compact support, and v is zero on S_ρ for $t = 0$ and for t large, one obtains (4).

It remains to prove (9) whose long proof will take up the rest of the section.

Suppose $r \geq \rho$. Then,

$$\begin{aligned} 0 &= \int_0^\infty \int_{|p|\leq r} tw(w_{tt} - \Delta w)(p,t)\,dp\,dt \\ &= \int_0^\infty \int_{|p|\leq r} tww_{tt}\,dp\,dt - \int_0^\infty \int_{|p|\leq r} t\nabla\cdot(w\nabla w)\,dp\,dt + \int_0^\infty \int_{|p|\leq r} t|\nabla w|^2\,dp\,dt \\ &= \int_0^\infty \int_{|p|\leq r} tww_{tt}\,dp\,dt - \int_0^\infty \int_{|p|=r} tw(p,t)w_r(p,t)\,dS_p\,dt + \int_0^\infty \int_{|p|\leq r} t|(\nabla w)(p,t)|^2\,dp\,dt, \end{aligned}$$

where $w_r(p,t)$ represents the radial derivative of w, i.e. $w_r(x,t) = x\cdot\nabla w(x,t)/|x|$. Hence

$$\int_0^\infty \int_{|p|=r} tw(p,t)w_r(p,t)\,dS_p\,dt = \int_0^\infty \int_{|p|\leq r} tw(p,t)w_{tt}(p,t)\,dp\,dt + \int_0^\infty \int_{|p|\leq r} t|(\nabla w)(p,t)|^2\,dp\,dt \ .$$

Differentiating this with respect to r, we obtain

$$\frac{dA(r)}{dr} = \int_0^\infty \int_{|p|=r} tw(p,t)w_{tt}(p,t)\,dS_p\,dt + \int_0^\infty \int_{|p|=r} t|(\nabla w)(p,t)|^2\,dS_p\,dt, \tag{20}$$

where

$$A(r) := \int_0^\infty \int_{|p|=r} t\,w(p,t)\,w_r(p,t)\,dS_p\,dt \ . \tag{21}$$

If we can show that $A(r)$ is independent of r for $r \geq \rho$, then (9) will follow from (20) if we take $g_1 = g_2 = g$ in (19).

To prove that $A(r)$ is independent of r for $r \geq \rho$, we will need some of the ideas used in [4] and the details may be found there. Let $\{\phi_m\}_{m=1}^{\infty}$ be spherical harmonics which form an orthonormal basis for $L^2(S_1(0))$ - see Chapter 4 of [10]. These are restrictions to $S_1(0)$ of some harmonic homogeneous polynomials on R^n. If ϕ_m is the restriction of a homogeneous polynomial of degree $k(m)$ then that homogeneous harmonic polynomial is $r^{k(m)}\phi_m(\theta)$ where $r = |x|$ and $\theta = x/|x|$.

Suppose g is a smooth function on R^n supported in $\overline{B_\rho(0)}$. We have a decomposition of g of the form (convergence in L^2)

$$g(r\theta) = \sum_{m=1}^{\infty} g_m(r)\, r^{k(m)}\, \phi_m(\theta), \qquad r \geq 0,\ |\theta| = 1$$

with

$$r^{k(m)} g_m(r) = \int_{|\theta|=1} g(r\theta)\, \phi_m(\theta)\, d\theta\ . \tag{22}$$

One may show that $g_m(r)$ is a smooth, even function, supported in $[-\rho, \rho]$. One may also show that

$$w(r\theta, t) = \sum_{m=1}^{\infty} a_m(r,t) r^{k(m)}\, \phi_m(\theta), \qquad r \geq 0,\ |\theta| = 1$$

where $a_m(r,t)$ is the solution of the Darboux equation

$$a_{mtt} - a_{mrr} - \frac{\nu(m)-1}{r} a_{mr} = 0, \qquad r \in (-\infty, \infty),\ t \geq 0,$$

$$a_m(.,t{=}0) = 0,\ a_{mt}(.,t{=}0) = g_m(.)\ ,$$

with $\nu(m) = n + 2k(m)$.

Substituting the above expansion for w into (21), the definition of $A(r)$, and using the orthonormality of the ϕ_m, we have (here $p = r\theta$)

$$\begin{aligned}
A(r) &= \int_0^\infty \int_{|p|=r} t \left(\sum_{m=1}^{\infty} a_m(r,t) r^{k(m)}\, \phi_m(\theta) \right) \left(\sum_{m=1}^{\infty} \left(a_m(r,t) r^{k(m)} \right)_r\, \phi_m(\theta) \right) dS_p\, dt \\
&= \int_0^\infty \int_{|\theta|=1} t\, r^{n-1} \left(\sum_{m=1}^{\infty} a_m(r,t) r^{k(m)}\, \phi_m(\theta) \right) \left(\sum_{m=1}^{\infty} \left(a_m(r,t) r^{k(m)} \right)_r\, \phi_m(\theta) \right) d\theta\, dt \\
&= \sum_{m=1}^{\infty} \int_0^\infty t\, r^{n-1+k(m)}\, a_m(r,t)\, \left(a_m(r,t) r^{k(m)} \right)_r\, dt \\
&= \sum_{m=1}^{\infty} \int_0^\infty t\, r^{n-2+2k(m)}\, a_m(r,t)\, (r\, a_{mr}(r,t) + k(m) a_m(r,t))\, dt \\
&= \sum_{m=1}^{\infty} \int_0^\infty t\, \left(r^{\nu(m)-1}\, a_m(r,t)\, a_{mr}(r,t) + k(m) r^{\nu(m)-2} a_m^2(r,t) \right)\, dt\ .
\end{aligned}$$

