Approximate reconstruction formulae for the cone beam transform I

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Abstract

In this paper we present a method for deriving approximate inversion formulae for the divergent beam x-ray transform in three-space when the source set is a curve. When the source set is a planar convex curve, we obtain new approximate inversion formulae which are quasi-local. Finally we compare these methods with existing convolution- backprojection algorithms for 3-D reconstruction.

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1. Introduction

Several authors have studied reconstruction algorithms for the divergent beam x-ray transform in R^3 when the set of sources comprises a curve. Feldkamp et. al. [FDK] have presented a reconstruction method for source set a plane circle, but acknowledge that their formula is not the result of a rigorous development. Nonetheless, they obtain fairly good reconstructions and their method has been adopted by other workers as well, e.g. Webb et. al. [WSBH]. P. Grangeat [Gr] presents an approximate inversion formula using a new relation between the divergent beam transform and the Radon transform. His method has been implemented, but it appears that to compute the reconstruction at any point in space requires the use of all the data. B. D. Smith [Sm] has shown that if a source curve satisfies some strong geometric conditions, then there is an exact reconstruction formula in which the reconstruction at some point x requires for each source γ only those line integrals over lines through γ lying in the plane determined by γ, x, and the tangent to the curve at γ. Unfortunately, as we show in an appendix, there are no bounded curves satisfying his geometric hypotheses. When the source set is a circle, Smith also suggests an approximate reconstruction method based on his full derivation. This approximate method inherits the quasi- local nature of his exact formula. This approximate method coincides with that of Feldkamp et. al., although that connection is not made in the paper by Smith.

In this paper we study a different class of approximate reconstruction formulae. For certain space curves, these are mathematically exact, in that as the radius of the point spread function tends to zero, the approximate reconstruction converges to the true function. For convex planar curves, we produce approximations which have nearly as good localization properties as the methods of Feldkamp et. al. and B. D. Smith.

In section II, we present a formula, found first by P. Grangeat relating the divergent beam transform and the Radon transform. We also review the method of deriving approximate inversion formulae for the Radon transform due to Leahy, Smith, Solmon, and Wagner [LSSW]. In section III we show how the two results of section II may be combined to provide an approximate inversion formula for the divergent beam x-ray transform with sources on a Tuy-Kirillov curve. In section IV we show that for the more severely ill-posed problem of 3-D reconstruction from sources lying on a plane curve a class of computationally tractable formulae arise from applying the methods of section III. In section V we compare the algorithm of Feldkamp et. al. and those proposed in section IV. Both are theoretically equivalent (although perhaps not in numerical implementation) to limited data inversion for the 3-D Radon transform. In the appendix we show that no bounded curve exists which satisfies the geometric hypotheses of B. Smith.

Notation: We will be concerned with reconstruction of a function f supported in a bounded open set Ω in R^3. We will usually take Ω to be the unit ball. Γ is a piecewise C^1 curve lying outside of Ω. When h is a distribution and g lies in an appropriate space of testing functions we may use $\int hg$ for $\langle h, g\rangle$. We use the Fourier transform convention of

$$\hat{f}(\xi) \;=\; (2\pi)^{\frac{-n}{2}} \int e^{-i\langle x,\xi\rangle}\, f(x)dx.$$

The divergent beam x-ray transform of a function f on R^n in the direction $\theta \in\ S^{n-1}$ is defined by

$$\mathcal{D}f(a,\theta) = \mathcal{D}_a f(\theta) = \int_0^\infty f(a+t\theta)\ dt.$$

The Radon transform of a function f in direction θ is defined by

$$\mathcal{R}f(t,\theta) = \mathcal{R}_\theta f(t) = \int_{\langle x,\theta\rangle = t} f(x)dx.$$

If $\eta \in S^{n-1}$ we denote by $\eta^\perp$ the subspace of R^n orthogonal to η. We will use E_η to denote the orthogonal projection on $\eta^\perp$ and π_η to denote the orthogonal projection on the subspace spanned by η.

