The spherical mean value operator with centers on a sphere

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Abstract

Let B represent the ball of radius ρ in R^n and S its boundary; consider the map $\mathcal{M} : C_0^\infty(\overline{B}) \to C^\infty(S \times [0, \infty))$ where

$$(\mathcal{M}f)(p, r) = \frac{1}{\omega_{n-1}} \int_{|\theta|=1} f(p + r\theta)\, d\theta$$

represents the mean value of f on a sphere of radius r centered at p. We summarize and discuss the results concerning the injectivity of $\mathcal{M}$, the characterization of the range of $\mathcal{M}$, and the inversion of $\mathcal{M}$. There is a close connection between mean values over spheres and solutions of initial value problems for the wave equation. We also summarize the results for the corresponding wave equation problem.

Key words: spherical mean values, wave equation

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1 Introduction

Recovering a function from its mean values over a family of spheres has a long history. John in [22] studied the case when the centers of the spheres are restricted to a plane; a very nice analysis of this problem was done by Bukhgeim and Kardakov in [9] and additional results are available in [13], [7] and [33]. Cormack and Quinto in [10] and Yagle in [37] studied the recovery of f from the mean values of f over spheres passing through a fixed point. The problem of inverting the spherical means transform restricted to the variety of spheres tangent to a hypersurface was approached by Goncharov in [20] using techniques from D-module theory. A more detailed study of this problem can be found in the book [30] by Palamodov. Recovering a function from its mean values over general families of surfaces has also received a lot of attention and we direct the interested reader to [30] and [23].

Motivated by applications in thermoacoustic tomography, the problem of recovering a function, supported in a ball, from its mean values over spheres centered on the boundary of the ball, has attracted considerable attention in the last decade with results appearing in various articles over this decade. Our goal in this article is to summarize and discuss the main theoretical results for this problem. We leave it to more knowledgeable people to review the results for the important issue of the design and implementation of numerical schemes for the inversion.

For any integer $n > 1$ and positive real number ρ, B_ρ will denote the origin centered open ball of radius ρ in R^n and S_ρ will denote its boundary; also ∂_n will denote the (outward pointing) normal derivative for a region. Let $C_0^\infty(\overline{B_\rho})$ denote the set of smooth functions on R^n with support in $\overline{B_\rho}$. Let $\tilde{C}(S_\rho \times [0,\infty))$ denote the set of smooth functions $F(p,t)$ on $S_\rho \times [0,\infty)$ which *are zero to infinite order in t at $t=0$*. Define the mean value operator

$$\begin{aligned} \mathcal{M} : C_0^\infty(\overline{B_\rho}) &\to \tilde{C}(S_\rho \times [0,\infty)) \\ (\mathcal{M}f)(p,t) &= \frac{1}{\omega_{n-1}} \int_{|\theta|=1} f(p+t\theta)\, d\theta, \qquad p \in S_\rho,\ t \geq 0, \end{aligned}$$

where ω_{n-1} represents the surface area of the unit sphere in R^n. So $(\mathcal{M}f)(p,t)$ represents the mean value of f on a sphere of radius t centered at $p \in S_\rho$. The definition makes sense even if t is negative and $(\mathcal{M}f)(p,t)$ is an even function of t with this extended definition.

For any $f \in C_0^\infty(\overline{B_\rho})$, let $v(x,t)$ be the solution of the following initial value problem for the wave equation -

$$v_{tt} - \Delta v = 0 \qquad (x,t) \in R^n \times R \tag{1}$$

$$v(x,0) = 0, \quad v_t(x,0) = f(x), \qquad x \in R^n. \tag{2}$$

Since this problem is well posed, the solution is smooth, and v is odd in t. We define a map

$$\begin{aligned} \mathcal{V} : C_0^\infty(\overline{B_\rho}) &\to \tilde{C}(S_\rho \times [0,\infty)) \\ (\mathcal{V}f)(p,t) &= v(p,t), \qquad p \in S_\rho,\ t \geq 0. \end{aligned}$$

In this article, we summarize the results regarding the injectivity, inversion and the range of $\mathcal{M}$ and $\mathcal{V}$. In the sections on inversion formulas, we have not attempted to state the minimum regularity of f required for the formulas to be valid. We have proved the inversion formulas for the case when f is C^∞ but the derivations clearly go through for less regular f. Also, the trace identities in [17], which are equivalent to the inversion formulas, suggest a much weaker regularity requirement on f for the inversion formulas to be valid.

$\mathcal{M}$ and $\mathcal{V}$ are closely related and results for one will imply results for the other. In fact from page 682 in [11] we have

$$(\mathcal{V}f)(p,t) = v(p,t) = \frac{1}{(n-2)!}\left(\frac{\partial}{\partial t}\right)^{n-2}\int_0^t (t^2-r^2)^{(n-3)/2} r\,(\mathcal{M}f)(p,r)\,dr. \tag{3}$$

From (3) it is clear that knowledge of $(\mathcal{M}f)(p,t)$ on a subset $\Gamma \times [0,T]$ of $S_\rho \times R$ will determine $(\mathcal{V}f)(p,t)$ on $\Gamma \times [0,T]$ (and vice versa). Hence, to a large extent, the uniqueness results for $\mathcal{M}$ and $\mathcal{V}$ are identical. The injectivity of $\mathcal{M}$ and $\mathcal{V}$ are consequences of several more general results about solutions of the wave equation; we state one of them.