Suppose $g(r)$ is a smooth even function supported in $[-\rho, \rho]$, ν an integer, and $a(r,t)$ is the solution of the Darboux equation

$$a_{tt} - a_{rr} - \frac{\nu - 1}{r} a_r = 0, \qquad r \in (-\infty, \infty),\ t \geq 0, \tag{23}$$

$$a(., t{=}0) = 0,\ a_t(., t{=}0) = g(r)\ . \tag{24}$$

Define

$$B(r, \nu, g) \equiv r^{\nu-1} \int_0^\infty t\, a(r,t)\, a_r(r,t)\, dt, \qquad C(r, \nu, g) \equiv r^{\nu-2} \int_0^\infty t\, a^2(r,t)\, dt\ . \tag{25}$$

Then $A(r)$ will be independent of r for all $r \geq \rho$ if we can show that $B(r, \nu, g)$ and $C(r, \nu, g)$ are independent of r for all $r \geq \rho$, for all smooth even g supported in $[-\rho, \rho]$, for all odd integers $\nu \geq 3$.

However, note that

$$\begin{aligned} \frac{d}{dr}\left(rC(r, \nu, g)\right) &= 2r^{\nu-1} \int_0^\infty t\, a(r,t)\, a_r(r,t)\, dt, \ + \ (\nu - 1)\, r^{\nu-2} \int_0^\infty t\, a^2(r,t)\, dt \\ &= 2B(r, \nu, g) + (\nu - 1)C(r, \nu, g)\ . \end{aligned} \tag{26}$$

So, if we could show that $C(r, \nu, g)$ is independent of r for $r \geq \rho$, then from (26) we would have $B(r, \nu, g) = (2 - \nu)C(r, \nu, g)/2$ and $B(r, \nu, g)$ would also be independent of r for $r \geq \rho$.

Let us also define

$$D(r, \nu, g) \equiv r^{\nu-2} \int_0^\infty t\, |a_r(r,t)|^2\, dt\ . \tag{27}$$

Given a smooth, even function $g(r)$ which is supported in $[-\rho, \rho]$, we define

$$h(r) = \int_{-\infty}^{r} s\, g(s)\, ds\ .$$

Then $h(r)$ is also a smooth, even function of r and is also supported in $[-\rho, \rho]$. Let $b(r,t)$ be the solution of

$$b_{tt} - b_{rr} - \frac{\nu - 3}{r} b_r = 0, \qquad r \in (-\infty, \infty),\ t \geq 0,$$

$$b(., t{=}0) = 0,\ b_t(., t{=}0) = h(.)\ .$$

Then one may verify that b_r/r is a solution of (23), (24). Hence $a = b_r/r$. So

$$C(r, \nu, g) = r^{\nu-4} \int_0^\infty t\, |b_r(r,t)|^2\, dt = D(r, \nu - 2, h)\ . \tag{28}$$

We will prove that $C(r, \nu, g)$ is independent of $r \geq \rho$ by induction on ν. The proof will be based on the following Lemma.

Lemma 1 *Suppose g is a smooth, even function, supported in $[-\rho, \rho]$, and $\nu \geq 1$ is odd.*

(a) If $D(r,\nu,g)$ is independent of r for $r \geq \rho$ then

$$D(r,\nu,g) = \frac{1}{2}\int_0^\infty s^{\nu-1}|g(s)|^2\,ds.$$

(b) For $r \geq \rho$,

$$B_r(r,\nu,g) = r\left(D(r,\nu,g) - \frac{1}{2}\int_0^\infty s^{\nu-1}|g(s)|^2\,ds\right) .$$

So from (a), for $r \geq \rho$, $B(r,\nu,g)$ is independent of r iff $D(r,\nu,g)$ is independent of r.

We now complete the proof of the independence of $C(r,\nu,g)$ from r. When $\nu = 1$, (23) is the one dimensional wave equation and

$$a(r,t) = \frac{1}{2}\int_{r-t}^{r+t} g(s)\,ds\ .$$

Hence, using the support of g, for $r \geq \rho$,

$$a_r(r,t) = \frac{g(r+t) - g(r-t)}{2} = \frac{-g(r-t)}{2}$$

if $t \geq 0$. Hence, using the support of g and that g is even, we have

$$\begin{aligned}
D(r,\nu{=}1,g) &= \frac{1}{r}\int_0^\infty t\,|a_r(r,t)|^2\,dt \\
&= \frac{1}{4r}\int_0^\infty t\,|g(r-t)|^2\,dt \;=\; \frac{1}{4r}\int_{-\infty}^{r}(r-s)\,|g(s)|^2\,ds \\
&= \frac{1}{4r}\int_{-r}^{r}(r-s)\,|g(s)|^2\,ds \;=\; \frac{1}{4}\int_{-r}^{r}|g(s)|^2\,ds \\
&= \frac{1}{2}\int_0^{r}|g(s)|^2\,ds \;=\; \frac{1}{2}\int_0^{\rho}|g(s)|^2\,ds\ .
\end{aligned}$$

So $D(r,\nu{=}1,g)$ is independent of r for all g which are smooth, even and supported in $[-\rho,\rho]$. Applying this to h and using (28), we obtain that $C(r,\nu{=}3,g)$ is independent of r, for all smooth, even g which are supported in $[-\rho,\rho]$.