2. Review

Proposition 1.1. *Suppose that $f \in C_0^1(\Omega)$, $a \in R^3$, and $\theta \in S^2$. Then*

$$\int_{S^2} \mathcal{D}_a f(\eta)\ \delta'(\langle \eta,\theta\rangle)d\eta = -(\mathcal{R}_\theta f)'(\langle a,\theta\rangle). \tag{1.1}$$

Proof. Choose spherical coordinates on the unit sphere so that φ is the polar angle measured from direction θ and ω is longitude measured from any ray in $\theta^\perp \cap S^2$. We suppose also that a Cartesian frame is chosen so that the z-axis coincides with the ray through θ. Then

$$\begin{aligned}
\int_{S^2} \mathcal{D}_a f(\eta)\delta'(\langle \eta,\theta\rangle)d\eta &= -\int_0^{2\pi} -\frac{\partial}{\partial\varphi}\mathcal{D}_a f\begin{pmatrix} cos\omega sin\varphi \\ sin\omega sin\varphi \\ cos\varphi \end{pmatrix}\Bigg|_{\varphi=\frac{\pi}{2}} d\omega \\
&= \int_0^{2\pi} \frac{\partial}{\partial\varphi}\int_0^\infty f(a+t\begin{pmatrix} cos\omega sin\varphi \\ sin\omega sin\varphi \\ cos\varphi \end{pmatrix})dt\Bigg|_{\varphi=\frac{\pi}{2}} d\omega \\
&= -\int_0^{2\pi}\int_0^\infty f_z(a+t\begin{pmatrix} cos\omega \\ sin\omega \\ 0 \end{pmatrix})\ tdtd\omega \\
&= -\int\int f_z(a+\begin{pmatrix} x \\ y \\ 0 \end{pmatrix})\ dxdy \\
&= -(\mathcal{R}_\theta f)'(\langle a,\theta\rangle).
\end{aligned}$$

We also use the approach to approximate reconstruction formulae emphasized by K. T. Smith and coworkers. Let e be an approximate δ-function in R^n; that is, e is a smooth function with integral equal to unity, and let $e_\rho(x) = \rho^{-n}e(\frac{x}{\rho})$. Then $e_\rho \to \delta$ as ρ tends to zero, with convergence as distributions. We shall also require that e, and thus e_ρ

be compactly supported. (Point spread functions which are not approximate δ-functions are treated by Smith and Keinert in [SK].)

We shall need the following formula from Leahy, Smith, Solmon, and Wagner [LSSW] (see also K. T. Smith [S]). If e_ρ is an approximate δ-function in R^n and $\mathcal{R}$ is the Radon transform, then

$$e_\rho * f(x) = \int_{S^{n-1}} \int_{-\infty}^{\infty} \mathcal{R}_\theta f(t) k_\theta(\langle x, \theta \rangle - t) dt d\theta \tag{1.3}$$

where

$$k_\theta(t) = \frac{1}{2}(2\pi)^{1-n}\Lambda^{n-1}\mathcal{R}_\theta e(t). \tag{1.4}$$

Here Λ is the square root of the Laplacian on R^1 defined via Fourier transform by

$$(\Lambda g)^\wedge(\tau) = |\tau|\hat{g}(\tau). \tag{1.5}$$

We shall need (1.3) and (1.4) for n=3.

3. Approximate Inversion for Kirillov-Tuy Curves

In an earlier paper [Fi], the author showed that the degree of ill-posedness for reconstruction from divergent beam data for sources on a curve depends on geometric properties of the curve. We recall that Γ is called a Kirillov-Tuy curve if every hyperplane which meets Ω intersects Γ transversely in at least one point. In this section we use the approximate inversion formula (1.3) and formula (1.2) relating the Radon transform and the divergent beam x-ray transform to produce an approximate inversion formula for the divergent beam transform when the source set is a Kirillov-Tuy curve.

We suppose throughout this section that Ω is the unit ball in R^3, $f \in C_0^1(\Omega)$, and that Γ is a Kirillov-Tuy curve lying in $R^3 \backslash \overline{\Omega}$.

