The X-ray transform for a non-abelian connection in two dimensions

David Finch
Department of Mathematics
Oregon State University
Corvallis, OR, 97331, USA

Gunther Uhlmann*
Department of Mathematics
University of Washington
Seattle, WA 98195, USA

October 29, 2003

Abstract

We show that in two dimensions we can determine a connection, from the line integrals of the connection with values in skew-hermitian $m \times m$ matrices provided that the connection has small curvature in an appropriate sense.

1 Introduction

In this section we motivate the integral geometry problem considered in section 2. It is well known that integral geometry problems arise naturally when considering inverse problems for hyperbolic equations.

Let's consider $m \times m$ smooth skew-hermitian matrices $A_i, i = 1, ..., n$. Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary. We investigate the following boundary value problem for the $m \times m$ system given by the wave equation associated to the Schrödinger equation with external Yang-Mills potential $A = (A_1, A_2, ..., A_n)$.

$$\begin{cases} (\partial_t^2 - \sum_{j=1}^n (-\partial_{x_j} + A_j)^2)u = 0 & \text{in } (0,T) \times \Omega, \\ u|_{t=0} = \partial_t u|_{t=0} = 0 & \text{in } \Omega, \\ u|_{(0,T)\times\partial\Omega} = f. & \end{cases} \tag{1}$$

where $f = (f_1, ..., f_m) \in H^2_{loc}((0,T) \times \partial\Omega)$, $f = 0$ for $t < 0$. Denote by $\nu = \nu(x)$ the outer normal to $\partial\Omega$ at $x \in \partial\Omega$. Let $\alpha := \sum_{i=1}^n A_i dx_i$ be the one form which represents the non-abelian connection defined by the matrices A_i. We define the hyperbolic Dirichlet-to-Neumann (DN) map Λ_A by

*Partly supported by NSF Grant DMS–0070488

$$\Lambda_A f := \left(\frac{\partial u}{\partial \nu} + \sum_{i=1}^{n} A_i \nu_i f\right)|_{(0,T)\times\partial\Omega} \tag{2}$$

where u is a solution to (1) and ν_i are the components of ν.

The inverse boundary value problem is whether we can determine the matrices A_i from the $m \times m$ operator-valued matrix Λ_A.

It is clear that one can not hope to uniquely determine A for if $G(x)$ is a smooth function taking values in the set of unitary matrices, $U(m)$, which is the identity to first order on $\partial\Omega$ and $y(t,x)$ is defined by $u(t,x) = G(x)y(t,x)$ with u satisfying (1), then y also satisfies (1) with A replaced by

$$A' = G^{-1}AG - G^{-1}\nabla G. \tag{3}$$

The argument given above shows that with A, A' as in (3), $\Lambda_{A'} f = \Lambda_A f$.

The transformation $A \to A'$ with A, A' as in (3) is called a *gauge transformation*, a terminology arising in physics. The simplest case occurs when the A_j are scalar, and pure imaginary. That form of the equation arises in electromagnetism where $\frac{1}{i}A_j$ represents a component of a magnetic potential. Since the product of scalars is commutative, the gauge transformation (3) reduces to $A \to A + G^{-1}\nabla G$. Since G would be a 1×1 unitary matrix in the scalar case, $G = e^{if(x)}$ so $G^{-1}\nabla G$ reduces to $i\nabla f$ and it is elementary that a magnetic potential could only be determined up to a gradient. It was noticed years ago that many of the gauge theories of physics could be expressed in the language of vector bundles over manifolds and connections on such bundles. (A connection defines a covariant derivative or an idea of parallel translation in a vector bundle. But for each smooth choice of an orthonormal basis for the fiber over an open set in the base, there is an associated set of matrices such as our A which encode the covariant derivatives of the basis elements, and which transform as in (3) when another orthonormal basis is selected. The change of basis corresponds to our G above. Thus another view of a connection is that it is a family of matrices A_j for each local frame which transform by (3) under changes of frame.) We don't make use substantial use of ideas from differential geometry, but it does provide a convenient language for stating our results. In geometric terms A defines the connection given by the one form $\alpha = \sum_{i=1}^{n} A_i dx_i$, and the indeterminancy manifested in (3) says that the best we could hope to reconstruct from Λ_A is the connection α. We recommend the book [1] for the language of connections and its relation with physics.