Theorem 1 (Injectivity). *Suppose Γ is a relatively open subset of S_ρ, $T > 2\rho$, $f \in C_0^\infty(\overline{B_\rho})$ and v is the solution of (1), (2). If $v(p,t)$ is zero on $\Gamma \times [0,T]$ then $f = 0$.*

Theorem 1, for $T = \infty$ is a consequence of a general result in [3] about injectivity sets for solutions of the wave equation - see also [26] and [4]. Theorem 1 was proved in [18] for $T = \infty$ (for any strictly convex domain) but the proof goes through for any T more than the diameter of the domain. [18] has another uniqueness theorem which asserts that $f \in C_0^\infty(\overline{B_\rho})$ may be recovered from a knowledge of $(\mathcal{M}f)(p,t)$ or $(\mathcal{V}f)(p,t)$ for all $(p,t) \in S_\rho \times [0,\rho]$ (instead of $[0,2\rho]$). There are many interesting results regarding injectivity sets for solutions of the wave equation and [6] is an excellent reference for these results.

Norton in [27] derived an inversion formula for $\mathcal{M}$, for the $n = 2$ case, involving a series expansion in Bessel functions. Norton and Linzer in [28] give an inversion formula for the $n = 3$ case for $\mathcal{M}$ and hence for $\mathcal{V}$, again, as a series expansion in spherical harmonics. Inversion formulas for $\mathcal{M}$ and $\mathcal{V}$ which do not involve series but are instead given as integrals (and are of back projection type) were derived in [18] for all odd n and in [15] for all even n. Kunyansky in [14] has also derived an inversion formula of back projection type for $\mathcal{M}$, valid for all $n > 1$, and the formula seems quite different from the inversion formulas in [18] and [15]. For the $n = 3$ case, in [36], Xu and Wang give another inversion formula of the back projection type (Kunyansky's formula is equivalent to theirs for $n = 3$), and in [19] a formula is stated to recover a function f from traces of the normal derivative $\partial_n v$ on $S_\rho \times [0,2\rho]$.

Some necessary conditions for the ranges of $\mathcal{M}$ and $\mathcal{V}$ are present in [24], [25], [2] (also see [8]) and were explicitly formulated in [31] for $n = 3$. When $n = 2$, the range of $\mathcal{M}$ was characterized in [5]; for odd n, the range of $\mathcal{V}$ (and implicitly the range of $\mathcal{M}$ - see section 4.3) was characterized in [16]. Guided by the results and some of the techniques in [5] and

[16], the range of $\mathcal{M}$ was explicitly characterized *for all* n in [1]. The explicit characterization of the range of $\mathcal{V}$, for even n, is still an open question.

We define the operator

$$\begin{aligned} \mathcal{D} : \tilde{C}(S_\rho \times [0,\infty)) \to \tilde{C}(S_\rho \times [0,\infty)) \\ (\mathcal{D}F)(p,t) = \left(\frac{1}{2t}\frac{\partial}{\partial t}\right) F(p,t), \qquad p \in S_\rho,\ t \geq 0. \end{aligned}$$

Relations more useful than (3) are (see page 682 in [11] and page 80 in [12])

$$(n \text{ odd}) \qquad (\mathcal{V}f)(p,t) = v(p,t) = \frac{\sqrt{\pi}}{2\Gamma(n/2)} \mathcal{D}^{(n-3)/2} t^{n-2} (\mathcal{M}f)(p,t), \tag{4}$$

$$(n \text{ even}) \qquad (\mathcal{V}f)(p,t) = v(p,t) = \frac{1}{\Gamma(n/2)} \mathcal{D}^{(n-2)/2} \int_0^t \frac{r^{n-1}(\mathcal{M}f)(p,r)}{\sqrt{t^2-r^2}}\, dr. \tag{5}$$

Because of the relationship between $\mathcal{V}$ and $\mathcal{M}$, it will be convenient to define $\mathcal{N} = t^{n-2}\mathcal{M}$, that is

$$\begin{aligned} \mathcal{N} : C_0^\infty(\overline{B_\rho}) \to \tilde{C}(S_\rho \times [0,\infty)) \\ (\mathcal{N}f)(p,t) = t^{n-2}(\mathcal{M}f)(p,t), \qquad p \in S_\rho,\ t \geq 0 \end{aligned}$$

whose formal L^2 adjoint (see [18]) is

$$\begin{aligned} \mathcal{N}^* : \tilde{C}(S_\rho \times [0,\infty)) \to C^\infty(R^n) \\ (\mathcal{N}^*F)(x) = \frac{1}{\omega_{n-1}} \int_{|p|=\rho} \frac{F(p,|p-x|)}{|p-x|}\, dS_p. \end{aligned}$$

Also, from chapter 4 in [34] we know that $L^2(S_1)$ has an orthonormal basis $\{\phi_m(.)\}_{m=1}^\infty$ consisting of homogeneous harmonic polynomials - let $k(m)$ be the degree of homogeneity of $\phi_m(.)$.

2 Results for $\mathcal{M}$

2.1 Inversion

The inversion formulas are different for the odd and even n cases and we state them separately.

Theorem 2 (Inversion for odd n). *If $n > 1$ is odd and $f \in C_0^\infty(\overline{B_\rho})$, then*

$$f(x) = \frac{c_n}{\rho}\,(\mathcal{N}^*\, t\, \mathcal{D}^{(n-3)/2}\, t^{-1}\, \partial_t^2\, t\, D^{(n-3)/2}\, t^{n-2}\, \mathcal{M}f)(x), \qquad x \in B_\rho,$$

$$f(x) = \frac{c_n}{\rho}\,(\mathcal{N}^*\, t\, \mathcal{D}^{(n-3)/2}\, t^{-1}\, \partial_t\, t\, \partial_t\, D^{(n-3)/2}\, t^{n-2}\, \mathcal{M}f)(x), \qquad x \in B_\rho,$$

$$f(x) = \frac{c_n}{\rho}\,\Delta_x\,(\mathcal{N}^*\, t\, \mathcal{D}^{n-3}\, t^{n-2}\, \mathcal{M}f)(x), \qquad x \in B_\rho,$$

where $c_n = \frac{(-1)^{(n-1)/2}\,\pi}{2\,\Gamma(n/2)^2}$.