First we show that integration over $S^2 \times I$ can be pulled back to integration over $S^2 \times \Gamma$. Let $\pi : \Gamma \times S^2 \to R \times S^2$ be given by $\pi(\gamma, \theta) = (\langle \gamma, \theta \rangle, \theta)$. Now each point $(t, \theta) \in I \times S^2$ determines a hyperplane, $\langle x, \theta \rangle = t$, which meets Γ transversely in at least one point. Let γ_0 be such a point. By the inverse function theorem, a neighborhood of (γ_0, θ) is mapped diffeomorphically onto a neighborhood of (t, θ) which we denote $V_{(t,\theta)}$. The family of open sets $V_{(t,\theta)}$ provides an open cover of $I \times S^2$ which admits a locally finite refinement $\{V_j\}$ where each V_j is diffeomorphic to an open U_j in $\Gamma \times S^2$, the diffeomorphism given by the restriction of π to U_j. Let $\{\varphi_j\}$ be a partition of unity subordinate to $\{V_j\}$ and let $\kappa_j = \varphi_j \circ (\pi|_{U_j})$. Let $m(\gamma, \theta) = \sum_j \kappa_j(\gamma, \theta)$. Then for any integrable g on $S^2 \times I$

$$\int_{S^2} \int_I g(t, \theta)\ d\theta dt = \int_{S^2} \int_\Gamma\ g \circ \pi\ J(\gamma, \theta)\ d\gamma d\theta \tag{3.1}$$

where $J(\gamma, \theta) = m(\gamma, \theta)|\langle \gamma', \theta \rangle|$, γ' being the unit tangent to Γ at γ.

Theorem 3.2. *Let e be an approximate δ-function. Then for* $x \in \Omega$

$$e * f(x) = \frac{1}{8\pi^2} \int_{S^2} \int_{\Gamma} \mathcal{D}_\gamma f(\eta) K(\eta, x, \gamma, \gamma')\ d\gamma d\eta \tag{3.3}$$

where

$$K(\eta, x, \gamma, \gamma') = \int_{S^2} J(\theta, \gamma)(\mathcal{R}_\theta e)'(\langle x - \gamma, \theta\rangle)\delta'(\langle \eta, \theta\rangle)\ d\theta. \tag{3.4}$$

Proof. By (1.3) and (1.4)

$$\begin{aligned} e * f(x) &= -\frac{1}{8\pi^2} \int_{S^2} \int_R \mathcal{R}_\theta f(t)(\mathcal{R}_\theta e)''(\langle x, \theta\rangle - t)\ dt d\theta \\ &= -\frac{1}{8\pi^2} \int_{S^2} \int_R (\mathcal{R}_\theta f)'(t)(\mathcal{R}_\theta e)'(\langle x, \theta\rangle - t)\ dt d\theta. \end{aligned} \tag{3.5}$$

Since

$$supp\big((\mathcal{R}_\theta f)'\big) \subseteq supp(\mathcal{R}_\theta f) \subseteq \pi_\theta(supp(f)) \subseteq \pi_\theta(\Omega) = [-1, 1]$$

we have, also using (3.1),

$$\begin{aligned} e * f(x) &= -\frac{1}{8\pi^2} \int_{S^2} \int_I (\mathcal{R}_\theta f)'(t)(\mathcal{R}_\theta e)'(\langle x, \theta\rangle - t)\ dt d\theta \\ &= -\frac{1}{8\pi^2} \int_{S^2} \int_\Gamma (\mathcal{R}_\theta f)'(\langle \gamma, \theta\rangle)(\mathcal{R}_\theta e)'(\langle x - \gamma, \theta\rangle) J(\gamma, \theta)\ d\gamma d\theta \\ &= \frac{1}{8\pi^2} \int_{S^2} \int_\Gamma \Big(\int_{S^2} \mathcal{D}_\gamma f(\eta)\delta'(\langle \eta, \theta\rangle) d\eta\Big)(\mathcal{R}_\theta e)'(\langle x - \gamma, \theta\rangle) J(\gamma, \theta\rangle) d\gamma d\theta \\ &= \frac{1}{8\pi^2} \int_{S^2} \int_\Gamma \mathcal{D}_\gamma f(\eta) \int_{S^2} (\mathcal{R}_\theta e)'(\langle x - \gamma, \theta\rangle) J(\gamma, \theta)\delta'(\langle \eta, \theta\rangle)\ d\theta d\gamma d\eta \end{aligned}$$

using Fubini's theorem and the self-adjointness of the operator with kernel $\delta'(\langle \eta, \theta\rangle)$.

