The range of the spherical mean value operator for functions supported in a ball

David Finch
Department of Mathematics
Oregon State University
Corvallis, OR 97331-4605
Email: finch@math.oregonstate.edu

Rakesh
Department of Mathematics
University of Delaware
Newark, DE 19716
Email: rakesh@math.udel.edu

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Abstract

Suppose $n > 1$ is an odd integer, f is a smooth function supported in a ball B with boundary S, and u is the solution of the initial value problem

$$u_{tt} - \triangle_x u = 0, \qquad (x,t) \in R^n \times [0,\infty);$$
$$u(x,t{=}0) = 0, \qquad u_t(x,t{=}0) = f(x), \qquad x \in R^n.$$

We characterize the range of the map $f \mapsto u|_{S\times[0,\infty)}$ and give a stable scheme for the inversion of this map. This also characterizes the range of the map sending f to its mean values over spheres centered on S.

1 Introduction

Below n will be an odd integer greater than or equal to 3, B_ρ will represent the open ball (in R^n) of radius ρ centered at the origin, $\overline{B_\rho}$ its closure, and S_ρ its boundary. Further, all functions will be real valued. Let $\tilde{C}^\infty(S_\rho \times [0,\infty))$ consist of the restriction to $S_\rho \times [0,\infty)$ of the smooth functions on $S_\rho \times (-\infty,\infty)$ which are supported in $S_\rho \times [0, 2\rho]$.

Define the mean value operator $\mathcal{M} : C_0^\infty(\overline{B_\rho}) \to C^\infty(S_\rho \times [0,\infty))$ where

$$(\mathcal{M}f)(x,r) = \frac{1}{w_n} \int_{|\theta|=1} f(x + r\theta)\, d\theta$$

for all $(x,r) \in S_\rho \times [0,\infty)$, where w_n is the surface area of the unit sphere in R^n. Then applications in thermo-acoustic tomography (see [11] and the references there) motivate the construction of the inverse of $\mathcal{M}$ and the characterization of the range of of $\mathcal{M}$. In [6] we derive several inversion formulas for $\mathcal{M}$ but we do not give a characterization of the range of $\mathcal{M}$. Here we characterize the range of an operator which should lead to a characterization of the range of $\mathcal{M}$.

For $f \in C_0^\infty(\overline{B_\rho})$, let $u(x,t)$ be the solution of the initial value problem

$$u_{tt} - \triangle u = 0, \qquad (x,t) \in R^n \times [0,\infty), \tag{1}$$

$$u(x,t{=}0) = 0, \;\; u_t(x,t{=}0) = f(x), \qquad x \in R^n, \tag{2}$$

Then, from page 682 of [5]

$$u(x,t) = \frac{\sqrt{\pi}}{2\,\Gamma(n/2)} \left(\frac{1}{2t} \frac{\partial}{\partial t} \right)^{(n-3)/2} \left(t^{n-2} (\mathcal{M}f)(x,t) \right) \;. \tag{3}$$

Hence the range of $\mathcal{M}$ may be determined from the range of the map $\mathcal{S} : C_0^\infty(\overline{B_\rho}) \to C^\infty(S_\rho \times [0,\infty))$, where

$$(\mathcal{S}f)(x,t) = u(x,t), \qquad (x,t) \in S_\rho \times [0,\infty). \tag{4}$$

In this article we will characterize and derive some properties of the range of $\mathcal{S}$ and this should lead to a characterization and properties of the range of $\mathcal{M}$. This transition is fairly simple when $n = 3$.

The problem of recovering a function from its mean values has a long history with contributions from many authors. The early results are best found in the book of F. John [9]. Recent interest has been spurred by its application in reflectivity tomography [13], thermoacoustic tomography [10, 12, 17], and the uniqueness results of [1]. Further references can be found in [5], and in [2] which has an extensive bibliography. This latter paper has a range characterization for $\mathcal{M}$ with centers on a circle in dimension two. In odd dimensions, [4] gives a nice and complete analysis of the inverse mean value problem when the mean values are taken over a family of spheres centered on a plane; this corresponds to studying the map from initial data to boundary traces for solutions of the wave equation where the boundary is the boundary of a half-plane. The goal in [6] and the current article was to do a complete analysis, in the odd dimensional case, for traces on the boundary of a sphere.

From Huygen's principle for the wave equation in odd dimensions, it is clear that the range of $\mathcal{S}$ is a subspace of $\tilde{C}^\infty(S_\rho \times [0,\infty))$. Keeping this in mind, we have the following characterizations of the range of $\mathcal{S}$.

Theorem 1. *A function $p \in \tilde{C}^\infty(S_\rho \times [0,\infty))$ is in the range of $\mathcal{S}$ iff $v(x,t{=}0) = 0$ for all $x \in \overline{B_\rho}$ where $v(x,t)$ is the solution of the backward IBVP*

$$v_{tt} - \triangle v = 0, \qquad (x,t) \in \overline{B_\rho} \times [0, 2\rho], \tag{5}$$

$$v(x,t{=}2\rho) = 0, \;\; v_t(x,t{=}2\rho) = 0, \qquad x \in \overline{B_\rho}, \tag{6}$$

$$v(x,t) = p(x,t), \qquad (x,t) \in S_\rho \times [0,2\rho]. \tag{7}$$

We also study a map closely related to $\mathcal{S}$. Define $\mathcal{T} : C_0^\infty(\overline{B_\rho}) \to C^\infty(S_\rho \times [0,\infty))$ where

$$(\mathcal{T} f)(x,t) = U(x,t), \qquad (x,t) \in S_\rho \times [0,\infty) \tag{8}$$

where $U(x,t)$ is the solution of the IVP

$$U_{tt} - \triangle U = 0, \qquad (x,t) \in R^n \times [0,\infty), \tag{9}$$

$$U(x,t{=}0) = f(x),\ U_t(x,t{=}0) = 0, \qquad x \in R^n. \tag{10}$$

We prove the following theorem.

Theorem 2. *A function $P \in \tilde{C}^\infty(S_\rho \times [0,\infty))$ is in the range of $\mathcal{T}$ iff $V_t(x,t{=}0) = 0$ for all $x \in \overline{B_\rho}$ where $V(x,t)$ is the solution of the backward IBVP*

$$V_{tt} - \triangle V = 0, \qquad (x,t) \in \overline{B_\rho} \times [0,2\rho], \tag{11}$$

$$V(x,t{=}2\rho) = 0,\ V_t(x,t{=}2\rho) = 0, \qquad x \in \overline{B_\rho}, \tag{12}$$

$$V(x,t) = P(x,t), \qquad (x,t) \in S_\rho \times [0,2\rho]. \tag{13}$$

From the definitions of $\mathcal{S}$ and $\mathcal{T}$ we see that $U(x,t) = u_t(x,t)$ for all $(x,t) \in R^n \times [0,\infty)$, hence $\mathcal{T} = \partial_t \mathcal{S}$. We will show later that Theorem 1 follows quickly from Theorem 2 because of this relation. We also observe that the relation $\mathcal{T} = \partial_t \mathcal{S}$ implies that for all P in the range of $\mathcal{T}$

$$\int_0^\infty P(x,t)\, dt = 0, \qquad \text{for all } x \in S_\rho, \tag{14}$$

but this condition is not required in the sufficiency part of Theorem 2. It is not obvious that the sufficiency condition of Theorem 2 implies (14).