Next, using (26), we conclude that $B(r,\nu{=}3,g)$ is independent of r. Then from Lemma 1 we conclude that $D(r,\nu=3,g)$ is independent of r, for all appropriate g. Then using (28) we conclude that $C(r,\nu=5,g)$ is independent of r. In this fashion, by induction on ν, we may show that $C(r,\nu,g)$ is independent of r for all odd $\nu \geq 3$.

End of Proof of Theorems 1 and 2

2.1.1 Proof of Lemma 1

The proof of Lemma 1 uses three identities.

Lemma 2 *Suppose $\nu \geq 3$ is odd, g and g_i, $i = 1, 2$ are smooth even function with support in $[-\rho, \rho]$, and $a(r,t)$, $a_i(r,t)$ are the corresponding solutions of the Darboux equations (23), (24). Then for all $r \geq \rho$,*

$$\frac{1}{2}\int_0^\infty s^{\nu-1} g_1(s)\, g_2(s)\, ds = -\, r^{\nu-2}\int_0^\infty t\, a_1(r,t)\, a_{2tt}(r,t)\, dt\ . \tag{29}$$

$$\frac{1}{2}\int_0^\infty s^{\nu-1} |g(s)|^2\, ds = r^{\nu-2}\int_0^\infty t\, |a_t(r,t)|^2\, dt, \tag{30}$$

$$\frac{1}{2}\int_0^\infty s^{\nu-1} |g(s)|^2\, ds = -r^{\nu-1}\int_0^\infty a_t(r,t)\, a_r(r,t)\, dt\ . \tag{31}$$

Note that there is no t term in the RHS of (31).

Proof of Lemma 2
We will prove the three identities when $r = \rho$ - the general $r \geq \rho$ case will then follow because if g is supported in $[-\rho, \rho]$ then it is also supported in $[-r, r]$.

To prove (29), take $k = (\nu - 3)/2$ - note k is a non-negative integer because ν is odd. Choose $r^k \phi(\theta)$ a homogeneous, harmonic polynomial of degree k in R^3 with the L^2 norm of ϕ being one. Then $w_i(x,t) = a_i(|x|)|x|^k \phi(\theta)$ are solutions of (6), (7) with $g(x) = g_i(|x|)$ and $n = 3$. Hence (19) is valid and we rewrite it as it applies to our situation. We obtain

$$\begin{aligned}
\text{LHS of (19)} &= \frac{1}{2}\int_{R^3} g_1(r)\, r^k\, \phi(\theta)\, g_2(r)\, r^k\, \phi(\theta)\, dx \qquad (x = r\theta) \\
&= \frac{1}{2}\int_0^\infty r^{3-1} r^{2k} g_1(r)\, g_2(r)\, dr = \frac{1}{2}\int_0^\infty r^{2k+2} g_1(r)\, g_2(r)\, dr = \frac{1}{2}\int_0^\infty r^{\nu-1} g_1(r)\, g_2(r)\, dr\ ,
\end{aligned}$$

$$\begin{aligned}
\text{RHS of (19)} &= \frac{-1}{\rho}\int_0^\infty \int_{|p|=\rho} t\, w_1(p,t)\, w_{2tt}(p,t)\, dS_p\, dt \\
&= \frac{-1}{\rho}\int_0^\infty \int_{|\theta|=1} t\, \rho^{3-1}\, \rho^{2k}\, a_1(\rho,t)\, a_{2tt}(\rho,t)\, \phi^2(\theta)\, d\theta\, dt \\
&= -\, \rho^{\nu-2}\int_0^\infty t\, a_1(\rho,t)\, a_{2tt}(\rho,t)\, dt,
\end{aligned}$$

which proves (29).

(30) follows from (29) if we take $g_1 = g_2 = g$ and integrate by parts the RHS of (29) - note strong Huygen's principle is valid for a for odd $\nu \geq 3$. The proof of (31) is an imitation of the proof of the standard energy identity for the wave equation. In fact (31) is just the energy identity

rewritten. To prove (31), note that

$$\begin{aligned}
0 &= 2r^{\nu-1}\,a_t\left(a_{tt}-a_{rr}-\frac{\nu-1}{r}a_r\right)\\
&= 2r^{\nu-1}a_ta_{tt}-2a_t\left(r^{\nu-1}\,a_r\right)_r\\
&= r^{\nu-1}\left(a_t^2\right)_t-2\left(r^{\nu-1}\,a_t\,a_r\right)_r+2r^{\nu-1}\,a_{rt}\,a_r\\
&= r^{\nu-1}\left(a_t^2+a_r^2\right)_t-2\left(r^{\nu-1}\,a_t\,a_r\right)_r\ .
\end{aligned}$$

Integrating this over the region $[-\rho,\rho]\times[0,\infty)$ and noting that $a(.,t)=0$ on $[-\rho,\rho]$ for large t ($\nu\geq 3$ is odd),

$$\begin{aligned}
0 &= \int_0^\infty\int_0^\rho r^{\nu-1}\left(a_t^2+a_r^2\right)_t-2\left(r^{\nu-1}\,a_t\,a_r\right)_r\,dr\,dt\\
&= -\int_0^\rho r^{\nu-1}\left(a_t^2+a_r^2\right)(r,t{=}0)\,dr\ -\ 2\rho^{\nu-1}\int_0^\infty a_t(\rho,t)\,a_r(\rho,t)\,dt\\
&= -\int_0^\rho r^{\nu-1}|g(r)|^2\,dr\ -\ 2\rho^{\nu-1}\int_0^\infty a_t(\rho,t)\,a_r(\rho,t)\,dt
\end{aligned}$$

proving (31).