It would be desirable for efficiency of computation if $K(\cdot, x, \cdot, \cdot)$ were to have small support for each x. (If reconstruction were actually local, then K would be supported in just those points $(\eta, x, \gamma, \gamma')$ for which x lies on the line through γ in direction η.) This does not appear to be the case. Since K does not appear to be "sparse" the computational burden of implementing this formula would seem to be high. Reconstruction on a n^3 lattice from data sampled at $O(n^3)$ points would require $O(n^6)$ evaluations of K, each of which would require an exact (or numerical) integration.

4. Approximate Reconstruction From Plane Curves

Convex planar curves are clearly not Kirillov-Tuy curves, and for such curves the reconstruction problem is strongly ill-posed [Fi]. Nevertheless, we show in this section that the 'approximate reconstruction' obtained applying the method of the preceding section where data is available (and ignoring or setting missing data to zero) has several computationally desirable features.

We suppose that Γ is a plane convex curve in the x-y plane lying outside the unit disk and that f and Ω are as in section 3. Let I_θ denote the orthogonal projection of Γ on $R\theta$. For θ near the equatorial plane (x-y plane), $I \subset I_\theta$, but as $\theta \to \pm e_3$, $I_\theta \to [0]$. We note that integration over $\bigcup_{\theta \in S^2}(I_\theta \times \{\theta\})$ may be pulled back to $\Gamma \times S^2$ in a fashion analogous to (3.1). The convexity of Γ affords the additional simplification that, except for a set of planes of measure zero, every plane wich meets Γ meets transversely in two points. (Compare this with the discussion in the appendix about B. Smith's condition. The important difference is that this only holds for those planes which meet Γ.) Thus we may take the function $m(\gamma, \theta)$ appearing in the definition of $J(\gamma, \theta)$ (immediately after (3.1)) to be the constant function $\frac{1}{2}$. Recalling (3.5) we have

$$\begin{aligned} e * f(x) &= -\frac{1}{8\pi^2}\int_{S^2}\int_R (\mathcal{R}_\theta f)'(t)(\mathcal{R}_\theta e)'(\langle x, \theta\rangle - t)\ dtd\theta \\ &= -\frac{1}{8\pi^2}\int_{S^2}\int_{I_\theta} (\mathcal{R}_\theta f)'(t)(\mathcal{R}_\theta e)'(\langle x, \theta\rangle - t)\ dtd\theta \\ &\quad -\frac{1}{8\pi^2}\int_{S^2}\int_{R\setminus I_\theta} (\mathcal{R}_\theta f)'(t)(\mathcal{R}_\theta e)'(\langle x, \theta\rangle - t)\ dtd\theta \\ &= Af(x) + Bf(x) \end{aligned} \tag{4.1}$$

Af is computable from known information, whereas Bf can not be directly computed. We now study the approximate reconstruction provided by Af. By the remarks preceding (4.1) we have

$$\begin{aligned} Af(x) &= -\frac{1}{16\pi^2}\int_{S^2}\int_\Gamma (\mathcal{R}_\theta f)'(\langle \gamma, \theta\rangle)(\mathcal{R}_\theta e)'(\langle x-\gamma, \theta\rangle)|\langle \gamma', \theta\rangle|\ d\gamma d\theta \\ &= -\frac{1}{16\pi^2}\int_{S^2}\int_\Gamma \mathcal{D}_\gamma f(\eta)\int_{S^2} (\mathcal{R}_\theta e)'(\langle x-\gamma, \theta\rangle)|\langle \gamma', \theta\rangle|\delta'(\langle \eta, \theta\rangle)\ d\theta d\gamma d\eta \end{aligned} \tag{4.2}$$

where the last step results from the same argument used to prove Theorem 3.2. Let h be the reconstruction kernel in (4.2).