The next characterization of the range of $\mathcal{S}$ is somewhat odd in that it seems circular; however it gives an interesting characterizing property of the range of $\mathcal{S}$.

Theorem 3. *A function $p \in \tilde{C}^\infty(S_\rho \times [0,\infty))$ is in the range of $\mathcal{S}$ iff*

$$\int_{S_\rho \times [0,\infty)} p(x,t)\, q_t(x,t)\, dS_x\, dt = 0 \tag{15}$$

for all $q \in \tilde{C}^\infty(S_\rho \times [0,\infty))$ in the range of $\mathcal{S}$.

One may also interpret this as the statement that a function $p \in \tilde{C}^\infty(S_\rho \times [0,\infty))$ is in the range of $\mathcal{S}$ iff p is orthogonal (in the L^2 sense) to the range of $\mathcal{T}$. The orthogonality condition (15) does not uniquely pick out the range of $\mathcal{S}$. For example, any subspace of the intersection of $\tilde{C}^\infty(S_\rho \times [0,\infty))$ and the space of functions symmetric in t about the point $t = \rho$, has the orthogonality property (15). Using the notation in [6], Theorem 3 may be interpreted as saying that the ranges of $\mathcal{S}$ and $\mathcal{T}$ are the kernels of $\mathcal{N}^*\mathcal{D}^*\dfrac{\partial}{\partial t}$ and $\mathcal{N}^*\mathcal{D}^*$ respectively.

Theorem 2 leads to another characterization of the range of $\mathcal{T}$ (hence another characterization of the range of $\mathcal{S}$) which has some similarities with a characterization of the range of $\mathcal{M}$, when $n = 2$, in [3]. Let $\{\phi_m\}_{m=1}^\infty$ be spherical harmonics which form an orthonormal basis for $L^2(S_1)$ - see Chapter 4 of [16]. These are restrictions to S_1 of certain harmonic homogeneous polynomials on R^n; also see section 2.

Theorem 4. *A function $P(x,t) \in \tilde{C}^\infty(S_\rho \times [0,\infty))$ is in the range of $\mathcal{T}$ iff for the cosine transform*

$$Z_m(\omega) \equiv \int_0^\infty P_m(t)\cos(\omega t)\,dt, \qquad m = 1, 2, \cdots,$$

we have $Z_m(\omega) = 0$ for all positive roots $\omega > 0$ of $J_{k+(n-2)/2}(\omega\rho) = 0$. Here k is the degree of the polynomial $\phi_m(x)$ and

$$P_m(t) = \rho^{-k}\int_{|\theta|=1} P(\rho\theta, t)\,\phi_m(\theta)\,d\theta,$$

is the m-th coefficient in the spherical harmonic expansion of $P(x,t)$.

The characterization condition in Theorem 4 is similar to condition 3 of Theorem 4 in [3] (by Ambartsoumian and Kuchment) which characterizes the range of $\mathcal{M}$ when $n = 2$. In two (and other even dimensions) the solution of the initial value problem for the wave equation is given by a non-local operator applied to the spherical mean transform. In even space dimensions, solutions of the wave equation do not satisfy Huygens's principle and so it makes more sense to work directly with the spherical means operator. In the setting of [3], the cosine transform of our Theorem 4 is replaced by the Hankel transform of order 0. However, the characterization condition in [3], for the $n = 2$ case, also requires a vanishing of moments condition, which is not a requirement in our Theorem 4; of course our result is only for odd n. The vanishing of moments condition in our setting would be that

$$\int_0^{2\rho} t^{2j} P_m(t)\,dt = 0, \qquad j = 0, 1, \cdots, k + (n-3)/2, \quad m = 1, 2, \cdots, \tag{16}$$

where k is the degree of homogeneity of $\phi_m(x)$. Since

$$\frac{d^\sigma}{d\omega^\sigma}\left(Z_m(\omega)\right)|_{\omega=0} = \begin{cases} 0 & \text{if } \sigma \text{ is odd} \\ \pm\displaystyle\int_0^{2\rho} t^\sigma P_m(t)\,dt & \text{if } \sigma \text{ is even,} \end{cases}$$

the vanishing moment condition is equivalent to the statement that $Z_m(\omega)$ has a root of order at least $2k + n - 1$ at $\omega = 0$; note that the characterization condition in Theorem 4 involves only the positive roots of $Z_m(\omega)$. This observation leads to a corollary in harmonic analysis.

Corollary 5. *Suppose μ is a positive integer and $p(t) \in C^\infty(R)$ with support in $[0, 2\rho]$; and $P(\omega)$ is the cosine transform*

$$P(\omega) \equiv \int_0^{2\rho} p(t)\,cos(\omega t)\,dt.$$

If $P(\omega) = 0$ for all positive roots ω of $J_{\mu-1/2}(\omega\rho) = 0$, then $P(\omega)$ has a zero of order 2μ at $\omega = 0$.

The necessity of the vanishing of moments conditions (16), for $n = 3$, was first observed by Patch in [14]. She used the conditions to give a procedure for extrapolating data when the centers were confined to a hemisphere. In section 6, we show the necessity of (16) for all odd n, $n \geq 3$.

Theorem 1 provides a stable method for inverting $\mathcal{S}$ - even with inexact data. Given a function $p(x,t)$ close to a function in the range of $\mathcal{S}$, if $v(x,t)$ is the solution of the well posed IBVP (5)-(7), then a candidate for $\mathcal{S}^{-1}p$ is $v_t(.,t{=}0)$ and since the solution of an IBVP problem for the wave equation is a stable process, this provides a stable inversion algorithm. The inversion algorithms in Theorems 1 and 2 have obvious extensions to the inversion of the map $\mathcal{S}+\mathcal{T}$, that is to the map $(f,g) \mapsto u|_{S_\rho \times [0,2\rho]}$ where $u(x,t)$ is the solution of the wave equation but with the modified initial conditions $(u,u_t)|_{t=0} = (f,g)$. Because of Huygen's principle, the inverse of this map is $p \mapsto (v,v_t)|_{t=0}$ where $v(x,t)$ is the solution of the IBVP in Theorem 1. A characterization of the range of the modified map $(f,g) \mapsto u|_{S_\rho \times [0,2\rho]}$ is an unsolved problem at the moment.

A natural question is to study the ranges of $\mathcal{S}$ and $\mathcal{T}$ when their domains are extended to square integrable functions or to functions with square integrable derivatives. Natural norms for the ranges are suggested by the trace identities in [7]. Our range characterization proofs break down in these cases but perhaps our proofs could be modified to produce similar (to the smooth case) characterizations of the ranges.

We introduce a differential operator which will be used in several places in this article. For any positive integer μ, the operator $\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^\mu$ may be rewritten as $\sum_{i,j} \alpha_{i,j} r^{-i} \frac{\partial^j}{\partial r^j}$ for some constants $\alpha_{i,j}$; define

$$A_\mu(r,\tau) \equiv \sum_{i,j} \alpha_{i,j} r^{-i} \tau^j.$$

Then

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^\mu (f(t+r)) = A_\mu(r,\partial_t)(f(t+r)). \tag{17}$$

We will need properties of the polynomial $A_\mu(r,\tau)$ which may be derived from the explicit expression for $A_\mu(r,\tau)$ given in the following lemma.