End of proof of Lemma 2

Proof of Lemma 1(a)
We define

$$P(\nu,g):=\frac{1}{2}\int_0^\infty s^{\nu-1}|g(s)|^2\,ds\ .$$

From Lemma 2, the RHS of (30) is independent of $r\geq\rho$. So differentiating the RHS of (30) we have

$$\begin{aligned}
0 &= (\nu-2)r^{\nu-3}\int_0^\infty t|a_t(r,t)|^2\,dt+2r^{\nu-2}\int_0^\infty ta_ta_{tr}\,dt\\
&= \frac{\nu-2}{r}P+2r^{\nu-2}\int_0^\infty ta_t(a_r)_t\,dt, \qquad \text{using (30).}
\end{aligned}$$

Multiplying by r, and integrating by parts the second integral, we obtain

$$\begin{aligned}
0 &= (\nu-2)P+2r^{\nu-1}\left(ta_ta_r|_{t=0}^\infty-\int_0^\infty(a_t+ta_{tt})a_r\,dt\right)\\
&= (\nu-2)P-2r^{\nu-1}\int_0^\infty a_ta_r\,dt-2r^{\nu-1}\int_0^\infty t(a_{rr}+\frac{\nu-1}{r}a_r)a_r\,dt, \qquad \text{using (23)}\\
&= \nu P-r^{\nu-1}\int_0^\infty t(a_r^2)_r\,dt-2(\nu-1)r^{\nu-2}\int_0^\infty ta_r^2\,dt, \qquad \text{using (31)}\\
&= \nu P-\frac{d}{dr}(rD(r))-(\nu-1)D(r)\ .
\end{aligned}$$

Hence

$$D'+\frac{\nu}{r}D=\frac{\nu}{r}P$$

or

$$(r^\nu D)' = \nu r^{\nu-1} P \ .$$

Since P is independent of r, integrating we have

$$D(r) = P + cr^{-\nu}, \qquad r \geq \rho \ .$$

Thus, $D(r)$ is independent of r iff $c = 0$.

QED

Proof of Lemma 1(b)
Differentiating the expression for $B(r, \nu, g)$, for $r \geq \rho$, we have

$$\begin{aligned}
B'(r) &= (\nu-1)r^{\nu-2}\int_0^\infty taa_r\,dt + r^{\nu-1}\int_0^\infty t(a_r^2 + aa_{rr})\,dt \\
&= (\nu-1)r^{\nu-2}\int_0^\infty taa_r\,dt + r^{\nu-1}\int_0^\infty ta_r^2\,dt + r^{\nu-1}\int_0^\infty ta(a_{tt} - \frac{\nu-1}{r}a_r)\,dt \qquad \text{using (23)} \\
&= r^{\nu-1}\int_0^\infty ta_r^2\,dt + r^{\nu-1}\int_0^\infty taa_{tt}\,dt \\
&= r(D(r) - P) \qquad \text{using (29)} \ .
\end{aligned}$$

QED

2.2 Proof of Corollary 1

The corollary is a consequence of Theorems 1, 2, and Theorem 6 in [4]. To see this, we need to show that

$$\lim_{\rho\to\infty} \frac{1}{\rho}\int_0^\infty \int_{|p|=\rho} tv_t w_t dS_p\,dt = 0, \tag{32}$$

for v and w as in Theorems 1 and 2. Accepting this, as the initial data has compact support, by Huygens' principle (in odd dimensions) the solution $u(p,t)$ is supported in a fixed t-intervals about $t = \rho$ when $|p| = \rho$, and so $|t|/\rho = 1 + O(1/\rho)$ on the support of the solution.

So it remains to prove (32). By Theorem 6 of [4]

$$-\int_0^\infty \int_{|p|=\rho} t\omega_1\omega_{2tt} dS_p dt = \int_0^\infty \int_{|p|=\rho} t\omega_{1t}\omega_{2t} dS_p dt$$

when the ω_i, $i = 1, 2$ are solutions of $\Box\omega_i = 0$, $\omega(\cdot, t = 0) = 0$, $\omega_{it}(\cdot, t = 0) = h_i$ for smooth functions h_i supported in the ball of radius R, and $\rho \geq R$. Performing the integration by parts on the left, and observing that the boundary terms contribute nothing, we obtain that

$$0 = \int_0^\infty \int_{|p|=\rho} \omega_1\omega_{2t} dS_\rho dt.$$

Now $\omega_1 = v_t$, for v as in (2), satisfies $\Box\omega_1 = 0$, $\omega(\cdot, t=0) = 0$, $\omega_t(\cdot, t=0) = \triangle f$ and with $\omega_2 = w$ we then have that

$$0 = \int_0^\infty \int_{|p|=\rho} v_t w_t dS_p dt.$$

Applying this, we obtain

$$\begin{aligned}
\int_0^\infty \int_{|p|=\rho} t v_t w_t dS_p dt &= \int_0^\infty \int_{|p|=\rho} (t-\rho) v_t w_t dS_p dt + \rho \int_0^\infty \int_{|p|=\rho} v_t w_t dS_p dt \\
&= \int_0^\infty \int_{|p|=\rho} (t-\rho) v_t w_t dS_p dt.
\end{aligned}$$