A closely related problem is the inverse scattering problem associated to the Schrödinger equation with external Yang-Mills potentials. Assume that the skew-hermitian matrices A_i have compact support in a ball of radius R. Then it is easy to see that if we know the scattering operator we can determine the hyperbolic Dirichlet-to-Neumann map in the ball of radius R [3], Proposition 3.5,[8], Theorem 12.14 (the proof given in those papers is for the Schrödinger equation associated to a scalar potential but it also applies in this case).

We follow the method indicated in [8] (see the proof of Theorem 11.3) to prove that the hyperbolic DN map for the Schrödinger equation for a scalar potential determines the X-ray transform of the potential, a result which was proved first in [6].

We first extend the A_i outside Ω so that they are zero outside the ball of radius R. Let $\theta \in S^{n-1}$. Let us consider $u(t,x,\theta)$, the $m \times m$ matrix solution of the wave equation with data a plane wave in the far past,

$$(\partial_t^2 - \sum_{j=1}^{n} (-\partial_{x_j} + A_j)^2) u = 0; \quad u|_{t<<0} = \delta(t - x \cdot \theta). \tag{4}$$

Using a similar argument to [8] one shows that if we know Λ_A then we can determine $u(t,x,\theta)$ for $x \cdot \theta$ sufficiently large. We write the solution of the continuation problem (4) in the form

$$u(t,x,\theta) = C(x,\theta)\delta(t - x\cdot\theta) + D(x,\theta)H(t - x\cdot\theta) + E(t,x,\theta) \tag{5}$$

where H denotes the Heaviside function and E is a continuous function.

The coefficient matrix C in (5) solves the transport equation

$$\theta \cdot \nabla C = \sum_{i=1}^{n} A_i \theta_i C(x,\theta); \quad C(x,\theta) = I_m \text{ for } x\cdot\theta < -R \tag{6}$$

where I_m denotes the $m \times m$ identity matrix.

Therefore if we know the DN map Λ_A we know $u(t,x,\theta)$ for $x\cdot\theta$ sufficiently large and thus we know $C(x,\theta)$ for $x\cdot\theta$ sufficiently large. The inverse boundary problem can be reduced to the problem of whether we can recover the matrices $A_i, i = i, ..., n$ if we know $C(x,\theta)$ for $x\cdot\theta$ sufficiently large.

In section 2 we state our results for the integral geometry problem mentioned above. We prove that if the curvature of the connection is sufficiently small we can determine the connection A if we know C satisfying (6) for $x\cdot\theta$ sufficiently large. The proof of this result is carried out in sections 3 and 4.

2 The results on the integral geometry problem

We state a local uniqueness theorem in two dimensions for the following problem which we motivated in the previous section. Suppose that

$$\theta \cdot \nabla C(x,\phi) = A(x)\cdot\theta C(x,\phi) \tag{7}$$

where $A(x) = (A_1(x), A_2(x))$ has compact support and each A_i is skew-hermitian matrix of size m, $\theta = (\cos(\phi), \sin(\phi))$, with $C(x,\theta) = I_m$ for $x\cdot\theta << 0$. Does the knowledge of $C(x,\theta)$ for $x\cdot\theta >> 0$ uniquely determine the coefficient matrices?

It is clear again that one can not hope to uniquely determine A for if $G(x)$ is a smooth function taking values in $U(m)$ which is the identity outside the support of A and Y is defined by $C(x,\theta) = G(x)Y(x,\theta)$ then Y satisfies (7) with A replaced by $G^{-1}AG - G^{-1}\nabla G$, but $C = Y$ when the norm of x is sufficiently large. Geometrically, this is the transformation law of a connection, so at best one could hope to recover the connection defined by A. (In geometric terms, C is the parallel translation operator of the connection $-A$ along straight lines.) This problem has been previously treated by other authors. V. Sharafutdinov [7] proved a more general local uniqueness result on the determination of a connection on a vector bundle over a compact Riemannian manifold, geodesically convex and with a strictly convex boundary, from the parallel translation along geodesics between all pairs of points in the boundary. The ideas at the foundation of his paper and this section are similar, but his development requires a lot of machinery which is avoided here. In late 1999, R. Novikov [5] proved a local uniqueness theorem for the determination of a connection from parallel translation along straight lines in R^2 which permitted him to conclude global uniqueness in higher dimensions, using a reduction to the local result in two dimensions, similar to the one used in [2] to prove global uniqueness for the attenuated X-ray transform in dimension 3. It too has a lengthy and technical proof. The following result was obtained at the same time as Sharafutdinov's, in fall 1997, but it was not published (Sharafutdinov mentioned the result at conferences in Oberwolfach and Saint Petersburg in August 1998).