Lemma 6. *If μ is a positive integer then*

$$A_\mu(r,\tau) = \sum_{j=0}^{\mu-1} (-1)^j \frac{(\mu+j-1)!}{j!\,(\mu-j-1)!\,2^j} \frac{\tau^{\mu-j}}{r^{\mu+j}}, \qquad r,\tau \in (-\infty,\infty),\ r \neq 0.$$

Also, as a polynomial in τ, the roots of $A_\mu(r,\tau)$ are simple and $A_\mu(r,\tau)$, $A_\mu(r,-\tau)$ have no common non-zero root.

2 Proof of Theorem 2

The necessity part of the Theorem is obvious. It remains to show the sufficiency part.

If the spherical harmonic ϕ_m is the restriction of a homogeneous polynomial of degree $k(m)$ then that homogeneous harmonic polynomial is $x \mapsto r^{k(m)}\phi_m(\theta)$ where $r=|x|$ and $\theta = x/|x|$. Since

$r^{k(m)}\phi(\theta)$ is harmonic, if $\triangle_S$ is the Laplace-Beltrami operator on $S_1(0)$, then noting that

$$\triangle = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\triangle_S$$

one may show that

$$\triangle_S \phi_m = -k(m)(k(m)+n-2)\phi_m \qquad \text{on } S_1 \ . \tag{18}$$

If $f(x)$ is a smooth function on R^n which is supported in $\overline{B_\rho}$ then the spherical harmonics expansion of f is

$$f(x) = \sum_{m=1}^{\infty} f_m(r) r^{k(m)} \phi_m(\theta), \qquad x \in R^n$$

where

$$r^{k(m)} f_m(r) = \int_{|\theta|=1} f(r\theta)\, \phi_m(\theta)\, d\theta \ ;$$

$f_m(r)$ is a smooth even function on $(-\infty, \infty)$ supported on $[-\rho, \rho]$. Below, we will be lax about the convergence of the series arising from the various spherical harmonics expansions; the convergence may be shown by deriving decay estimates for $r^{k(m)} f_m(r)$ by repeated applications of Stokes's theorem for the unit sphere - see [15].

Consider the spherical harmonics expansion of a $P(x,t) \in \tilde{C}^\infty(S_\rho \times [0, \infty))$;

$$P(x,t) = \sum_{m=1}^{\infty} p_m(t) \rho^{k(m)} \phi_m(\theta), \qquad (x,t) \in S_\rho \times (-\infty, \infty),$$

where $p_m(t)$ is a smooth function on $(-\infty, \infty)$ which is supported on $[0, 2\rho]$. Then the solution $V(x,t)$ of (11)-(13) is

$$V(x,t) = \sum_{m=1}^{\infty} b_m(r,t) r^{k(m)} \phi_m(\theta)$$

where $b_m(r,t)$ is the solution of the backward initial value problem

$$b_{m,tt} - b_{m,rr} - \frac{\nu - 1}{r} b_{m,r} = 0, \qquad (r,t) \in [-\rho, \rho] \times [0, 2\rho], \tag{19}$$

$$b_m(r, t{=}2\rho) = 0, \ b_{m,t}(r, t{=}2\rho) = 0 \qquad r \in [-\rho, \rho], \tag{20}$$

$$b_m(r{=}\rho, t) = p_m(t), \qquad t \in [0, 2\rho], \tag{21}$$

with $\nu = n + 2k(m)$. Since $V_t(x, t{=}0) = 0$ for all x in $\overline{B_\rho}$, from the uniqueness of the coefficients in the spherical harmonic expansion we have $b_{m,t}(r, t{=}0) = 0$ for all $r \in [-\rho, \rho]$. Then the sufficiency part of Theorem 2 will follow if we can show that $b_m(r,t)$ has an extension to a smooth, even (in r) function on $(-\infty, \infty) \times [0, 2\rho]$ which satisfies the two conditions that (19) holds on this larger domain and

$$b_m(r, t{=}0) = 0, \ b_{m,t}(r, t{=}0) = 0, \qquad \text{if } |r| \geq \rho.$$

So Theorem 2 will follow if we prove the following proposition.

Proposition 7. *Suppose $\nu \geq 3$ is an odd integer and $p(t)$ is the restriction to $[0,\infty)$ of a smooth function on $(-\infty,\infty)$ which is supported on $[0,2\rho]$. Let $b(r,t)$ be the solution of the backward IBVP*

$$\mathcal{L}_\nu b \equiv b_{tt} - b_{rr} - \frac{\nu-1}{r} b_r = 0, \qquad (r,t) \in [-\rho,\rho] \times [0,\infty), \tag{22}$$
$$b(r,t) = 0, \qquad (r,t) \in [-\rho,\rho] \times [2\rho,\infty), \tag{23}$$
$$b(r{=}\pm\rho,t) = p(t), \qquad t \in [0,\infty). \tag{24}$$

If $b_t(r,t{=}0) = 0$ for all $r \in [-\rho,\rho]$ then $b(r,t)$ has a smooth, even (in r) extension to $(-\infty,\infty) \times [0,\infty)$ so that $b(r,t)$ satisfies (22) on $(-\infty,\infty)\times[0,\infty)$, and $b(r,t{=}0) = 0$, $b_t(r,t{=}0) = 0$ if $|r| \geq \rho$.

Proof of Proposition 7

For any smooth even function $h(r)$ on $[-\rho,\rho]$ we define the operator D by $(Dh)(r) = h'(r)/r$, $r \in [-\rho,\rho]$. We may verify that

$$\mathcal{L}_\nu (Dc)(r,t) = D(\mathcal{L}_{\nu-2} c)(r,t) \tag{25}$$

for any smooth function $c(r,t)$ which is even in r. Since $\nu \geq 3$ is odd, we may write $\nu = 2\mu + 1$ for some positive integer μ. Then repeated application of (25) gives us

$$\mathcal{L}_\nu (D^\mu c)(r,t) = D^\mu(\mathcal{L}_{\nu-2\mu} c)(r,t) = D^\mu(\mathcal{L}_1 c)(r,t) \tag{26}$$

for any smooth function $c(r,t)$ which is even in r.

Suppose $c(r,t)$ is the solution of the IBVP

$$\mathcal{L}_1 c \equiv c_{tt} - c_{rr} = 0, \qquad (r,t) \in [-\rho,\rho] \times [0,\infty), \tag{27}$$
$$c(r,t) = 0, \qquad (r,t) \in [-\rho,\rho] \times [2\rho,\infty), \tag{28}$$
$$c(r{=}\pm\rho,t) = q(t), \qquad t \in [0,\infty). \tag{29}$$

for some smooth function $q(t)$ on $[0,\infty)$ which is zero for $t \geq 2\rho$. We claim that $b(r,t) = (D^\mu c)(r,t)$ on $[-\rho,\rho]\times[0,\infty)$ for a suitably chosen function $q(t)$. It is clear enough that $(D^\mu c)(r,t)$ satisfies (22) and (23). It remains to show that there is a smooth function $q(t)$ on $[0,\infty)$, supported in $[0,2\rho]$ so that $(D^\mu c)(r{=}\rho,t) = p(t)$ for $t \geq 0$. Note that (22)-(24) has a unique solution.

The solution of (27) - (29) is

$$c(r,t) = q(t-r+\rho) + q(t+r+\rho), \qquad (r,t) \in [-\rho,\rho]\times[0,\infty).$$

To check this, note that our expression for c is clearly an even function of r which satisfies (27). Further, for $(r,t) \in [-\rho,\rho]\times[2\rho,\infty)$ we have $t \pm r + \rho \geq 2\rho$ and hence (28) is satisfied because $q(t)$ is zero for $t \geq 2\rho$. Finally $c(r{=}\rho,t) = q(t) + q(2\rho+t) = q(t)$ for $t \geq 0$.