These were derived in [18] by an explicit computation when $n = 3$ and then the higher n cases were deduced by the use of spherical harmonic expansions. Corollary 3.5 in [21] discusses, for $n = 3$, some situations when the above inversion formulas are true for less smoother f.

When $n = 3$ another interesting inversion formula was derived by Xu and Wang in [36]. A different derivation, based on the inversion formulas in Theorem 2, is given in subsection 4.1.

Theorem 3 (Inversion for $n = 3$). *If $n = 3$ and $f \in C_0^\infty(\overline{B_\rho})$ then*

$$f(x) = \frac{1}{2\pi\rho}\nabla_x \cdot \int_{|p|=\rho} p\,\frac{(\partial_t\, t\, \mathcal{M}f)(p, |x-p|)}{|x-p|}\, dS_p, \qquad x \in B_\rho.$$

When n is even, the inversion formulas are more complicated. For the odd n case, inversion requires an integration only over the surface S_ρ (but uses the values of $\mathcal{M}f$ on $S_\rho \times [0, 2\rho]$) where as even dimensional inversion formulas require an integration over $S_\rho \times [0, 2\rho]$ and with a kernel different from the one used for the odd dimensional case.

Theorem 4 (Inversion for even n). *If n is even and $f \in C_0^\infty(\overline{B_\rho})$ then for all $x \in B_\rho$*

$$f(x) = \frac{1}{c_n\rho}\Delta_x \int_{|p|=\rho}\int_0^{2\rho} \log\left|t^2 - |x-p|^2\right| \left(t\mathcal{D}^{n-2}t^{n-2}\mathcal{M}f\right)(p,t)\, dt\, dS_p,$$
$$f(x) = \frac{2}{c_n\rho}\int_{|p|=\rho}\int_0^{2\rho} \log\left|t^2 - |x-p|^2\right| \left(t\mathcal{D}^{n-1}t^{n-1}\partial_t\,\mathcal{M}f\right)(p,t)\, dt\, dS_p,$$

where

$$c_n = (-1)^{(n-2)/2}\, 2((n-2)/2)!\, \pi^{n/2} = (-1)^{(n-2)/2}\left[((n-2)/2)!\right]^2 \omega_{n-1}.$$

These formulas may be simplified a little when $n = 2$.

Theorem 5 (Inversion for $n = 2$). *Suppose $n = 2$ and $f \in C_0^\infty(\overline{B_\rho})$ and, for $t < 0$, redefine $(\mathcal{M}f)(p,t)$ to be an odd function of t. Then for all $x \in B_\rho$*

$$f(x) = \frac{1}{2\pi\rho} \int_{|p|=\rho} \int_{-2\rho}^{2\rho} \frac{(t\,\partial_t \mathcal{M}f)(p,t)}{|x-p|-t}\, dt\, ds_p,$$
$$f(x) = \frac{1}{2\pi\rho} \int_{|p|=\rho} |x-p| \int_{-2\rho}^{2\rho} \frac{(\partial_t \mathcal{M}f)(p,t)}{|x-p|-t}\, dt\, ds_p,$$

where the inner integrals are to be computed as principal values.

Theorems 4 and 5 were proved in [15] where first Theorem 4 was proved for $n = 2$ via an integral identity, the higher n case was proved by a reduction using a spherical harmonic expansion, and Theorem 5 was derived from Theorem 4.

Kunyansky in [14] has derived an inversion formula for $\mathcal{M}$ valid for all $n > 1$. Let $J(t) = t^{(2-n)/2} J_{(n-2)/2}(t)$ and $N(t) = t^{(2-n)/2} N_{(n-2)/2}(t)$ where $J_{(n-2)/2}(t)$ and $N_{(n-2)/2}(t)$ are the Bessel and Neumann functions of order $(n-2)/2$. For any function $F \in \tilde{C}(S_\rho \times [0,\infty))$ which is zero for large t, define

$$(\mathcal{L}F)(p,t) = \int_0^\infty s^{2n-3} \left(N(st) \int_0^{2\rho} J(st')\, t'^{n-1}\, F(p,t')\, dt' - J(st) \int_0^{2\rho} N(st')\, t'^{n-1}\, F(p,t')\, dt' \right) ds.$$

Theorem 6 (Inversion for all $n > 1$). *Suppose $n > 1$ and $f \in C_0^\infty(\overline{B_\rho})$. Then*

$$f(x) = \frac{1}{4(2\pi)^{n-1}\rho} \nabla_x \cdot \int_{|p|=\rho} p\,(\mathcal{L}\mathcal{M}f)(p, |x-p|)\, dS_p, \qquad \text{for all } x \in B_\rho.$$

2.2 Range descriptions

For any f in $C_0^\infty(\overline{B_\rho})$, let $F(x,t)$ be the mean value of f on a sphere of radius t, centered at x, that is

$$F(x,t) = \frac{1}{\omega_{n-1}} \int_{|\theta|=1} f(x+t\theta)\, d\theta, \qquad (x,t) \in R^n \times R.$$

Then $\mathcal{M}f$ is the restriction of $F(x,t)$ to $S_\rho \times [0,\infty)$. It is well known (see [11]) that $F(x,t)$ satisfies the Euler-Poisson-Darboux equation

$$F_{tt} + \frac{n-1}{t} F_t = \Delta_x F, \qquad x \in R^n,\ t > 0.$$

Also, $F(x,0) = f(x)$, and because of the support of f, we have $F(x,t)$ is zero for $t \geq 2\rho$. Finally, $F(x,t)$ is even in t so $F_t(x,0) = 0$ on R^n. These properties suggest the following inversion scheme for $\mathcal{M}$.