$$h(x, \gamma, \gamma', \eta) = \int_{S^2} (\mathcal{R}_\theta e)'(\langle x-\gamma, \theta\rangle)|\langle \gamma', \theta\rangle|\delta'(\langle \eta, \theta\rangle)\ d\theta \tag{4.3}$$

Theorem 4.4. *Let $e = e_\rho$ be a radial approximate δ-function with support in the ball of radius ρ centered at the origin. Then $h(x, \gamma, \gamma', \cdot)$ has support in the set of η obtained by* *intersecting the unit sphere with those planes parallel to γ' whose distance from $x-\gamma$ is not greater than ρ.*

Proof. The proof is computational. Let $g_\rho = (\mathcal{R}_\theta e_\rho)'$. Since e_ρ is radial, $\mathcal{R}_\theta e_\rho$ is even and thus g_ρ is odd and independent of θ. Carrying out the differentiation in (4.3)

$$\begin{aligned} h(x,\gamma,\gamma',\eta) &= -\int_{S^2} \nabla_\eta \big[\, g_\rho(\langle x-\gamma,\theta\rangle)|\langle\gamma',\theta|\big]\delta(\langle\eta,\theta\rangle)\; d\theta \\ &= -\langle\gamma',\eta\rangle \int_{S^2} g_\rho(\langle x-\gamma,\theta\rangle)\; sgn(\langle\gamma',\theta\rangle)\delta(\langle\eta,\theta\rangle) d\theta \\ &\quad - \langle x-\gamma,\eta\rangle \int_{S^2} g'_\rho(\langle x-\gamma,\theta\rangle)|\langle\gamma',\theta\rangle|\delta(\langle\eta,\theta\rangle)d\theta \\ &= -\langle\gamma',\eta\rangle \int_{S^2\cap\eta^\perp} g_\rho(\langle x-\gamma,\varphi\rangle) sgn(\langle\gamma',\varphi\rangle)\; d\varphi \\ &\quad - \langle x-\gamma,\eta\rangle \int_{S^2\cap\eta^\perp} g'_\rho(\langle x-\gamma,\varphi\rangle)|\langle\gamma',\varphi\rangle| d\varphi \end{aligned} \tag{4.5}$$

Let I denote the first integral on the right hand side of (4.5) and II the second. We will simplify these integrals using angular coordinates in $\eta^\perp$ measured from the rays containing the orthogonal projections of γ' and $(x-\gamma)$. Since Γ is convex and exterior to Ω, it is clear that no η for which $\mathcal{D}_\gamma f(\eta)$ is non-zero can be parallel to γ'. If $x-\gamma$ is parallel to η then $\langle x-\gamma,\varphi\rangle = 0$ for all $\varphi \in \eta^\perp$. In this case, since g_ρ is odd, $g_\rho(0)=0$ and the first integral vanishes, whereas the second evaluates to $-4\|x-\gamma\| g'_\rho(0)[1-\langle\gamma',\eta\rangle^2]^{\frac{1}{2}}$. In I, both terms are odd in φ, so

$$I = 2\int_H g_\rho(\langle x-\gamma,\varphi\rangle)\; d\varphi$$

where $H = (S^2\cap\eta^\perp)\cap\{\langle\gamma',\varphi\rangle \geq 0\}$. Let ν be the intersection of the line generated by $E_\eta(x-\gamma)$ with H, and let α_0 be the acute angle between ν and $E_\eta\gamma'$. Then using that g_ρ is odd, it is easy to show that H decomposes into two sectors, each subtending an angle of $\frac{\pi}{2}-\alpha_0$, which contribute equally to I and two sectors, each subtending an angle of α_0, whose contributions to I cancel one another.