Now $q(t)$ is to be chosen as a smooth function on $[0,\infty)$, supported on $[0,2\rho]$, so that for $t\geq 0$,

$$\begin{aligned}
p(t) &= (D^{\mu}c)(r{=}\pm\rho,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\mu}(q(t-r+\rho)+q(t+r+\rho))|_{r=\pm\rho} \\
&= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\mu}(q(t+r+\rho))|_{r=-\rho} \\
&= A_{\mu}(r,\partial_t)(q(t+r+\rho))|_{r=-\rho} \\
&= A_{\mu}(-\rho,\partial_t)q(t). \qquad (30)
\end{aligned}$$

We will actually find a smooth function $q(t)$ on $(-\infty,\infty)$, supported on $(-\infty,2\rho]$, so that

$$A_{\mu}(-\rho,\partial_t)q(t) = p(t), \qquad \text{for all } t\in(-\infty,\infty). \qquad (31)$$

Note, we are given that $p(t)$ is a smooth function on $(-\infty,\infty)$ which is supported in $[0,2\rho]$. Since $p(t)$ is zero for $t\geq 2\rho$, there is a unique solution of the backward initial value problem for (31) with the initial condition $q(t)=0$ for $t\geq 2\rho$ and this completes the construction of $q(t)$. *Note that we do not know the form of $q(t)$ for $t\leq 0$ even though $p(t)$ is zero for $t\leq 0$ - this is going to be the crucial issue.*

From the hypothesis of Proposition 7, $p(t)$ is such that $b_t(r,t{=}0)=0$ for $|r|\leq\rho$. Hence

$$0 = \partial_t\, D^{\mu}\left(q(t-r+\rho)+q(t+r+\rho)\right)|_{t=0} = D^{\mu}\left(q'(-r+\rho)+q'(r+\rho)\right), \qquad |r|\leq\rho. \qquad (32)$$

Formally one may verify that on even functions $D=\dfrac{1}{2}\dfrac{\partial}{\partial r^2}$. Hence $q'(-r+\rho)+q'(r+\rho)$ is an even function whose μth derivative, with respect to r^2, is zero on $[-\rho,\rho]$ which implies

$$q'(-r+\rho)+q'(r+\rho) = \sum_{i=0}^{\mu-1}\tilde{\alpha}_i r^{2i}, \qquad |r|\leq\rho,$$

for some constants $\tilde{\alpha}_i$. Integrating this and using the value at $r=0$ we have

$$q(r+\rho)-q(-r+\rho) = \sum_{i=0}^{\mu-1}\alpha_i r^{2i+1} \equiv \alpha(r), \qquad |r|\leq\rho, \qquad (33)$$

for some constants α_i. We claim that $q(t)$ is zero for $t\leq 0$ and we justify this claim next.

Differentiating (33) and noting that $q(t)$ is zero for $t\geq 2\rho$ we have $q^k(0)=\pm q^k(2\rho)=0$ for $k\geq 2\mu$. From (31) and the fact that $p(t)$ is zero for $t\leq 0$, we have

$$A_{\mu}(-\rho,\partial_t)q(t) = 0, \qquad \text{for } t\leq 0. \qquad (34)$$

Hence $q(t)$ is the solution of a homogeneous, constant coefficient ODE, over the region $t\leq 0$. Now, from Lemma 6, the operator $A_{\mu}(-\rho,\partial_t)$ has zero as a characteristic root of multiplicity 1 and all the non-zero roots have multiplicity 1. Hence

$$q(t) = \beta_0 + \sum_j \beta_j e^{z_j t} \equiv \beta(t), \qquad \text{for all } t\leq 0, \qquad (35)$$

for some constants β_j and $z_j \neq 0$. Now $\beta(z)$ is an entire analytic function and $\beta^k(0) = q^k(0) = 0$ for all $k \geq 2\mu$; hence $\beta(t)$ must be a polynomial in t. So from the definition of $\beta(t)$ in (35) and linear independence arguments we conclude that $\beta(t) = \beta_0$; hence $q(t)$ is constant for $t \leq 0$. So $q^k(0) = 0$ for all $k \geq 1$; also $q^k(2\rho) = 0$ for all $k \geq 0$ because $q(t)$ is zero for $t \geq 2\rho$. Taking high order derivatives of (33) and substituting $r = \rho$, and repeating this for successively lower order derivatives, we can conclude that $\alpha_i = 0$ for all $i \geq 0$, implying $q(r+\rho) - q(-r+\rho) = 0$ for $|r| \leq \rho$. Substituting $r = \rho$ and noting $q(2\rho) = 0$, we obtain $q(0) = 0$. Hence $q(t)$, as a function on $(-\infty, \infty)$, is supported in $[0, 2\rho]$.

Now

$$b(r,t) = D^{\mu}(q(t - r + \rho) + q(t + r + \rho)), \qquad \text{for all } (r,t) \in [-\rho, \rho] \times [0, \infty)$$

for some integer $\mu \geq 1$. However, the RHS of the above expression is well defined for all $(r,t) \in (-\infty, \infty) \times [0, \infty)$ and let us call this extended function $\tilde{b}(r,t)$. Then $\tilde{b}(r,t)$ is an even function of r and satisfies (22) over the larger region $(-\infty, \infty) \times [0, \infty)$ because $q(t - r + \rho) + q(t + r + \rho)$ satisfies $\mathcal{L}_1 c = 0$ on $(-\infty, \infty) \times [0, \infty)$. Further, because $q(.)$ is supported in $[0, 2\rho]$, one may check that $\tilde{b}(r, t{=}0)$ and $\tilde{b}_t(r, t{=}0)$ are zero if $|r| \geq \rho$. Also, $\tilde{b}_t(r, t{=}0) = 0$ if $|r| \leq \rho$ by our construction of q. Hence $\tilde{b}(r,t)$ is the extension of $b(r,t)$ we sought. This completes the proof of Proposition 7.

QED

3 Proof of Theorem 1

The necessity part follows from Huygen's principle for the wave equation in odd dimensions. The sufficiency part follows from Theorem 2 as shown below.

Suppose $p \in \tilde{C}^{\infty}(S_\rho \times [0, \infty))$ and satisfies the hypothesis of the theorem. So $v(x, t{=}0) = 0$ for $x \in \overline{B_\rho}$ where $v(x,t)$ is the solution of the the IBVP (5)-(7). Define $w(x,t) = v_t(x,t)$; then w is the solution of the backward IBVP

$$\begin{gathered} w_{tt} - \triangle w = 0, \qquad (x,t) \in \overline{B_\rho} \times [0, 2\rho], \\ w(x, t{=}2\rho) = 0,\ w_t(x, t{=}2\rho) = 0, \qquad x \in \overline{B_\rho}, \\ w(x,t) = p_t(x,t), \qquad (x,t) \in S_\rho \times [0, 2\rho]. \end{gathered}$$

Further, for $x \in \overline{B_\rho}$

$$w_t(x, t{=}0) = v_{tt}(x, t{=}0) = \triangle v(x, t{=}0) = 0$$

because $v(x, t{=}0) = 0$; also $p_t \in \tilde{C}^{\infty}(S_\rho \times [0, \infty))$ because $p \in \tilde{C}^{\infty}(S_\rho \times [0, \infty))$. Hence Theorem 2 implies that p_t is in the range of $\mathcal{T}$. But $\mathcal{T} = \partial_t \mathcal{S}$, the range of $\mathcal{S}$ is a subset of $\tilde{C}^{\infty}(S_\rho \times [0, \infty))$, and the only function in $\tilde{C}^{\infty}(S_\rho \times [0, \infty))$ which is the integral of p_t is p, hence p must be in the range of $\mathcal{S}$.