Then using that v_t and w_t have support near $t = \rho$, after dividing by ρ, the limit, as $\rho \to \infty$, is equal to zero.

End of Proof of Corollary 1.

3 Proof of Theorem 3

The existence of a solution u in $H^1(R^n \times [0,T])$ to the IVP (13), (14), is guaranteed by the standard theory for initial value problems for hyperbolic PDE. Bringing the first and zeroth order terms in the LHS of (13) to the RHS and absorbing it in F will still keep F in L^2. Further, standard energy estimates bound the H^1 norm of u on $R^n \times [0,T]$ by the RHS of (15). So it is enough to prove Theorem 3 for the special case when $a_i = 0$ and $b = 0$.

We next show that Theorem 3 follows if we prove (15) for all *smooth* f, g, F satisfying the hypothesis of Theorem 3. Given f, g, F satisfying the hypothesis of the theorem, we can find a sequence of smooth functions f_k, g_k, F_k which also satisfy the hypothesis of the theorem so that $\|f_k - f\|_{H^1} \to 0$, $\|g_k - g\|_{L^2} \to 0$, and $\|F_k - F\|_{L^2} \to 0$. Let u_k be the unique solution corresponding to f_k, g_k, F_k and u the unique solution corresponding to f, g, F - note that the u_k are smooth, u is in $H^1(R^n \times [0,T])$, and u_k converges to u in the H^1 norm (from standard energy estimates). Hence the trace of u_k converges to the trace of u on $S_\rho \times [0,T]$ in the L^2 norm (at least).

Now applying (15) to the functions $f_k - f_m$, $g_k - g_m$, and $F_k - F_m$, we may conclude that the traces of u_k on $S_\rho \times [\epsilon, T]$ form a Cauchy sequence in $H^1(S_\rho \times [\epsilon, T])$. Hence it is convergent to an H^1 function on $S_\rho \times [\epsilon, T]$. But the traces of u_k on $S_\rho \times [\epsilon, T]$ converge to the trace of u in the L^2 norm. Hence the trace of u on $S_\rho \times [\epsilon, T]$ is in $H^1(S_\rho \times [\epsilon, T])$. Now applying the standard theory for the well posedness of the IBVP for hyperbolic PDE with Dirichlet boundary conditions (see [9]) to the region $\overline{B_\rho} \times [\epsilon, T]$, and noting that the trace of u on $S_\rho \times [\epsilon, T]$ is in H^1, we have that the normal derivative of u has L^2 trace on $S_\rho \times [\epsilon, T]$ and the normal derivative of u_k approaches the normal derivative of u in $L^2(S_\rho \times [\epsilon, T])$. Finally, since (15) is valid for u_k, letting k approach infinity proves (15) for u.

So it remains to prove (15) when f, g, F are smooth and satisfy the support hypothesis of the theorem. There is no loss of generality in assuming that $F(., t)$ vanishes to infinite order at $t = T$, and hence is still a smooth function when F is defined to be zero for $t \geq T$. We will take $u(x,t)$ to be the solution of (13) and (14) over the region $R^n \times [0, \infty)$. Noting that $t \geq \epsilon$ for all $t \in [\epsilon, T]$, from Theorems 1 and 2, and the standard trace theory (to estimate the L^2 norm of the trace of u), one may easily derive (15) when $F = 0$. So from linearity, we must now establish (15) when $f = 0$, $g = 0$, and $\mathcal{L}$ is just the wave operator.

So suppose u is the solution of

$$\Box u = F \qquad R^n \times [0,T], \tag{33}$$

$$u(.,t{=}0) = 0, \;\; u_t(.,t{=}0) = 0 \tag{34}$$

with $F \in C^\infty(R^n \times [0,T])$ and $F(.,t)$ supported in $\overline{B_{\rho_1}}$ for all t. For each $s \geq 0$, define $w(x,t;s)$ to be the solution of the IVP

$$\Box_{x,t} w(x,t;s) = 0 \qquad R^n \times [s, \infty), \tag{35}$$

$$w(.,t{=}s;s) = 0, \;\; w_t(.,t{=}s;s) = F(.,s)\ . \tag{36}$$

Then, from Duhamel's principle we know that

$$u(x,t) = \int_0^t w(x,t;s)\, ds\ . \tag{37}$$

Now, from the hypothesis about the support of F and the speed of propagation, we have $w(.,.;s)$ is zero on a neighborhood of $S_\rho \times [s, s + \rho - \rho_1)$. So, from Theorem 2 applied to $w(x,t;s)$ we have (note the $t - s$ term below instead of the t term in Theorem 2 because $w(.,.;s)$ starts at $t{=}s$ instead of $t{=}0$)