For any function $F(p,t) \in \tilde{C}(S_\rho \times [0,\infty))$ with $F(.,t)=0$ for $t \geq 2\rho$, let $u(x,t)$ be the solution of the backward initial boundary value problem

$$u_{tt} + \frac{n-1}{t} u_t = \Delta_x u, \qquad (x,t) \in \overline{B_\rho} \times (0,2\rho], \tag{6}$$

$$u(x,2\rho) = 0,\ \ u_t(x,2\rho) = 0, \qquad x \in \overline{B_\rho}, \tag{7}$$

$$u(x,t) = F(x,t), \qquad (x,t) \in S_\rho \times (0,2\rho]. \tag{8}$$

Since the problem is well posed, there is a unique smooth solution of (6)-(8) in $\overline{B_\rho} \times (0,2\rho]$.

If $F(p,t) = (\mathcal{M}f)(p,t)$ for some $f \in C_0^\infty(\overline{B_\rho})$ then clearly $F(x,t)$ - the mean value of f for $(x,t) \in R^n \times R$ - is the solution of (6)-(8), and $F(x,0) = f(x)$ for $x \in B_\rho$. Also $F_t(x,0) = 0$ for $x \in B_\rho$ because $F(x,t)$ is even in t - this simple observation turns out to be the key condition in the characterization of the range of $\mathcal{M}$.

For a general $F \in \tilde{C}(S_\rho \times [0,\infty))$, if $u(x,t)$ is the solution of (6)-(8) then $\lim_{t\to 0^+} u_t(x,0)$ may not be zero for $x \in B_\rho$ but it is zero if F is in the range of $\mathcal{M}$. The surprising result is that this condition, combined with a moment condition (not needed when n is odd), is also sufficient for F to be in the range of $\mathcal{M}$. A result similar to this was proved for the operator $\mathcal{V}$, for odd n, in [16]. This, combined with a range characterization of $\mathcal{M}$ in [5] for $n=2$ and an observation in [31], suggested some of the tools and motivated the characterization of the range of $\mathcal{M}$ for all $n>1$ in [1].

Theorem 7 (Range description for odd n). *Suppose $n>1$ is an odd integer. A function $F(p,t) \in \tilde{C}(S_\rho \times [0,\infty))$ is in the range of $\mathcal{M}$ iff $F(.,t)=0$ for $t \geq 2\rho$ and one of the following two conditions is satisfied.*

1. *For any Dirichlet eigenfunction $\psi(x)$ of the Laplacian on B_ρ,*

$$\lim_{t\to 0^+} \int_{B_\rho} u_t(x,t)\, \psi(x)\, dx = 0.$$

2. *$\hat{F}_m(\tau) = 0$ for every $m = 1,2,\cdots$ and every non-zero root τ of $J_{(n+2k(m)-2)/2}(\rho\tau) = 0$, where*

$$\hat{F}_m(\tau) = \int_0^{2\rho} \int_{|\theta|=1} F(\rho\theta,t)\, t^{n/2}\, J_{n/2-1}(\tau t)\, \phi_m(\theta)\, d\theta\, dt$$

represents the Hankel transform of the mth coefficient in the spherical harmonic expansion of F.

Functions in the range of $\mathcal{M}$ satisfy a moment condition observed by Patch for $n=3$ (see [31]). When n is odd this condition is indirectly implied by the conditions in Theorem 7. When n is even, it is not known whether the moment condition is implied by the conditions in Theorem 7 though it is believed that the moment condition is not implied. We state the moment condition in a form equivalent to the one stated in [31] and [1].

A function $F(p,t) \in \tilde{C}(S_\rho \times [0,\infty))$ which is zero for $t \geq 2\rho$, is said to satisfy the **moment condition** if

$$\int_0^{2\rho} \int_{|\theta|=1} t^{2l+n-1} \phi_m(\theta) F(\rho\theta, t)\, d\theta\, dt = 0 \tag{9}$$

for all $m = 1, 2, \cdots$ and all non-negative integers l strictly less than $k(m)/2$.

Theorem 8 (Range description for even n). *Suppose n is an even integer. A function $F(p,t) \in \tilde{C}(S_\rho \times [0,\infty))$ is in the range of $\mathcal{M}$ iff $F(.,t) = 0$ for $t \geq 2\rho$, F satisfies the moment condition, and F satisfies one of the two conditions in Theorem 7.*

Palamodov in [29] has given an interesting analysis of the range of $\mathcal{M}$ using the theory of Fourier Integral Operators (FIO) - see Corollary 8.1 in [29]. Suppose K is a compact subset of the open ball B_ρ and $f \in C_0^\infty(\overline{B_\rho})$ has support in K. Then for any $(p,t) \in S_\rho \times (0,\infty)$,

$$(\mathcal{M}f)(p,t) = \frac{1}{\omega_{n-1}\, t^{n-1}} \int_{R^n} f(x)\, \delta(t - |p-x|)\, dx \tag{10}$$

may be written as a FIO; note that because f is supported in $K \Subset B_\rho$, $(\mathcal{M}f)(p,t) = 0$ for t near *zero*. From the L^2 theory of FIO, the fact that $\mathcal{M}^*\mathcal{M}$ is a pseudodifferential operator, and the injectivity of $\mathcal{M}$, one may obtain an estimate

$$\|f\|_{H^\alpha} \leq C_{1,\alpha,K} \|\mathcal{M}f\|_{H^{\alpha+(n-1)/2}} \leq C_{2,\alpha,K} \|f\|_{H^\alpha}$$

for all $f \in C_0^\infty(\overline{B_\rho})$ with support in K and for all real α; here H^α represents the Sobolev norm. This proves that for $\mathcal{M}$ defined on functions in $H^\alpha(B_\rho)$ with support in K, the range is a closed subset of $H^{\alpha+(n-1)/2}(S_\rho \times (0,\infty))$. Hence the range of $\mathcal{M}$ will be the orthogonal complement of the kernel of $\mathcal{M}^*$.