QED

4 Proof of Theorem 3

Suppose $f_i \in \tilde{C}^\infty(S_\rho \times [0,\infty))$, $i = 1,2$, and let u_i be the solution of the IVP

$$\begin{aligned} u_{i,tt} - \triangle u_i = 0, & \qquad (x,t) \in R^n \times [0,\infty), \\ u_i(x,t{=}0) = 0, \ u_{i,t}(x,t{=}0) = f_i(x), & \qquad x \in R^n, \end{aligned}$$

In [6] (see Theorem 6), we proved the following identities

$$\frac{1}{2}\int_{R^n} f_1(x)\, f_2(x)\, dx = \frac{-1}{\rho}\int_0^\infty \int_{|p|=\rho} t\, u_1(p,t)\ u_{2tt}(p,t)\, dS_p\, dt, \tag{36}$$

$$\frac{1}{2}\int_{R^n} f_1(x)\, f_2(x)\, dx = \frac{1}{\rho}\int_0^\infty \int_{|p|=\rho} t\, u_{1t}(p,t)\ u_{2t}(p,t)\, dS_p\, dt. \tag{37}$$

If we integrate by parts the RHS of (36) and use (37) we obtain

$$\int_{S_\rho \times [0,\infty)} u_1(x,t)\ u_{2,t}(x,t)\ dS_x\ dt = 0.$$

This proves the necessity part of Theorem 3. We now establish the sufficiency part.

Suppose $p(x,t) \in \tilde{C}^\infty(S_\rho \times [0,\infty))$ which satisfies (15) for all $q(x,t)$ in the range of $\mathcal{S}$. Fix a positive integer m, and let $f(r)$ be a smooth even function on $(-\infty,\infty)$ which is supported in $[-\rho,\rho]$. Let $a(r,t)$ be the solution of the IVP

$$\begin{aligned} a_{tt} - a_{rr} - \frac{\nu-1}{r} a_r = 0, & \qquad (r,t) \in (-\infty,\infty) \times [0,\infty), \\ a(r,t{=}0) = 0, \ a_t(r,t{=}0) = f(r), & \qquad r \in (-\infty,\infty). \end{aligned}$$

Here $\nu = n + 2k(m)$ where $k(m)$ is the degree of homogeneity of $\phi_m(\theta)$ where $\phi_m(.)$ are the spherical harmonic functions introduced in the Introduction just before the statement of Theorem 4 - also see the beginning of Section 2. Then, as seen before,

$$\mathcal{S}(f(r) r^{k(m)} \phi_m(\theta)) = a(r,t) r^{k(m)} \phi_m(\theta)|_{r=\rho}.$$

Hence, from the hypothesis, we have

$$\begin{aligned} 0 &= \int_{S_\rho \times [0,\infty)} p(x,t)\ \partial_t \mathcal{S}(f(r) r^{k(m)} \phi_m(\theta))(x,t)\ dS_x\, dt \\ &= \int_{S_\rho \times [0,\infty)} p(x,t)\ a_t(\rho,t)\ \rho^{k(m)} \phi_m(\theta)\ dS_x\, dt \\ &= \text{constant} \int_0^\infty a_t(\rho,t)\, p_m(t)\, dt \end{aligned} \tag{38}$$

where

$$p(x,t) = \sum_{m=1}^\infty p_m(t) \rho^{k(m)} \phi_m(\theta), \qquad (x,t) \in S_\rho \times (-\infty,\infty);$$

is the spherical harmonic expansion of $p(x,t)$; here $p_m(t)$ is a smooth function on $(-\infty,\infty)$ which is supported on $[0,2\rho]$.

Since ν is odd, we define the integer $\mu=(\nu-1)/2$. Taking $f=D^\mu g$ where $g(r)$ is an arbitrary, even, smooth function on $(-\infty,\infty)$ with support in $[-\rho,\rho]$, we define

$$b(r,t)=\frac{1}{2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\mu}(g(r+t)+g(r-t)).$$

Then from usual arguments, $b(r,t)$ is the solution of the IVP

$$\begin{aligned} b_{tt}-b_{rr}-\frac{\nu-1}{r}b_r=0, \qquad & (r,t)\in(-\infty,\infty)\times[0,\infty),\\ b(r,t{=}0)=f(r),\ b_t(r,t{=}0)=0, \qquad & r\in(-\infty,\infty). \end{aligned}$$

Hence $a_t=b$ implying

$$2a_t(r,t)=\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\mu}(g(r+t)+g(r-t)).$$

Since $g(r)$ is supported in $[-\rho,\rho]$ and is even in r, for $t\geq 0$ we have

$$\begin{aligned} 2a_t(\rho,t) &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\mu}(g(r+t)+g(r-t))|_{r=\pm\rho}\\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\mu}(g(r+t))|_{r=-\rho}\\ &= A_\mu(-\rho,\partial_t)(g(t-\rho)) \end{aligned}$$

using the definition of A_μ in (17). Hence from (38), we have

$$\begin{aligned} 0 &= \int_0^\infty p_m(t)\,A_\mu(-\rho,\partial_t)(g(t-\rho))\,dt\\ &= \int_0^\infty g(t-\rho)\,A_\mu(-\rho,-\partial_t)(p_m(t))\,dt\\ &= \int_0^\infty g(t-\rho)\,A_\mu(\rho,\partial_t)(p_m(t))\,dt\\ &= \int_{-\rho}^\infty g(t)\,A_\mu(\rho,\partial_t)(p_m(t+\rho))\,dt, \end{aligned}$$

for every smooth even function $g(r)$ on $(-\infty,\infty)$ with support in $[-\rho,\rho]$. Hence $A_\mu(\rho,\partial_t)(p_m(t+\rho))$ is an odd function of t on $[-\rho,\rho]$.

So, noting the form of the solutions of (1)-(2), the sufficiency part of Theorem 3 will follow from the following proposition.

Proposition 8. *Suppose ν is an odd positive integer, μ the positive integer $\mu\equiv(\nu-1)/2$, and $p(t)$ is the restriction to $[0,\infty)$ of a smooth function on $(-\infty,\infty)$ which is supported on $[0,2\rho]$. If $A_\mu(\rho,\partial_t)(p(t+\rho))$ is an odd function of t on $[-\rho,\rho]$ then there is a smooth even function $f(r)$ on*

$(-\infty, \infty)$, *supported in* $[-\rho, \rho]$, *so that* $a(\rho, t) = p(t)$ *on* $[0, \infty)$ *where* $a(r, t)$ *is the solution of the IVP*

$$a_{tt} - a_{rr} - \frac{\nu - 1}{r} a_r = 0, \qquad (r, t) \in (-\infty, \infty) \times [0, \infty), \tag{39}$$

$$a(r, t{=}0) = 0, \;\; a_t(r, t{=}0) = f(r), \qquad r \in (-\infty, \infty). \tag{40}$$