Theorem 2.1 *Suppose that C_1 and C_2 satisfy (7) with A replaced by $A^{(1)}$ and $A^{(2)}$ respectively, and $C_1(x,\theta) = C_2(x,\theta) = I$ for $x \cdot \theta << 0$ and $C_1(x,\theta) = C_2(x,\theta)$ for $x \cdot \theta >> 0$. Then if $\|A^{(i)}\|_{C^1}, i = 1,2$ is sufficiently small, then there is a gauge transformation G so that $A^{(2)} = G^{-1}A^{(1)}G - G^{-1}\nabla G$.*

In this paper $\|M\|^2 = \mathrm{tr} M^*M$ is used as norm of the matrix M. When needed, $\|M\|_{op}$ will denote the operator norm.

More recently we have proven the following variation of this result, in which there is no smallness or closeness requirement on $A^{(i)}, i = 1,2$, but instead on their curvature operators. We recall that the curvature of the connection given by A is given by the two form $\omega_A = (\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} + [A_2, A_1])dx_1 \wedge dx_2$. We call the curvature operator K_A the matrix coefficient of the two form.

This version does not appear to be a corollary of either Sharafutdinov's or Novikov's results.

Theorem 2.2 *Suppose that C_1 and C_2 satisfy (7) with A replaced by $A^{(1)}$ and $A^{(2)}$ respectively, with the support of $A^{(i)}$ in an open set Ω with smooth boundary. Assume further that $C_1(x,\theta) = C_2(x,\theta) = I$ for $x \cdot \theta << 0$ and $C_1(x,\theta) = C_2(x,\theta)$ for $x \cdot \theta >> 0$. If*

$$\frac{1}{2} diam(\Omega)^2 (\sup \|K_{A^{(1)}}\|_{op} + \sup \|K_{A^{(2)}}\|_{op}) < 1,$$

then $A^{(1)}$ is gauge equivalent to $A^{(2)}$.

The proof of Theorem 2.1 proceeds in two steps. First G is defined, so that the connection equation is satisfied in a single direction. Let $\tilde{A} = G^{-1}AG - G^{-1}\nabla G$. Then computations similar to those of [9] are used to prove $\tilde{A}_2^{(1)} = A_2^{(2)}$, provided that the size of the coefficient matrices times the diameter of a ball containing both supports is sufficiently small.

3 Step 1: Fixing the connection

We first note that since the components of $A^{(i)}, i = 1, 2$ are skew-hermitian, then each C_i is unitary. Let $G(x) = C_1(x, e_1)C_2^*(x, e_1)$. Since C_1 and C_2 are equal when the norm of x is sufficiently large, then G is the identity when the norm of x is sufficiently large. With Y defined by $C_1 = GY$ then Y satisfies (7), with $A^{(1)}$ replaced by $\tilde{A}^{(1)} = G^{-1}A^{(1)}G - G^{-1}\nabla G$, and has the same boundary values as C_1. Moreover, $Y(x, e_1) = G^{-1}C_1(x, e_1) = C_2(x, e_1)$ so clearly $\tilde{A}^{(1)} \cdot e_1 = A^{(2)} \cdot e_1$. Thus we have fixed a gauge, and have now reduced the problem to showing that $\tilde{A}_2^{(1)} = A_2^{(2)}$.