$$I = 4sgn\langle E_\eta\gamma', E_\eta(x-\gamma)\rangle \int_0^{\frac{\pi}{2}-\alpha_0} g_\rho(b\; cos\sigma)\; d\sigma$$

where $b = \|E_\eta(x-\gamma)\|$. To study II we use the evenness of the integrand to write

$$II = 2\|E_\eta\gamma'\| \int_H g'_\rho(\langle x-\gamma,\varphi\rangle)\big|\langle \frac{E_\eta\gamma'}{\|E_\eta\gamma'\|},\varphi\rangle\big|\; d\varphi$$

where H is now $(S^2\cap\eta^\perp)\cap\{\langle E_\eta(x-\gamma,\varphi)\rangle \geq 0\}$. Now if $\nu = (cos\alpha_0, sin\alpha_0), \alpha_0 \in (-\frac{\pi}{2},\frac{\pi}{2})$ is the point of intersection of H and the line generated by $E_\eta\gamma'$ we have

$$II = 2\|E_\eta\gamma'\| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g'_\rho(b\; cos\alpha)\; \cos(\alpha-\alpha_0)\; d\alpha$$

where again $b = \|E_\eta(x-\gamma)\|$. Breaking the integral into pieces according to the sign of $cos(\alpha - \alpha_0)$, using cosine addition, integrating one of the resulting terms, and observing some cancellation of terms, there results

$$II = 4\|E_\eta\gamma'\|\{cos(\alpha_0)\int_0^{\frac{\pi}{2}-|\alpha_0|} g'_\rho(b\ cos\alpha)\ cos\alpha\ d\alpha \quad + \quad \frac{|sin\alpha_0|}{b} g_\rho(b\ |sin\alpha_0|)\ \}.$$

Combining I and II and changing variable ($\alpha \to \frac{\pi}{2} - \alpha$) give

$$\begin{aligned} h(x,\gamma,\gamma',\eta) = &-4sgn\langle E_\eta\gamma', E_\eta(x-\gamma)\rangle\langle\gamma',\eta\rangle \int_{\alpha_0}^{\frac{\pi}{2}} g_\rho(b\ sin\alpha)\ d\alpha \\ &- 4\langle x-\gamma,\eta\rangle\|E_\eta\gamma'\|\{cos\alpha_0 \int_{\alpha_0}^{\frac{\pi}{2}} g'_\rho(b\ sin\alpha) sin\alpha\ d\alpha + \frac{sin\alpha_0}{b} g_\rho(b\ sin\alpha_0)\} \end{aligned} \tag{4.6}$$

where α_0 is here the acute angle between $E_\eta\gamma'$ and $E_\eta(x-\gamma)$ and $b = \|E_\eta(x-\gamma)\|$. Now recalling that g_ρ and g'_ρ have support in $[-\rho,\rho]$ we see that h vanishes if $b\ sin\alpha_0 \geq \rho$, or, equivalently, if the distance from $x-\gamma$ to the plane through γ parallel to η and γ' is greater than ρ.

A first consequence of Theorem(4.4) is that the computation of Af using a radial approximate δ-function on an n^3 grid from $n \times n^2$ data points on $\Gamma \times S^2$ is at most an order $O(n^5)$ computation. This results from the quasi-local nature of the reconstruction kernel h. To reconstruct at x requires for each source γ only the data from directions near the plane containing $x-\gamma$ and the tangent to Γ at γ. If the kernel is pre-computed and stored, it will require also $O(n^5)$ storage locations. However, if Γ is a circle and if the reconstruction grid and sources are invariant under a group of rotations about the axis of the circle, then the storage requirements for h may be reduced to $O(n^4)$, since h will transform covariantly under the group of rotations.

More importantly, the quasi-local nature of h means that the reconstruction algorithm can be parallelized. The data and kernel values needed to reconstruct at x from a given source γ are independent from those needed to reconstruct at y, if the distance from x to the plane through γ and y parallel to γ' exceeds the diameter of the point spread function. Finding what parallel architecture is best suited to this algorithm is a question which deserves study.

Finally, there is an approximation to Af, when the source set is a circle, which can be calculated more effectively than what is presented above. A sequel to this paper will discuss this approximation, which involves an $O(n^4)$ computation, and some numerical results.