From (10), for any smooth function $F(p,t)$ on $S_\rho \times (0,\infty)$ supported away from $t = 0$, we have

$$(\mathcal{M}^*F)(x) = \frac{1}{\omega_{n-1}} \int_{|p|=\rho} \int_0^\infty \frac{F(p,t)}{t^{n-1}}\, \delta(t - |p-x|)\, dt\, dS_p = \frac{1}{\omega_{n-1}} \int_{|p|=\rho} \frac{F(p, |p-x|)}{|p-x|^{n-1}}\, dS_p, \qquad x \in K.$$

This motivates the following result of Palamodov.

Theorem 9 (Range characterization). *Suppose $n > 1$, α a real number, and K is a compact subset of the open ball B_ρ. Let $\mathcal{M}$ be restricted to functions in $H^\alpha(B_\rho)$ with support in K. Then the range of $\mathcal{M}$ is the set of functions in $H^{\alpha+(n-1)/2}(S_\rho \times (0,\infty))$ which are supported away from $t = 0$ and which are orthogonal (in the $L^2(S_\rho \times (0,\infty))$ sense) to all compactly supported functions $F \in C^\infty(S_\rho \times (0,\infty))$ for which $(\mathcal{M}^*F)(x) = 0$ for all $x \in K$.*

3 Results for $\mathcal{V}$

3.1 Inversion

For n odd, the inversion formulas for $\mathcal{V}$ are closely related to the inversion formulas for $\mathcal{M}$ because of the fairly simple relationship between $\mathcal{V}$ and $\mathcal{M}$ given in (4).

Theorem 10 (Inversion for odd n). *If $n > 1$ is odd and $f \in C_0^\infty(\overline{B_\rho})$, then*

$$f(x) = c_n \, (\mathcal{N}^* \, \mathcal{D}^{(n-3)/2} \, t^{-1} \, \partial_t^2 \, t \, \mathcal{V} f)(x), \qquad x \in B_\rho,$$

$$f(x) = c_n \, (\mathcal{N}^* \, t \, \mathcal{D}^{(n-3)/2} \, t^{-1} \, \partial_t \, t \, \partial_t \, \mathcal{V} f)(x), \qquad x \in B_\rho,$$

$$f(x) = c_n \, \Delta_x \, (\mathcal{N}^* \, t \, \mathcal{D}^{(n-3)/2} \mathcal{V} f)(x), \qquad x \in B_\rho$$

where $c_n = \frac{(-1)^{(n-1)/2}\sqrt{\pi}}{\rho \, \Gamma(n/2)}$.

For $n = 3$, the inversion formula of Theorem 3 may be expressed as the following result.

Theorem 11 (Inversion for $n = 3$). *If $n = 3$ and $f \in C_0^\infty(\overline{B_\rho})$ then*

$$f(x) = \frac{1}{2\pi\rho} \nabla_x \cdot \int_{|p|=\rho} p \, \frac{(\partial_t \mathcal{V} f)(p, |x-p|)}{|x-p|} \, dS_p.$$

The above results recovered the initial data, f, from the trace of the solution v, of (1), (2), on $S_\rho \times [0, 2\rho]$. [19] announced an interesting formula, in the $n = 3$ case, for recovering f from the trace of the normal derivative of v on $S_\rho \times [0, 2\rho]$. We give a proof of this formula in section 4.1.

Theorem 12 (Inversion for $n = 3$ from Neumann traces). *If $n = 3$, $f \in C_0^\infty(\overline{B_\rho})$ and $v(x,t)$ is the solution of (1), (2) then*

$$|x| \, f(x) = -\frac{\rho}{2\pi} \int_{|x|}^{\rho} \int_{|p|=\rho} (\partial_t^2 \partial_n v)(p, |\sigma\theta - p|) \, dS_p \, d\sigma, \qquad \forall x \in B_\rho, \; x \neq 0.$$

Here $\theta = x/|x|$ and ∂_n stands for the (outward pointing) normal derivative.

When n is even, the inversion formulas take a slightly different form and because of the more complicated nature of the relationship between $\mathcal{V}$ and $\mathcal{M}$ - see (5) - the connection between the inversion formulas for $\mathcal{V}$ and $\mathcal{M}$ is not very direct. We state inversion formulas (derived in [15]) coming from trace identities for the wave equation. If f_i, $i = 1, 2$ are in $C_0^\infty(\overline{B_\rho})$ and v_i is the solution of (1), (2) with $f = f_i$ then

$$\int_{B_\rho} f_1(x)\, f_2(x)\, dx = \frac{-2}{\rho} \int_{S_\rho} \int_0^\infty t v_{1tt}(p,t)\, v_2(p,t)\, dt\, dS_p = \frac{2}{\rho} \int_{S_\rho} \int_0^\infty t v_{1t}(p,t)\, v_{2t}(p,t)\, dt\, dS_p.$$

These lead to inversion formulas using $\mathcal{V}^*$ - the formal L^2 adjoint of $\mathcal{V}$ - where (see subsection 4.4)

$$(\mathcal{V}^* F)(x) = \frac{(-1)^{(n-2)/2}}{\omega_{n-1}\,\Gamma(n/2)} \int_{|p|=\rho} \int_{|x-p|}^\infty \frac{1}{\sqrt{t^2 - |x-p|^2}}\, (t\mathcal{D}^{(n-2)/2} t^{-1})(F(p,t))\, dt\, dS_p$$

for any $F \in \tilde{C}(S_\rho \times [0, \infty))$ with adequate decay at infinity.