We now relabel $\tilde{A}^{(1)}$ by $A^{(1)}$ again, and suppose that C_1 and C_2 are as above, with $A_1^{(1)} = A_1^{(2)}$. Since

$$\theta \cdot \nabla(C_1 - C_2) = A \cdot \theta \ (C_1 - C_2) + (A^{(2)} - A^{(1)}) \cdot \theta \ C_2$$

we have that

$$C_1 - C_2 = C_1 \int^x C_1^{-1}(A^{(2)} - A^{(1)}) \cdot \theta C_2.$$

In the formula above we use the notation $\int^x V(\cdot, \theta)$ to mean the integral of $V(\cdot, \theta)$ with respect to arc length along the ray $L_{x,\theta} = \{x + t\theta : -\infty < t \leq 0\}$ oriented in direction θ. The above formula is just variation of parameters since C is a fundamental matrix solution of the differential equation of (7).

Using that $C_1 = C_2$ outside of a sufficiently large ball and $A^{(i)}, i = 1, 2$ is supported on Ω, we obtain

$$0 = \int_l C_1^{-1}(A^{(2)} - A^{(1)}) \cdot \theta C_2$$

over every line l, passing through Ω with direction θ. Since $A_1^{(1)} = A_1^{(2)}$ This simplifies to

$$0 = \int_l C_1^{-1}(A_2^{(1)} - A_2^{(2)})C_2. \tag{8}$$

4 Step 2: Energy inequalities

We shall prove in this section that the identity (8) implies that $A_2^{(1)} = A_2^{(2)}$. In order to do this, we follow Vertgeim's [9] adaptation (to the matrix setting) of the method of Mukhometov [4], which is based on a very ingenious energy integral method for the kinetic equation. Letting

$$U(x,y,\phi)=\int_{-\infty}^{0} C_1^{-1}((x,y)+t\theta,\phi)\left[A_2^{(2)}((x,y)+t\theta)-A_2^{(1)}((x,y)+t\theta)\right]C_2((x,y)+t\theta,\phi)\,dt$$

and $\mathcal{K}=\theta\cdot\nabla$ we have

$$\mathcal{K}U=C_1^{-1}(A_2^{(2)}-A_2^{(1)})C_2$$

so

$$\mathcal{L}U:=\frac{\partial}{\partial\phi}(C_1\mathcal{K}UC_2)=0.$$

Letting $\mathcal{G}=\theta^{\perp}\cdot\nabla$ we have the following version of equation (7) of [9].

$$\begin{aligned}2\mathcal{R}e(\mathcal{G}U)^*C_1^{-1}(\mathcal{L}U)C_2 &= 2\mathcal{R}e(\mathcal{G}U)^*\left[C_1^*C_{1,\phi}\,\mathcal{K}U+\mathcal{K}U\,C_{2,\phi}^*C_2\right]+\mathcal{R}e\frac{\partial}{\partial\phi}\left[(\mathcal{G}U)^*(\mathcal{K}U)\right]\\ &\quad \mathcal{R}e(\mathcal{K}U)^*(\mathcal{K}U)+\mathcal{R}e(\mathcal{G}U)^*(\mathcal{G}U)-\mathcal{R}e\{\mathcal{G}U_\phi^*\mathcal{K}U-\mathcal{G}U^*\mathcal{K}U_\phi\}.\end{aligned}$$

Taking the trace of the preceding identity, integrating over Ω and S^1, and using that U and its derivatives are zero on the boundary of Ω gives

$$0=\int 2\mathcal{R}e\,\mathrm{tr}\left[(\mathcal{G}U)^*C_1^*C_{1,\phi}\mathcal{K}U+(\mathcal{G}U)^*(\mathcal{K}U)C_{2,\phi}^*C_2)\right]+\sum\|\nabla u_{i,j}\|^2. \tag{9}$$

However, if $U\neq 0$, the right hand side is positive provided that the norm of the term involving the C_i and their derivatives is sufficiently small. By Cauchy-Schwarz, one may readily estimate

$$\begin{aligned}&\int 2\mathcal{R}e\,\mathrm{tr}\left[(\mathcal{G}U)^*C_1^*C_{1,\phi}\mathcal{K}U+(\mathcal{G}U)^*(\mathcal{K}U)C_{2,\phi}^*C_2)\right]\le\\ &\qquad(\sup\|C_1^*C_{1,\phi}\|_{op}+\sup\|C_{2,\phi}^*C_2\|_{op})\int(\|\mathcal{K}U\|^2+\|\mathcal{G}U\|^2),\end{aligned}$$

where $\|\ \ \|_{op}$ represents the operator norm of the matrix. Thus if

$$\sup\|C_1^*C_{1,\phi}\|_{op}+\sup\|C_{2,\phi}^*C_2\|_{op}<1$$

formula (9) shows that $U=0$ and thus $A_2^{(2)}=A_2^{(1)}$.