Proof of Proposition 8

Let $q(t)$ be the solution of the ODE $A_\mu(-\rho, \partial_t) q(t) = p'(t)$ with $q(t)$ zero for $t \leq 0$; note that $p(t)$ is zero for $t \leq 0$. Then

$$q(t) = \int_0^t p'(s)\, G(t - s)\, ds$$

where $G(t)$ is the solution of the homogeneous ODE $A_\mu(-\rho, \partial_t) G(t) = 0$ on $(-\infty, \infty)$ with $\partial_t^i G(0) = 0$ for $i = 0, \cdots, \mu - 2$ and $\partial_t^{\mu-1} G(0) = 1$. If λ_i, $i = 1, \cdots, \mu - 1$ are the non-zero roots of the characteristic polynomial $A_\mu(-\rho, \tau)$ (which are distinct - see Lemma 6), then $G(t) = c_0 + \sum_{i=1}^{\mu-1} c_i e^{\lambda_i t}$ for suitably chosen constants c_i. Hence

$$\begin{aligned} q(t) &= c_0 \int_0^t p'(s)\, ds + \sum_{i=1}^{\mu-1} c_i\, e^{\lambda_i t} \int_0^t p'(s)\, e^{-\lambda_i s}\, ds \\ &= p(t) \sum_{i=0}^{\mu-1} c_i + \sum_{i=1}^{\mu-1} \lambda_i c_i\, e^{\lambda_i t} \int_0^t p(s)\, e^{-\lambda_i s}\, ds \\ &= \sum_{i=1}^{\mu-1} \lambda_i c_i\, e^{\lambda_i t} \int_0^t p(s)\, e^{-\lambda_i s}\, ds \end{aligned} \tag{41}$$

because $\sum_{i=0}^{\mu-1} c_i = G(0) = 0$. We now show that $q(t)$ is supported in $[0, 2\rho]$; note we know that $q(t)$ is zero for $t \leq 0$.

Using integration by parts and the fact that $p(t)$ is supported in $[0, 2\rho]$, we have

$$\begin{aligned} \int_{-\rho}^{\rho} A_\mu(\rho, \partial_t) p(t + \rho)\, e^{\lambda_i t} dt &= \int_{-\rho}^{\rho} p(t + \rho) A_\mu(\rho, -\partial_t) e^{\lambda_i t}\, dt \\ &= \int_{-\rho}^{\rho} p(t + \rho) A_\mu(\rho, -\lambda_i) e^{\lambda_i t}\, dt \\ &= 0 \end{aligned}$$

because $A_\mu(\rho, -\lambda_i) = A_\mu(-\rho, \lambda_i) = 0$ by the definition of λ_i. Since $A_\mu(\rho, \partial_t) p(t + \rho)$ is an odd function of t on $[-\rho, \rho]$ and $p(t)$ is supported in $[0, 2\rho]$, we now have

$$\int_{-\rho}^{\rho} A_\mu(\rho, \partial_t) p(t + \rho)\, e^{-\lambda_i t} dt = - \int_{-\rho}^{\rho} A_\mu(\rho, \partial_t) p(t + \rho)\, e^{\lambda_i t} dt = 0.$$

Therefore

$$\begin{aligned} 0 &= \int_{-\rho}^{\rho} A_\mu(\rho,\partial_t) p(t+\rho)\, e^{-\lambda_i t} dt = \int_{-\rho}^{\rho} p(t+\rho) A_\mu(\rho,-\partial_t) e^{-\lambda_i t}\, dt \\ &= A_\mu(\rho,\lambda_i) \int_{-\rho}^{\rho} p(t+\rho) e^{-\lambda_i t}\, dt = A_\mu(\rho,\lambda_i)\, e^{\lambda_i \rho} \int_0^{2\rho} p(t) e^{-\lambda_i t}\, dt. \end{aligned}$$

From Lemma (6), $A_\mu(r,\tau)$ and $A_\mu(r,-\tau)$ do not have any common non-zero roots when $r \neq 0$, so $A_\mu(\rho,\lambda_i) \neq 0$. Hence

$$\int_0^{2\rho} p(t) e^{-\lambda_i t}\, dt = 0, \qquad i = 1, 2, \cdots, \mu - 1.$$

Hence from (41), and the fact that $p(.)$ is supported in $[0, 2\rho]$, we have $q(.)$ is zero for $t \geq 2\rho$.

Define the (even in r) function

$$b(r,t) = D^\mu (q(t+r+\rho) + q(t-r+\rho)), \qquad (r,t) \in (-\infty,\infty) \times [0,\infty).$$

Then, for $t \geq 0$,

$$b(r{=}\rho, t) = D^\mu (q(t+r+\rho) + q(t-r+\rho))|_{r=\pm\rho} = D^\mu (q(t+r+\rho)|_{r=-\rho} = A_\mu(-\rho, \partial_t) q(t) = p'(t).$$

Next we show that $b_t(r, t{=}0) = 0$ for all $r \in (-\infty,\infty)$. Since $A_\mu(\rho,\partial_t)p(t+\rho)$ is an odd function of t, the function $A_\mu(\rho,\partial_t)p'(t+\rho)$ is an even function of t; hence $A_\mu(\rho,\partial_t)A_\mu(-\rho,\partial_t)q(t+\rho)$ is an even function of t. Now $A_\mu(\rho,\tau) = A_\mu(-\rho,-\tau)$, hence

$$A_\mu(\rho,\tau) A_\mu(-\rho,\tau) = c\,\tau^2 \prod_{i=1}^{\mu-1} (-\tau - \lambda_i)(\tau - \lambda_i) = c(-1)^{\mu-1} \tau^2 \prod_{i=1}^{\mu-1} (\tau^2 - \lambda_i^2).$$

Hence $A_\mu(\rho,\partial_t)A_\mu(-\rho,\partial_t)$ is a sum of even order derivatives with respect to t and hence preserves parity of functions. Let $q_e(t)$ and $q_o(t)$ be the even and odd parts of $q(t+\rho)$; so $q(t+\rho) = q_e(t) + q_o(t)$. Hence $A_\mu(\rho,\partial_t)A_\mu(-\rho,\partial_t)q_o(t)$ is the odd part of $A_\mu(\rho,\partial_t)A_\mu(-\rho,\partial_t)q(t+\rho)$ and hence must be zero. Hence $q_o(t)$ satisfies a homogeneous differential equation; since $q_o(t)$ is of compact support, it must be zero. Hence $q(t+\rho)$ is an even function of t implying

$$b_t(r, t{=}0) = D^\mu (q'(r+\rho) + q'(-r+\rho)) = 0, \qquad \text{for all } r \in (-\infty,\infty).$$

Hence $b(r,t)$ is the solution of

$$\begin{gathered} b_{tt} - b_{rr} - \frac{\nu - 1}{r} b_r = 0, \qquad (r,t) \in (-\infty,\infty) \times [0,\infty), \\ b(r, t{=}0) = f(r),\ b_t(r, t{=}0) = 0, \qquad r \in (-\infty,\infty), \end{gathered}$$

where

$$f(r) = D^\mu (q(r+\rho) + q(-r+\rho))$$

is supported in $[-\rho, \rho]$. Then

$$a(r,t) = \int_0^t b(r,s)\, ds$$

is the solution we sought. **QED**

5 Proof of Theorem 4

Given a $P(x,t) \in \tilde{C}^\infty(S_\rho \times [0,\infty))$, let $V(x,t)$ be the solution of the IBVP (11) - (13) in Theorem 2. Then P is in the range of $\mathcal{T}$ iff $V_t(.,t{=}0) = 0$ on $\overline{B_\rho}$, that is iff

$$\int_{B_\rho} V_t(x,t{=}0)\, g(x)\, dx = 0 \tag{42}$$

for some dense family of smooth functions g in $L^2(B_\rho)$. Let $w(x,t)$ be the solution of the IBVP

$$w_{tt} - \triangle w = 0 \qquad (x,t) \in \overline{B_\rho} \times [0,\infty) \tag{43}$$

$$w(x,t) = 0, \qquad (x,t) \in S_\rho \times [0,\infty), \tag{44}$$

$$w(\cdot,t{=}0) = g,\ \ w_t(\cdot,t{=}0) = 0 \quad x \in \overline{B_\rho}. \tag{45}$$