$$\begin{aligned}
\frac{\rho}{2}\int_{R^n} |F(x,s)|^2\, dx &= \int_s^\infty \int_{|p|=\rho} (t-s)|w_t(p,t;s)|^2\, dS_p\, dt \;\geq\; (\rho - \rho_1)\int_s^T \int_{|p|=\rho} |w_t(p,t;s)|^2\, dS_p\, dt \\
\frac{\rho}{2}\int_{R^n} |F(x,s)|^2\, dx &= \int_s^\infty \int_{|p|=\rho} (t-s)|\nabla w(p,t;s)|^2\, dS_p\, dt \;\geq\; (\rho - \rho_1)\int_s^T \int_{|p|=\rho} |\nabla w(p,t;s)|^2\, dS_p\, dt\ .
\end{aligned}$$

Integrating these over the interval $[0,T]$, we have

$$\int_0^T \int_s^T \int_{|p|=\rho} |w_t(p,t;s)|^2\, dS_p\, dt\, ds \;\leq\; C \int_0^T \int_{R^n} |F(x,s)|^2\, dx\, ds \tag{38}$$

$$\int_0^T \int_s^T \int_{|p|=\rho} |\nabla w(p,t;s)|^2\, dS_p\, dt\, ds \;\leq\; C \int_0^T \int_{R^n} |F(x,s)|^2\, dx\, ds\ . \tag{39}$$

From (37), (38), (39), and using $w(p,t;t)$=0, we have

$$\begin{aligned}
&\int_0^T \int_{|p|=\rho} \left(|u_t(p,t)|^2 + |\nabla u(p,t)|^2\right)\, dS_p\, dt \\
&= \int_{|p|=\rho} \int_0^T \left(\left| \int_0^t w_t(p,t;s)\, ds \right|^2 + \left| \int_0^t \nabla w(p,t;s)\, ds \right|^2 \right) dt\, dS_p \\
&\leq T \int_{|p|=\rho} \int_0^T \int_0^t \left(|w_t(p,t;s)|^2 + |\nabla w(p,t;s)|^2\right)\, ds\, dt\, dS_p \qquad \text{(Cauchy-Schwarz inequality)} \\
&= T \int_{|p|=\rho} \int_0^T \int_s^T \left(|w_t(p,t;s)|^2 + |\nabla w(p,t;s)|^2\right)\, dt\, ds\, dS_p \qquad \text{(reversing order of integration)} \\
&\leq C \int_0^T \int_{R^n} |F(x,s)|^2\, dx\, ds\ .
\end{aligned}$$

The $|u(p,t)|^2$ term on the LHS of (15) is bounded by the standard trace theorem and we have proved (15).

QED

4 Proof of Theorem 4

Below, u_i will represent the partial derivative of u with respect to x_i. In the $n = 1$ case $\int_{H\times[0,T]} f(x,t) dS_x$ will just mean $\int_0^T f(0,t)\, dt$. Arguing as in the case of Theorem 3, to prove Theorem 4 it is enough to establish (18) for *smooth* f, g, F which satisfy the hypothesis of the theorem and when $\mathcal{L}$ is the wave operator. Since the solution u of (13), (14) depends linearly on f, g, F, it is enough to prove (18) in the three separate cases when two out of the three of f, g, F are zero.

As in the case of Theorem 3, a crucial role is played by trace identities for the halfplane, similar to Theorems 1 and 2. For the halfplane case the important trace identity was derived by Bukhgeim and Kardakov in [3]. The following crucial proposition follows quickly from their trace identity.

Proposition 1 *Suppose $\delta > 0$, n is an odd integer, $n \geq 1$, and $w(x,t)$ is the solution of*

$$\Box w = 0, \qquad \text{on } R^n \times [0,T],$$

$$w(.,t{=}0) = f,\ \ w_t(.,t{=}0) = 0$$

where $f \in C_0^\infty(R^n)$, f even in x_n, and f supported in $|x_n| \geq \delta$. Then

$$\int_0^T \int_H w^2(p,t)\, dS_p\, dt \leq \frac{T}{2\delta} \|f\|_0^2\ . \tag{40}$$

Proof of Proposition 1
Extending the solution past $t = T$, we have $w(x,t)$ is the solution of the IVP

$$\Box w = 0, \qquad \text{on } R^n \times [0,\infty),$$

$$w(.,t{=}0) = f, \;\; w_t(.,t{=}0) = 0$$

where f is even. Hence w is even in x_n and so $w_n = 0$ on $H \times [0,\infty)$. For odd $n \geq 3$, in [3], it is shown that

$$\int_0^\infty \int_H \frac{w^2(p,t)}{t} \, dS_p \, dt = \int_{R^n_+} \frac{f^2(x)}{x_n} \, dx \; .$$

This is also valid when $n = 1$ as may be verified by using the explicit solution - here $\int_H h(p)\, dS_p$ is to mean $h(0)$. Since f is supported in $|x_n| \geq \delta$, we have

$$\begin{aligned} \int_0^T \int_H w^2(p,t) \, dS_p \, dt \;\; &\leq \;\; T \int_0^T \int_H \frac{w^2(p,t)}{t} \, dS_p \, dt \;\leq\; T \int_0^\infty \int_H \frac{w^2(p,t)}{t} \, dS_p \, dt \\ &= \;\; T \int_{R^n_+} \frac{f^2(x)}{x_n} \, dx \;\leq\; \frac{T}{\delta} \int_{R^n_+} f^2(x) \, dx \;=\; \frac{T}{2\delta} \|f\|_0^2, \end{aligned}$$

proving the proposition. QED

We will need some energy identities for solutions of IBVP for the wave equation in the region $R^n_+ \times [0,T]$. These estimate the L^2 norm of u_t and u_n on $H \times [0,T]$ in terms of all other boundary and initial data, and the L^2 norm of u_i, $i \neq n$, in terms of all other boundary and initial data. The proof uses the multipliers u_i and u_t and may be found in standard books dealing with IBVP for the wave equation.