Theorem 13 (Inversion for even n). *If n is even and $f \in C_0^\infty(\overline{B_\rho})$, then*

$$f(x) = \frac{-2}{\rho} (\mathcal{V}^* t\, \partial_t^2\, \mathcal{V} f)(x), \qquad x \in B_\rho$$
$$f(x) = \frac{-2}{\rho} (\mathcal{V}^* \partial_t\, t\, \partial_t\, \mathcal{V} f)(x), \qquad x \in B_\rho.$$

This formula is somewhat unsatisfactory since the computation of $\mathcal{V}^*$ needs an integration over an infinite interval. However, one may recover $(\mathcal{M} f)(p,t)$ for $(p,t) \in S_\rho \times [0, 2\rho]$ from $(\mathcal{V} f)(p,t)$ for $(p,t) \in S_\rho \times [0, 2\rho]$ (see subsection 4.3), hence one may invert $\mathcal{V}$ using the inversion formulas for $\mathcal{M}$ using only finite integration in t. We state such a formula for $n = 2$ (obtained in [15]) and we leave it to reader to obtain a formula for even $n > 2$ using the results in subsection 4.3.

Theorem 14 (Inversion for $n = 2$). *If $n = 2$ and $f \in C_0^\infty(\overline{B_\rho})$, then*

$$f(x) = \frac{1}{\rho \pi^2} \Delta_x \int_{|p|=\rho} \int_0^{2\rho} (\partial_t \mathcal{V} f)(p,t)\, K(t, |x-p|)\, dt\, ds_p$$

where

$$K(t,s) = \int_t^{2\rho} \frac{r}{\sqrt{r^2 - t^2}}\, \log|r^2 - s^2|\, dr.$$

3.2 Range descriptions

If $n > 1$ is odd, $f \in C_0^\infty(\overline{B_\rho})$, and $v(x,t)$ is the solution of (1), (2), then $v(x,t) = 0$ for $x \in B_\rho$ if $|t| \geq 2\rho$; of course the restriction of v to $S_\rho \times [0, \infty)$ is $\mathcal{V} f$. This suggests an inversion scheme for $\mathcal{V}$. For any $F \in \tilde{C}(S_\rho \times [0,\infty))$ which is zero for $t \geq 2\rho$, let $q(x,t)$ be the solution of the initial boundary value problem

$$
\begin{aligned}
q_{tt} - \Delta_x q &= 0 \qquad && (x,t) \in B_\rho \times [0, 2\rho], && (11)\\
q(x, 2\rho) = 0, \quad q_t(x, 2\rho) &= 0 && x \in B_\rho, && (12)\\
q(x,t) &= F(x,t) && (x,t) \in S_\rho \times [0, 2\rho]. && (13)
\end{aligned}
$$

If $F = \mathcal{V}f$ for some $f \in C_0^\infty(\overline{B_\rho})$ then one observes that $q = v$ in $B_\rho \times [0,T]$; hence $q(x,0) = 0$ for $x \in B_\rho$. This condition turns out, also, to be sufficient for F to be in the range of $\mathcal{V}$. In [16], we proved the following theorem.

Theorem 15 (Range description for odd n). *Suppose $n > 1$ is an odd integer. A function $F(p,t) \in \tilde{C}(S_\rho \times [0,\infty))$ is in the range of $\mathcal{V}$ iff $F(.,t) = 0$ for $t \geq 2\rho$ and one of the following three conditions is satisfied.*

1. *$q(x,0) = 0$ for all $x \in B_\rho$.*
2. *$\hat{F}_m(\tau) = 0$ for every $m = 1, 2, \cdots$ and every positive root τ of $J_{(n+2k(m)-2)/2}(\tau\rho) = 0$, where*
$$\hat{F}_m(\tau) = \int_0^{2\rho} \int_{|\theta|=1} F(\rho\theta, t)\, \phi_m(\theta)\, \sin(\tau t)\, d\theta\, dt$$
represents the sine transform of the mth coefficient in the spherical harmonic expansion of F.
3. *F is in the kernel of $\mathcal{N}^*(\mathcal{D}^*)^{(n-3)/2}\partial_t$ or equivalently in the kernel of $\mathcal{N}^* t \mathcal{D}^{(n-1)/2}$ (see Theorem 3 in [16]).*

The range descriptions for $\mathcal{V}$ for even n are not known in any nice explicit manner.

4 Proofs and other details

4.1 Proof of Theorem 11

We give a proof of Theorem 11, different from the one in [36]. When $n = 3$, the inversion formulas of Theorem 10 take the form

$$f(x) = \frac{-1}{2\pi\rho} \int_{|p|=\rho} \frac{\partial_t^2 (tv(p,t))_{t=|x-p|}}{|x-p|} \, dS_p, \tag{14}$$

$$f(x) = \frac{-1}{2\pi\rho} \int_{|p|=\rho} \frac{\partial_t (tv_t(p,t))_{t=|x-p|}}{|x-p|} \, dS_p, \tag{15}$$

$$f(x) = \frac{-1}{2\pi\rho} \Delta_x \int_{|p|=\rho} v(p, |x-p|) \, dS_p \tag{16}$$

for every $x \in B_\rho$. Taking the difference of (14), (15), we obtain

$$0 = \int_{|p|=\rho} \frac{v_t(p, |x-p|)}{|x-p|} \, dS_p, \qquad x \in B_\rho. \tag{17}$$

Then from (16), for any $x \in B_\rho$, we have

$$\begin{aligned} f(x) &= \frac{-1}{2\pi\rho} \nabla \cdot \int_{|p|=\rho} \nabla_x \left(v(p, |x-p|) \right) \, dS_p = \frac{-1}{2\pi\rho} \nabla \cdot \int_{|p|=\rho} v_t(p, |x-p|) \frac{x-p}{|x-p|} \, dS_p \\ &= \frac{-1}{2\pi\rho} \nabla \cdot \left(x \int_{|p|=\rho} \frac{v_t(p, |x-p|)}{|x-p|} \, dS_p - \int_{|p|=\rho} p \, \frac{v_t(p, |x-p|)}{|x-p|} \, dS_p \right) \\ &= \frac{1}{2\pi\rho} \nabla \cdot \int_{|p|=\rho} p \, \frac{v_t(p, |x-p|)}{|x-p|} \, dS_p, \qquad \text{using (17).} \end{aligned}$$