Then integrating the identity

$$0 = w(V_{tt} - \triangle V) - V(w_{tt} - \triangle w) = (wV_t - Vw_t)_t - \nabla \cdot (w\nabla V - V\nabla w)$$

over the region $B_\rho \times [0, 2\rho]$ and using Gauss's theorem and the initial and boundary conditions satisfied by V and w, we obtain

$$\int_{B_\rho} g(x)\, V_t(x,t{=}0)\, dx = \int_{S_\rho} \int_0^{2\rho} P(x,t)\, \partial_\nu w(x,t)\, dt\, dS_x; \tag{46}$$

here ∂_ν is the outward normal derivative. Hence, from (42), P belongs to the range of $\mathcal{T}$ if and only if the right side of (46) is zero for a family of smooth functions g on R^n, which are supported in $\overline{B_\rho}$, and which are dense in $L^2(B_\rho)$.

Consider the family of functions

$$g_{m,\omega}(x) = |x|^{(2-n)/2-k} \phi_m(x) J_{k+(n-2)/2}(\omega|x|), \qquad x \in \overline{B_\rho},$$

where m varies over the positive integers, k is the degree of homogeneity of $\phi_m(x)$ (defined earlier), and ω varies over the positive zeros of $J_{k+(n-2)/2}(\rho\omega) = 0$. One may check that $g_{m,\omega}(x)$ are all of the Dirichlet eigenfunctions of the Laplacian on B_ρ; $g_{m,\omega}$ corresponds to the eigenvalue $-\omega^2$; and hence the family $g_{m,\omega}$ is dense in $L^2(B_\rho)$. For such g, the solution $w(x,t)$ of (43) - (45) is

$$w(x,t) = g_{m,\omega}(x) \cos(\omega t), \qquad (x,t) \in \overline{B_\rho} \times [0, 2\rho].$$

For use later, we note that for $x = \rho\theta$ in S_ρ

$$(\partial_\nu w)(x,t) = \rho^{(2-n)/2} J'_{k+(n-2)/2}(\rho w)\, \phi_m(\theta)\, \cos(\omega t) = c\, \phi_m(\theta) \cos(\omega t) \tag{47}$$

for some non-zero constant c (the positive roots of $J_{k+(n-2)/2}(\rho\omega) = 0$ are of multiplicity 1).

Then, a function $P \in \tilde{C}^\infty(S_\rho \times [0,\infty))$ is in the range of $\mathcal{S}$ iff for $m = 1, 2, \cdots$ and all positive roots ω of $J_{k+(n-2)/2}(\rho\omega) = 0$, we have (for some non-zero constant c)

$$\begin{aligned} 0 &= \int_{S_\rho} \int_0^{2\rho} P(x,t)\, \partial_\nu w(x,t)\, dt\, dS_x = c \int_0^{2\rho} \int_{|\theta|=1} P(\rho\theta, t)\, \phi_m(\theta)\, \cos(\omega t)\, d\theta\, dt \\ &= c \int_0^{2\rho} P_m(t)\, \cos(\omega t)\, dt. \end{aligned}$$

This completes the proof of Theorem 4. **QED**

6 Moment conditions

Here we prove our claim in the introduction that if $U(x,t)$ is the solution of the IVP (9), (10) for some $f \in C_0^\infty(\overline{B_\rho})$ then

$$\int_0^\infty t^{2j} U_m(\rho,t)\, dt = 0, \qquad j = 0, 1, \cdots, k + (n-3)/2,$$

where k is the degree of homogeneity of $\phi_m(x)$ and

$$r^k U_m(r,t) = \int_{|\theta|=1} U(r\theta, t)\, \phi_m(\theta)\, d\theta.$$

We give two somewhat different proofs of our claim.

6.1 Proof using spherical harmonics expansions

Imitating the ideas in Section 2, we may show that

$$U_m(r,t) = \frac{1}{2} D^\mu (f_m(r+t) + f_m(r-t))$$

where $\mu = (n + 2k - 1)/2$ and

$$r^k f_m(r) = \int_{|\theta|=1} f(r\theta)\, \phi_m(\theta)\, d\theta.$$

Note that $f_m(r)$ is a smooth even function on $(-\infty, \infty)$ which is supported on $[-\rho, \rho]$. For future use, we note that for $t \geq 0$

$$\begin{aligned} U_m(\rho,t) &= \frac{1}{2} D^\mu (f_m(r+t) + f_m(r-t))|_{r=\pm\rho} = \frac{1}{2} D^\mu (f_m(r+t))|_{r=-\rho} \\ &= \frac{1}{2} A_\mu(r, \partial_t) f_m(r+t)|_{r=-\rho} = \frac{1}{2} A_\mu(-\rho, \partial_t) f_m(t-\rho) \\ &= \frac{1}{2} A_\mu(\rho, -\partial_t) f_m(t-\rho). \end{aligned} \tag{48}$$

Below c will represent a generic constant which could possibly be zero. From (48) and an integration by parts, we have

$$\begin{aligned} 2\int_0^{2\rho} t^{2j} U_m(\rho,t)\, dt &= \int_0^{2\rho} t^{2j}\, A_\mu(\rho, -\partial_t) f_m(t-\rho)\, dt \\ &= \int_0^{2\rho} f_m(t-\rho)\, A_\mu(\rho, \partial_t)(t^{2j})\, dt \\ &= \int_{-\rho}^{\rho} f_m(t)\, A_\mu(\rho, \partial_t)((t+\rho)^{2j})\, dt. \end{aligned} \tag{49}$$

However, from our definition of A_μ we have

$$\begin{aligned}
A_\mu(\rho,\partial_t)(t+\rho)^{2j} &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^\mu \left((t+r)^{2j}\right)|_{r=\rho} \\
&= \sum_{l=0}^{2j}\binom{2j}{l}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^\mu \left(r^l t^{2j-l}\right)|_{r=\rho} \\
&= \sum_{l=0}^{2j}\binom{2j}{l} t^{2j-l}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^\mu \left(r^l\right)|_{r=\rho} \qquad (50)
\end{aligned}$$

So if j is a non-negative integer between 0 and $k+(n-3)/2$ then $l \leq 2j \leq 2k+(n-3) < 2\mu$. Since $\frac{1}{r}\frac{\partial}{\partial r}$ corresponds to differentiation with respect to r^2, if l is even then

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^\mu \left(r^l\right) = 0.$$

Hence, in (50), the non-zero terms arise only when l is odd, and hence the RHS of (50) consists of odd powers of t. Combining this with (49) and the fact that $f_m(r)$ is an even function, we conclude that

$$\int_0^{2\rho} t^{2j} U_m(\rho,t)\,dt = 0, \qquad j = 0,1,\cdots,k+(n-3)/2.$$

6.2 Proof using solutions of the wave equation

For use below, we observe that for $x \in S_\rho$, $(Mf)(x,t)$ vanishes to infinite order at $t=0$ because f is supported in $\overline{B_\rho}$. Below, c represents a constant which could possibly be zero.