Proposition 2 *Suppose $n \geq 1$, $f, g \in C_0^\infty(R^n)$, $F \in C_0^\infty(R^n \times [0,T])$, and u is the solution of the IVP*

$$\Box u = F \qquad R^n \times [0,T],$$

$$u(.,t{=}0) = f, \;\; u_t(.,t{=}0) = g \; .$$

Then

$$\int_{H\times[0,T]} u_1^2 + \cdots + u_{n-1}^2 \; dS_p \, dt \leq C \left(\|f\|_1^2 + \|g\|_0^2 + \int_0^T \|F(.,t)\|_0^2 \, dt + \int_{H\times[0,T]} u_t^2 + u_n^2 \; dS_p \, dt \right), \tag{41}$$

$$\int_{H\times[0,T]} u_t^2 + u_n^2 \; dS_p \, dt \leq C \left(\|f\|_1^2 + \|g\|_0^2 + \int_0^T \|F(.,t)\|_0^2 \, dt + \int_{H\times[0,T]} u_1^2 + \cdots + u_{n-1}^2 \; dS_p \, dt \right), \tag{42}$$

with C independent of f, g, F.

Consider the three cases where $\mathcal{L} = \Box$ and two of the three functions f, g, F are zero. If f, g, F were odd functions of x_n, then $u(x,t)$ would also be an odd function of x_n. Hence u, u_t, and u_i, $i \neq n$, would be zero on $H \times [0,T]$ and (18) follows right away from (42). Since every function may be written as the sum of an odd and even function in a unique way with the norms of the even and odd parts bounded by the corresponding norms of the original function, we need only deal with the case where f, g, F are even in x_n and only one of the f, g, F is non-zero.

If f, g, and F are even, then u is even and hence $u_n = 0$ on $S \times [0,T]$. So to prove (18) it is enough to estimate the u_t and u_i, $i \neq n$ terms on the LHS of (18). Because of (42), it is actually enough to estimate just the u_i, $i \neq n$ terms on the LHS of (18).

We also note that the $|u|^2$ term on the LHS of (18) may be estimated by the RHS using standard estimates for the solutions of the wave operator and the estimates for the trace of an H^1 function on a hypersurface. So below we will focus on estimating the other terms on the LHS of (18).

Next, we show, using the method of descent, that the n even case follows from the n odd case. Suppose Theorem 4 is valid for all odd $n \geq 1$. Now, suppose $n \geq 2$ is an even integer and the hypothesis of Theorem 4 is valid and $u(x,t)$ is the solution of (16), (17). Choose $\chi(z) \in C_0^\infty(R)$ with $\chi(z) = 1$ for $|z| \leq T+1$. Let $v(x,z,t)$ be the solution of the IVP

$$v_{tt} - \Delta_x v - v_{zz} = F(x,t)\chi(z) \qquad (x,z) \in R^{n+1},\ t \in [0,T] \tag{43}$$

$$v(x,z,t{=}0) = f(x)\chi(z),\ v_t(x,z,t{=}0) = g(x)\chi(z)\ . \tag{44}$$

Since the initial data and the RHS are smooth and have compact support, and $n+1$ is odd, we have from Theorem 4 for the odd dimensional case

$$\begin{aligned}
&\int_0^T \int_{-\infty}^{\infty} \int_H \left(|v(p,z,t)|^2 + |v_t(p,z,t)|^2 + |\nabla_x v(p,z,t)|^2 + |v_z(p,z,t)|^2\right) dS_p\, dz\, dt \\
&\leq\ C \int_{-\infty}^{\infty} \left(\|f(.)\chi(z)\|_1^2 + \|g(.)\chi(z)\|_0^2 + \int_0^T \|F(.,t)\chi(z)\|_0^2\, dt \right) dz \\
&\leq\ C_1 \left(\|f(.)\|_1^2 + \|g(.)\|_0^2 + \int_0^T \|F(.,t)\|_0^2\, dt \right)\ .
\end{aligned} \tag{45}$$

Now the distance of the subset $|z| \leq 1$ of $R^{n+1}_{x,z}$ from the subset $|z| \geq T+1$ of $R^{n+1}_{x,z}$ is T. Hence, from a domain of dependence argument, $v(x,z,t)$ for $|z| \leq 1$, $x \in R^n$, and $t \in [0,T]$, would not change if $\chi(z) = 1$ for all $z \in R$. So $v(x,z,t) = u(x,t)$ for $|z| \leq 1$, $x \in R^n$, and $t \in [0,T]$. Hence, if, on the LHS of (45), the z integral is taken only over the interval $[-1,1]$, we obtain

$$2\int_0^T \int_H \left(|u(p,t)|^2 + |u_t(p,t)|^2 + |\nabla_x u(p,t)|^2\right) dS_p\, dt \ \leq\ C_1 \left(\|f(.)\|_1^2 + \|g(.)\|_0^2 + \int_0^T \|F(.,t)\|_0^2\, dt \right)\ .$$

So Theorem 4 is valid in the n even case.