4.2 Proof of Theorem 12

From (15) and (17), for every $x \in B_\rho$, we have

$$f(x) = \frac{-1}{2\pi\rho} \int_{|p|=\rho} v_{tt}(p, |x-p|) \, dS_p \tag{18}$$

$$= \frac{-\rho}{2\pi} \int_{|\theta|=1} v_{tt}(\rho\theta, |x-\rho\theta|) \, d\theta. \tag{19}$$

Applying $\rho^2 \frac{\partial}{\partial \rho}$ yields

$$
\begin{aligned}
0 = \rho^2 \int_{|\theta|=1} v_{tt}(\rho\theta, |x - \rho\theta|)\, d\theta + \rho^3 \int_{|\theta|=1} (\partial_n v_{tt})(\rho\theta, |x - \rho\theta|)\, d\theta \\
+ \rho^3 \int_{|\theta|=1} v_{ttt}(\rho\theta, |x - \rho\theta|)\, \frac{\langle x - \rho\theta, -\theta\rangle}{|x - \rho\theta|}\, d\theta.
\end{aligned}
$$

Using (19), this may be rewritten as

$$
0 = -2\pi\rho f(x) + \rho \int_{|p|=\rho} (\partial_n v_{tt})(p, |x - p|)\, dS_p - \int_{|p|=\rho} v_{ttt}(p|x - p|) \frac{\langle x - p, p\rangle}{|x - p|}\, dS_p. \tag{20}
$$

By differentiation under the integral in (18), we also have the relation

$$
-2\pi\rho x \cdot \nabla f(x) = \int_{|p|=\rho} v_{ttt}(p, |x - p|) \frac{\langle x, x - p\rangle}{|x - p|}\, dS_p. \tag{21}
$$

Subtracting (20) from (21) and noting that $\langle x, x - p\rangle + \langle x - p, p\rangle = |x|^2 - |p|^2$ gives

$$
\begin{aligned}
-2\pi\rho x \cdot \nabla f(x) = 2\pi\rho f(x) - \rho \int_{|p|=\rho} (\partial_n v_{tt})(p, |x - p|)\, dS_p \\
+ (|x|^2 - \rho^2) \int_{|p|=\rho} \frac{v_{ttt}(p, |x - p|)}{|x - p|}\, dS_p. \qquad (22)
\end{aligned}
$$

Now $w = v_{tt}$ satisfies (1) with initial conditions $w(\cdot, t = 0) = 0$ and $w_t(\cdot, t = 0) = \Delta f$ and so by (17) satisfies

$$
0 = \int_{|p|=\rho} \frac{v_{ttt}(p, |x - p|)}{|x - p|}\, dS_p.
$$

Thus the last term in (22) is zero, and the equation may be rewritten as

$$
x \cdot \nabla f(x) + f(x) = \frac{\rho}{2\pi} \int_{|p|=\rho} (\partial_n v_{tt})(p, |x - p|)\, dS_p.
$$

This may be rewritten as

$$
\frac{\partial}{\partial r}(r f(r\theta)) = \frac{\rho}{2\pi} \int_{|p|=\rho} (\partial_n v_{tt})(p, |r\theta - p|)\, dS_p.
$$

Integrating this over $[r, \rho]$ and noting the support of f we have

$$
r f(r\theta) = -\frac{\rho}{2\pi} \int_r^\rho \int_{|p|=\rho} (\partial_n v_{tt})(p, |\sigma\theta - p|)\, dS_p\, d\sigma
$$

which may be rewritten as

$$
|x|\, f(x) = -\frac{\rho}{2\pi} \int_{|x|}^\rho \int_{|p|=\rho} (\partial_t^2 \partial_n v)(p, |\sigma\theta - p|)\, dS_p\, d\sigma, \qquad \forall x \in B_\rho,\ x \neq 0.
$$

Here $\theta = x/|x|$.

4.3 From $\mathcal{V}f$ to $\mathcal{M}f$

In this subsection we

- suggest a relationship between the ranges of $\mathcal{V}$ and $\mathcal{M}$ for odd n;
- show, for even n, how $\mathcal{M}f$ may be recovered from $\mathcal{V}f$ on $S_\rho \times [0, 2\rho]$, for any $f \in C_0^\infty(\overline{B_\rho})$.

The map $\mathcal{D} : \tilde{C}(S_\rho \times [0, \infty)) \to \tilde{C}(S_\rho \times [0, \infty))$ is injective and its range is the subset of $\tilde{C}(S_\rho \times [0, \infty))$ consisting of functions F for which $\int_0^\infty F(p, t)\, dt = 0$ for all $p \in S_\rho$. Further, $\mathcal{D}$ maps functions which are zero for $t > 2\rho$ to such functions. Define the operator $\mathcal{I}$ for any smooth function on $S_\rho \times R$ via

$$(\mathcal{I}F)(p, t) = 2 \int_0^t sF(p, s)\, ds.$$

Then $(\mathcal{I}DF)(p, t) = F(p, t)$ for any smooth function $F(p, t)$ on $S_\rho \times R$ with $F(p, 0) = 0$ for all $p \in S_\rho$. So observing the relationship (4), one may obtain a relationship between the ranges of $\mathcal{V}$ and $\mathcal{M}$ for odd n.