If $U(x,t)$ is the solution of (9), (10) then $U = u_t$ where u is the solution of (1), (2). Using the formula (3) we have

$$U(x,t) = c\frac{\partial}{\partial t}\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{(n-3)/2}(t^{n-2}(\mathcal{M}f)(x,t)).$$

Hence, for a fixed $x \in S_\rho$, using integration by parts, we have

$$\begin{aligned}
\int_0^{2\rho} t^{2j}\, U(x,t)\,dt &= c\int_0^\infty t^{2j}\,\frac{\partial}{\partial t}\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{(n-3)/2}\left(t^{n-2}(\mathcal{M}f)(x,t)\right)\,dt \\
&= c\int_0^\infty t^{n-2}\,(\mathcal{M}f)(x,t)\,\left(\frac{\partial}{\partial t}\frac{1}{2t}\right)^{(n-3)/2}(t^{2j-1})\,dt. \\
&= c\int_0^\infty t^{n-2}\,(\mathcal{M}f)(x,t)\,\frac{\partial}{\partial t}\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{(n-5)/2}(t^{2j-2})\,dt. \qquad (51)
\end{aligned}$$

Since $\frac{1}{2t}\frac{\partial}{\partial t}$ corresponds to differentiation with respect to t^2, we have

$$\frac{\partial}{\partial t}\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{(n-5)/2}(t^{2j-2}) = \begin{cases} 0 & \text{if } 2j-2 \leq n-5 \\ c\,t^{2j-2-(n-5)-1} & \text{otherwise.}\end{cases}$$

Hence $\int_0^{2\rho} t^{2j}\, U(x,t)\, dt = 0$ if $2j \leq n-3$.

If $2j \geq n-1$ then continuing with (51) we have

$$\begin{aligned}
\int_0^{2\rho} t^{2j}\, U(x,t)\, dt &= c\int_0^{\infty} (\mathcal{M}f)(x,t)\, t^{n-2+2j-2-(n-5)-1}\, dt & (52)\\
&= c\int_0^{\infty} t^{2j}\, (\mathcal{M}f)(x,t)\, dt = c\int_0^{\infty} t^{2j}\int_{|\theta|=1} f(x+t\theta)\, d\theta\, dt & (53)\\
&= c\int_{R^n} |y-x|^{2j-(n-1)} f(y)\, dy & (54)\\
&= c\int_{R^n} (|y|^2+|x|^2-2y\cdot x)^{j-(n-1)/2} f(y)\, dy & (55)\\
&= c\int_{R^n} (|y|^2+\rho^2-2y\cdot x)^{j-(n-1)/2} f(y)\, dy & (56)
\end{aligned}$$

if $x \in S_\rho$. Hence, for $x \in S_\rho$, $\int_0^{2\rho} t^{2j}\, U(x,t)\, dt$ is 0 if $j < (n-1)/2$ and is the restriction to S_ρ of a polynomial of degree at most $j-(n-1)/2$ for all $j \geq (n-1)/2$.

If $\phi_m(x)$ is homogeneous (polynomial) of degree $k(m)$, then $j-(n-1)/2 < k(m)$ iff $2j < 2k(m)+n-1$. So from the orthogonality properties of the spherical harmonics (see [16]), for $2j < 2k(m)+n-1$ we have

$$0 = \int_{|\theta|=1}\int_0^{\infty} t^{2j}\, U(\rho\theta,t)\, \phi_m(\theta)\, dt\, d\theta = c\int_0^{\infty} t^{2j} U_m(\rho,t)\, dt,$$

for some non-zero constant c.

7 Proof of Lemma 6

To establish the lemma we use some known results about Hankel functions in [8], namely, that $H^{(1)}_{-1/2}(z) = \sqrt{2/(\pi z)}e^{iz}$, a differentiation formula, and a finite expansion for $H^{(1)}_{n-1/2}(z)$ when n is a positive integer. By Fourier inversion, for $r \neq 0$,

$$A_\mu(r,\partial_t) f(t+r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(t+r)\tau} A_\mu(r, i\tau)\hat{f}(\tau) d\tau;$$

also, using the definition, we have

$$\begin{aligned}
A_\mu(r,\partial_t) f(t+r) &= \left(\frac{1}{r}\frac{d}{dr}\right)^{\mu} f(t+r)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\tau}\left[\left(\frac{1}{r}\frac{d}{dr}\right)^{\mu} e^{ir\tau}\right]\hat{f}(\tau) d\tau.
\end{aligned} \qquad (57)$$

Now $e^{iz} = \sqrt{\pi/2}\,\sqrt{z}\,H^{(1)}_{-1/2}(z)$ and so

$$\begin{aligned}\left(\frac{1}{r}\frac{d}{dr}\right)^{\mu} e^{ir\tau} &= \tau^{2\mu}\left(\frac{1}{z}\frac{d}{dz}\right)^{\mu} e^{iz}|_{z=r\tau} \\ &= \tau^{2\mu}\sqrt{\pi/2}\,\left(\frac{1}{z}\frac{d}{dz}\right)^{\mu}(z^{1/2}\,H^{(1)}_{-1/2}(z))|_{z=r\tau} \\ &= \tau^{2\mu}\,\sqrt{\pi/2}\,(-1)^{\mu} z^{1/2-\mu}\,H^{(1)}_{\mu-1/2}(z)|_{z=r\tau} \\ &= \tau^{2\mu}\sqrt{\pi/2}\,(-1)^{\mu}(r\tau)^{1/2-\mu}\,H^{(1)}_{\mu-1/2}(r\tau).\end{aligned} \tag{58}$$

The closed form sum for Hankel function of index a natural number minus one-half takes the form

$$H^{(1)}_{\mu-1/2}(z) = \sqrt{\frac{2}{\pi z}}i^{-\mu}e^{iz}\sum_{j=0}^{\mu-1}(-1)^j\frac{(\mu+j-1)!}{j!(\mu-j-1)!}\frac{1}{(2iz)^j}. \tag{59}$$

Substituting this in the integral and observing that it should give the integrand to be $e^{i(t+r)\tau}A_\mu(r,i\tau)$ we get the expression given in the statement of the lemma.

To see that, for $r \neq 0$, $A_\mu(r,\tau)$ and $A_\mu(r,-\tau)$ have no common non-zero root, suppose that $z \neq 0$ is a common root. Then $\pm z$ are roots of $A_\mu(\rho,\tau)$ and so $A_\mu(\rho,\tau) = (\tau^2 - z^2)q(\tau)$ for some polynomial $q(\tau)$ of degree $\mu - 2$. However, in any such product the coefficient of $\tau^{\mu-1}$ is zero and this does not hold for A_μ. To show that the roots of $A_\mu(\rho,z) = 0$ are simple we note that by (59) if $A_\mu(\rho,z)$ has a multiple root at $z_0 \neq 0$, then $H^{(1)}_{\mu-1/2}(z)$ and its derivative have a common zero $z = -i\rho z_0$.. But this can not occur, since $H^{(1)}_{\mu-1/2}$ is a solution of the Bessel equation. Finally, since the coefficient of z in A_μ is non-zero, $z = 0$ is a simple root also. **QED**

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