Summarizing, to prove Theorem 4, we may assume that n is odd, $\mathcal{L} = \Box$, only one of f, g, F is non-zero, and f, g, F are smooth and even in x_n. Further, we only need to establish the inequality

$$\int_0^T \int_H u_1(p,t)^2 + \cdots + u_{n-1}(p,t)^2\, dS_p\, dt \leq C \left(\|f\|_1^2 + \|g\|_0^2 + \int_0^T \|F(.,t)\|_0^2\, dt \right), \tag{46}$$

with C independent of f, g, F, where u is the solution of the IVP (16), (17).

4.1 Case where f even and only $f \neq 0$

Now u_i, $i \neq n$ is also the solution of the wave equation except its initial data is

$$u_i(., t{=}0) = f_i, \; u_{it}(., t{=}0) = 0 \; .$$

Further, f_i, $i \neq n$ is also even in x_n because f is. Hence, Proposition 1 applied to u_i gives

$$\int_0^T \int_H |u_i|^2(p,t) \, dS_p \, dt \leq C \|f_i\|_0^2, \qquad i \neq n \; . \tag{47}$$

Since $u_n = 0$ on H (u is even in x_n), using (42) and (47), we obtain (18) for the case where only f is non-zero.

4.2 Case where g even and only $g \neq 0$

So $u(x,t)$ is the solution of the IVP

$$\Box u = 0, \qquad \text{on } R^n \times [0,T]$$

$$u(., t{=}0) = 0, \; u_t(., t{=}0) = g$$

where g is assumed to be even and supported in $|x_n| \geq \delta$. Define $w = u_t$. Then w satisfies the wave equation and further the IC for w are

$$w(., t{=}0) = u_t(., t{=}0) = g, \; w_t(., t{=}0) = u_{tt}(., t{=}0) = \Delta u(., t{=}0) = 0 \; .$$

Hence Proposition 1 applied to this w gives us

$$\int_0^T \int_H |u_t(p,t)|^2 \, dS_p \, dt \leq C \|g\|_0^2 \; .$$

Since g is even, then u is even in x_n and hence $u_n = 0$ on $H \times [0, \infty)$. So the above inequality combined with (41) allows us to estimate u_i, $i \neq n$, and hence proves (18) in the special case under consideration.

4.3 Case where F is even and only $F \neq 0$

So $u(x,t)$ is the solution of

$$\Box u = F(x,t) \qquad x \in R^n, \; t \in [0,T]$$

$$u(., t{=}0) = 0, \; u_t(., t{=}0) = 0 \; .$$

where $F(x,t)$ is even in x_n and $F(.,t)$ is supported in $|x_n| \geq \delta$ for all $t \in [0,T]$. Then u_t is the solution of the IVP

$$\Box u_t = F_t(x,t) \qquad x \in R^n, \ t \in [0,T]$$

$$u_t(.,t{=}0) = 0, \ (u_t)_t(.,t{=}0) = F(.,t{=}0) \ .$$

One may verify that

$$u_t(x,t) = \int_0^t w(x,t;s)\, ds \tag{48}$$

where $w(x,t;s)$, $t \geq s \geq 0$, is the solution of the IVP

$$\Box_{x,t} w = 0 \qquad x \in R^n, \ t \in [s,T],$$

$$w(.,s;s) = F(.,s), \ w_t(.,s;s) = 0 \ .$$

Note that $F(.,s)$ is even in x_n and supported in $|x_n| \geq \delta$. Hence Proposition 1 applied to $w(x,t;s)$ gives us

$$\int_s^T \int_H |w(p,t;s)|^2 \, dS_p \, dt \leq \frac{T-s}{2\delta} \|F(.,s)\|_0^2 \leq \frac{T}{2\delta} \|F(.,s)\|_0^2 \ . \tag{49}$$

Hence from (48), the Cauchy-Schwartz inequality, and (49)

$$\begin{aligned}
\int_H \int_0^T |u_t(p,t)|^2 \, dt \, dS_p &= \int_H \int_0^T \left| \int_0^t w(p,t;s)\, ds \right|^2 dt\, dS_p \\
&\leq \int_H \int_0^T t \int_0^t |w(p,t;s)|^2 \, ds\, dt\, dS_p \\
&\leq T \int_H \int_0^T \int_0^t |w(p,t;s)|^2 \, ds\, dt\, dS_p \\
&= T^2 \int_H \int_0^T \int_s^T |w(p,t;s)|^2 \, dt\, ds\, dS_p \\
&\leq \frac{T^3}{2\delta} \int_0^T \|F(.,s)\|_0^2 \, ds \ .
\end{aligned}$$

So we have estimated the u_t term in the LHS of (18); further u_n is zero on $H \times [0,T]$. Hence, using (41), we may estimate all the terms on the LHS of (18) thus proving Theorem 4.

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