5. Discussion and Comparison

From the derivation in section 4, the algorithm proposed in this paper is manifestly equivalent to a limited data reconstruction for the Radon transform (see eqs.(4.1) and (4.2)). The algorithm of Grangeat is similarly based, as he computes the derivative of the Radon transform as an intermediate step. In effect, the algorithm of Feldkamp et. al. also amounts to a limited data reconstruction for the Radon transform. Indeed, their method is an implementation of the approximate reconstruction given by

$$f^\sharp(x) = c \int_{S^2} \Lambda(\Lambda \mathcal{R}_\theta f \cdot \chi_\theta)(\langle x, \theta \rangle)\, d\theta \tag{5.1}$$

where Λ is the operator defined by eq. (1.5) and χ_θ is the characteristic function of I_θ. Here the outer Λ is filtered and the inner Λ results from an identity relating the Fourier transform of the extension, homogeneous of degree -1, of the even part of the cone beam transform and Λ applied to the Radon transform. This result is the basis of B. Smith's paper. Indeed, the formal derivation of sections 8 and 9 and appendix D of Smith's paper applied when the source set is a circle yields formula (5.1). (There is a small caveat to be mentioned. The computations rely upon homogeneity of certain distributions which is no longer exact after filtering. Similarly, Feldkamp et. al. arrive at their formulae making homogeneity assumptions. In the context of fan beam reconstruction, these amount to the trick of Lakshminarayanan [L]. Thus the filtered version of (5.1) and the filtered version of the algorithm of Feldkamp et. al. may not be identical.)

Only extensive numerical testing can decide in any particular context whether the algorithm of section 4 or that of Feldkamp et. al. is superior. Several comments of a theoretical flavor can be made about their respective Radon transform equivalents. The author sees several advantages for the limited data reconstruction given by Af over that given by (5.1). The operator Λ and its filtered version are non-local. Thus the reconstruction given by (5.1) will be strongly dependent on cancellation in the integral, since $\Lambda(\Lambda \mathcal{R}_\theta f \cdot \chi_\theta)$ will be non-zero well outside the support of $\mathcal{R}_\theta f$. Since the (truncated) convolution in Af of (4.1) does not spread the support of $\mathcal{R}_\theta f$ by more than ρ, the reconstruction so obtained may have fewer artifacts. Even for such a simple function as the characteristic function of a ball, $\Lambda \mathcal{R}_\theta f$ is unbounded ($\mathcal{R}_\theta f'$ is discontinuous, but bounded) so one may expect further numerical difficulties in the filtering and stronger edge effects. Thirdly, the approximate reconstruction provided by Af is more explicit in what is reconstructed, and so is more amenable to error analysis or to systematic variation in the pointspread function by variation of e. On the other side, the chief argument for the method of Feldkamp et. al. (not its Radon equivalent) is ease of implementation and computational efficiency. This still holds even when the method of Feldkamp et. al. is compared to the approximation to Af alluded to at the end of section 4.

Appendix

In this appendix we examine the question of when it is possible for a curve Γ to satisfy the condition that there exist an integer M so that almost every hyperplane meeting Ω meets Γ in M points. This is one of the conditions assumed by B. D. Smith [Sm] in developing an inversion formula for the cone beam transform. We show that if one assumes one order of differentiability greater than does Smith, then there are no compact curves satisfying this condition. (The author believes that the additional smoothness is only a technical condition which can be weakened.)

We assume that Ω is a compact convex subset in R^3 with non-empty interior and that Γ is a compact, connected, embedded, piecewise C^2 curve lying in $R^3\backslash\Omega$. We assume two conditions on Γ which are weakened versions of conditions C3 and C4 of Smith [Sm].

1. For almost every $\theta \in S^2$, $\pi_\theta : \Gamma \to R$, $\pi_\theta(\gamma) = \langle\gamma, \theta\rangle$, has only a finite number of critical points. (Smith assumes this for every direction.)