We now show, for even n, how to recover $\mathcal{M}f$ on $S_\rho \times [0, 2\rho]$ from a knowledge of $\mathcal{V}f$ on $S_\rho \times [0, 2\rho]$. For $f \in C_0^\infty(\overline{B_\rho})$, and any $p \in S_\rho$, $(\mathcal{M}f)(p, t)$ is smooth and vanishes to infinite order in t at $t = 0$, for $p \in S$, and hence so does the Abel transform of $t^{n-2}(\mathcal{M}f)(p, t)$. Thus, from (5),

$$\Gamma(n/2)\mathcal{I}^{(n-2)/2}(\mathcal{V}f)(p, t) = \int_0^t \frac{r^{n-1}}{\sqrt{t^2 - r^2}} (\mathcal{M}f)(p, r)\, dr.$$

The inverse of the Abel operator on the right is well-known. Indeed,

$$\begin{aligned}\frac{\pi}{2} \int_0^r \sigma^{n-1} \mathcal{M}f(p, \sigma)\, d\sigma, &= \int_0^r \frac{1}{\sqrt{r^2 - t^2}} \int_0^t \frac{1}{\sqrt{t^2 - \sigma^2}} \sigma^{n-1} \mathcal{M}f(p, \sigma)\, d\sigma\, dt \\ &= \Gamma(n/2) \int_0^r \frac{1}{\sqrt{r^2 - t^2}} \mathcal{I}^{(n-2)/2}(\mathcal{V}f)(p, t)\, dt\end{aligned}$$

which implies

$$(\mathcal{M}f)(p, r) = \frac{2\Gamma(n/2)}{\pi\, r^{n-1}} \frac{\partial}{\partial r} \int_0^r \frac{1}{\sqrt{r^2 - t^2}} \mathcal{I}^{(n-2)/2}(\mathcal{V}f)(p, t)\, dt.$$

4.4 Computation of $\mathcal{V}^*$ for even n

We note that the formal L^2 adjoint of $\mathcal{D}$ is $\mathcal{D}^* = -\frac{1}{2} \frac{\partial}{\partial t} \frac{1}{t}$ and for any positive integer l, $(D^*)^l = (-1)^l t\, D^l t^{-1}$.

For even n, using (5), for F in the range of $(D^*)^{(n-2)/2}$, we have

$$\begin{aligned}
\Gamma(n/2)\,\langle \mathcal{V}f, F\rangle &= \int_{S_\rho}\int_0^\infty F(p,t)\mathcal{D}_t^{(n-2)/2}\int_0^t \frac{r^{n-1}}{\sqrt{t^2-r^2}}(\mathcal{M}f)(p,r)\,dr\,dt\,dS_p \\
&= \int_{S_\rho}\int_0^\infty (\mathcal{D}_t^*)^{(n-2)/2}F(p,t)\int_0^t \frac{r^{n-1}}{\sqrt{t^2-r^2}}(\mathcal{M}f)(p,r)\,dr\,dt\,dS_p \\
&= \int_{S_\rho}\int_0^\infty r^{n-1}(\mathcal{M}f)(p,r)\int_r^\infty (t^2-r^2)^{-1/2}(\mathcal{D}_t^*)^{(n-2)/2}F(p,t)\,dt\,dr\,dS_p \\
&= \frac{1}{\omega_{n-1}}\int_{S_\rho}\int_{\mathbf{R}^n} f(p+y)\int_{|y|}^\infty \frac{1}{\sqrt{t^2-|y|^2}}(\mathcal{D}_t^*)^{(n-2)/2}F(p,t)\,dt\,dy\,dS_p \\
&= \frac{1}{\omega_{n-1}}\int_{\mathbf{R}^n} f(x)\int_{S_\rho}\int_{|x-p|}^\infty \frac{1}{\sqrt{t^2-|x-p|^2}}(\mathcal{D}_t^*)^{(n-2)/2}F(p,t)\,dt\,dS_p\,dx.
\end{aligned}$$

Thus, for $x \in B_\rho$,

$$\begin{aligned}
(\mathcal{V}^*F)(x) &= \frac{1}{\omega_{n-1}\,\Gamma(n/2)}\int_{S_\rho}\int_{|x-p|}^\infty \frac{1}{\sqrt{t^2-|x-p|^2}}(D_t^*)^{(n-2)/2}F(p,t)\,dt\,dS_p \\
&= \frac{(-1)^{(n-2)/2}}{\omega_{n-1}\,\Gamma(n/2)}\int_{S_\rho}\int_{|x-p|}^\infty \frac{1}{\sqrt{t^2-|x-p|^2}}(tD_t{}^{(n-2)/2}t^{-1})(F(p,t))\,dt\,dS_p.
\end{aligned}$$

5 Open problems

We describe some open problems closely related to the ones considered in this article.

The characterization of the range of $\mathcal{V}$ when n is even is incomplete. While the ranges of $\mathcal{M}$ and $\mathcal{V}$ are related and there is a complete characterization of the range of $\mathcal{M}$ for all n, there isn't a nice explicit characterization of the range of $\mathcal{V}$ for even n. Arguments similar to those used in [29] for our Theorem 9 will probably lead to a characterization but they will not be as explicit as the ones given in Theorem 8.

Sarah Patch has proposed an interesting question which remains open. Let H_ρ be the upper half of $\overline{B_\rho}$ and let S'_ρ be the part of the boundary of H_ρ which is in S_ρ. If f is a smooth function supported in H_ρ then Patch proposed the recovery of f from $(\mathcal{M}f)(p,t)$ for $(p,t) \in S'_\rho \times [0,\infty)$. Uniqueness follows from Theorem 1.

Suppose f and g are functions in $C_0^\infty(\overline{B_\rho})$ and $u(x,t)$ is the solution of the IVP

$$\begin{aligned}
u_{tt} - \Delta_x u = 0 \qquad & (x,t) \in R^n \times R, \\
u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad & x \in R^n.
\end{aligned}$$

The problem of recovering f and g from the traces of u or $\partial_n u$ on $S_\rho \times [0, \infty)$ and a characterization of the range of the map from (f, g) to traces are open problems. In particular, an extension Theorem 12 to all n or to all odd n remains unsolved.

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