This requires bounds on the derivative part only, since the C_i are unitary. Since the derivatives also satisfy a differential equation, this should be estimated in terms of the norm of $A^{(i)}, i=1,2$ and the diameter of the Ω. One such path occurs in Theorem 2.2, whose proof relies on a more detailed analysis of the terms CC_ϕ^* occurring on the right side of (9). In the following formulas we use again the notation $U(x,\theta)=\int^x V(\cdot,\theta)$ to mean that $U(x,\theta)$ is given as the integral with respect to arc length along the ray $L_{x,\theta}=\{x+t\theta:-\infty<t\le 0\}$

oriented in direction θ. We recall that formula (7) states that C is a fundamental matrix solution of the differential equation, now expressed in terms of the operator $\mathcal{K}$,

$$\mathcal{K}C = A(x)\cdot\theta C(x,\theta) \tag{10}$$

where we now write $C(x,\theta)$ interchangeably with $C(x,\phi)$. Differentiation of this formula with respect to ϕ gives

$$\mathcal{K}C_\phi + \mathcal{G}C = A(x)\cdot\theta^\perp + A\cdot\theta C_\phi \tag{11}$$

while applying $\mathcal{G}$ gives

$$\mathcal{K}\mathcal{G}C = \mathcal{G}(A(x)\cdot\theta)C + A(x)\cdot\theta\,\mathcal{G}C. \tag{12}$$

These are non-homogeneous linear equations for C_ϕ and $\mathcal{G}C$, respectively, for which the initial condition when $x\cdot\theta << 0$ is the 0 matrix. Variation of parameters applied to formula (12) gives the solution

$$\mathcal{G}C = C\int^x C^{-1}\mathcal{G}(A\cdot\theta)C. \tag{13}$$

Now expanding the derivative and a bit of algebra leads to

$$\mathcal{G}(A\cdot\theta) = \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} + \mathcal{K}\,A\cdot\theta^\perp$$

which is substituted into formula (13). Integration by parts is applied to the term $C^{-1}\mathcal{K}(A\cdot\theta)C$ in the integral, and expanding and using (10) and its adjoint gives

$$\mathcal{G}C = A\cdot\theta^\perp + C\int^x C^{-1}\left[\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} + A\cdot\theta^\perp A\cdot\theta - A\cdot\theta A\cdot\theta^\perp\right]C. \tag{14}$$

Expanding and rearranging the commutator term we obtain

$$A\cdot\theta^\perp A\cdot\theta - A\cdot\theta A\cdot\theta^\perp = [A_2, A_1],$$

hence

$$\mathcal{G}C = A\cdot\theta^\perp + C\int^x C^{-1}\left\{\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} + [A_2, A_1]\right\}C \tag{15}$$

where now the expression conjugated inside the integral is the curvature of $-A$ as a connection, which we denote by K_A. Now solving (11) by variation of parameters gives

$$C_\phi = C\int^x C^{-1}\left[A\cdot\theta^\perp - \mathcal{G}C\right]$$

and the term in brackets in the integral can be replaced using (15) to give

$$C_\phi = C\int C^{-1}[-C\int C^{-1}K_A C] = -C\int\int C^{-1}K_A C \tag{16}$$

Using this for C equal to C_1 or C_2, which are unitary, it follows upon reversing order of integration in the iterated integral that

$$C^* C_\phi = -\int^x (x-y) C^* K_A C. \tag{17}$$

This allows us to estimate the operator norm of $C^* C_\phi$ by

$$\|C^* C_\phi\|_{op} \le \sup \|K_A\|_{op} \int_{L(x,\theta)} (x-y)dy \le \sup \|K_A\|_{op} \frac{1}{2} (\mathrm{diam}(\Omega)^2). \tag{18}$$

This concludes the proof of Theorem 2.2.

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