2. There is an integer M so that almost every hyperplane which intersects both Γ and Ω meets Γ in M points.

Theorem A.1. *Let Γ satisfy conditions 1) and 2). Then either $M = 1$ and Γ is a straight line segment or $M = 2$, and in either case there is a non-empty open set of planes which intersect Ω but do not intersect Γ. If some chord of Γ meets Ω then Γ is a planar convex curve.*

Proof. The gist of the argument is to show that if Γ has non-vanishing curvature at some point, then a plane P and a point $\gamma_0 \in \Gamma$ may be found so that P meets Ω, P is tangent to Γ at γ_0, Γ locally lies to one side of P, and so that every other intersection of P with Γ is transverse. Then a small movement of P towards the center of curvature produces two intersections while a movement away breaks the intersection. Since the other intersections are transverse, the intersection number for the rest of Γ does not change. This gives two open sets of planes where the intersection number differs by two. By condition 2) this can only occur if $M = 2$ and one of the open sets of planes does not meet Γ. If Γ has zero curvature at every smooth point, then Γ is a union of straight line segments. A simple perturbation argument like that outlined above shows that if Γ has a vertex then $M = 2$, otherwise Γ is a straight line segment and $M = 1$, and in either case there is an open set of planes which meet Ω but do not meet Γ.

We now present a series of lemmas to show that a plane P and point γ_0 can be found which meet the conditions stated above.

Lemma A.2. *If A is an open subset of R^m, $B \subset A$, Y is a normed vector space, $f : A \to Y$ is a C^k mapping, $k \geq 1$, and if there is an integer ν with $0 \leq \nu < m$ so that for every $x \in B$ rank$(Df(x)) \leq \nu$ then $\mathcal{H}^{\nu+(m-\nu)/k}(f(B)) = 0$, where $\mathcal{H}^\alpha$ is Hausdorff α-measure.*

Proof. This is Theorem 3.4.3 in Federer [Fe].

At a point of non-zero curvature, any tangent plane other than the osculating plane will have the property that Γ locally lies on one side. Thus we have to show that we can find a point and a plane so that the only point of tangency between Γ and the plane is at the given point. If the tangent line to Γ at γ is tangent to Γ at some other point(s) (bitangent case) then no plane tangent at γ will satisfy the conditions. We will show that the directions of bitangents are rare among tangent directions. Assume that γ_0 is point such that the tangent line is not a bitangent. Let γ_0' be the unit tangent to Γ at γ_0. The planes tangent at γ_0 are parametrized by $S^2 \cap (\gamma_0')^\perp$ which can be identified with S^1. Under this identification we define $\rho : \Gamma \backslash \{\gamma_0\} \to S^1$ by

$$\rho(\gamma) = \frac{(\gamma - \gamma_0) \times \gamma_0'}{\|(\gamma - \gamma_0) \times \gamma_0'\|}.$$

Computing the differential of ρ, it is readily seen that γ is a critical point exactly when the tangent at γ is parallel to the plane spanned by γ_0' and $(\gamma - \gamma_0)$: that is, exactly when the plane tangent at γ_0 is not tranverse to Γ at γ. By Lemma A.2 the image of the critical set has $\mathcal{H}^{\frac{1}{2}}$ measure zero. ($\mathcal{H}^1$ measure zero if Γ is only C^1.) Thus almost every (in the sense of Hausdorff or Lebesgue measure) plane tangent to Γ at γ_0 is everywhere else transverse to Γ. We summarize this in a lemma.

Lemma A.3. *If γ_0 is a point on Γ such that the tangent line is nowhere else tangent then almost every plane tangent to Γ at γ_0 is everywhere else transverse to Γ.*

Finally it must be shown that bitangents are rare. Since 'almost all' in assumptions 1) and 2) does not exclude that Γ contains a straight line segment, rare will mean that the set of directions of bitangents is a thin subset of the set of tangent directions. Applying Theorem 3.2.3 of Federer [Fe] near a point where the curvature is non-zero, one sees that the image of Γ in S^2 under the unit tangent mapping has $\mathcal{H}^1$ measure greater than zero.

Let s be an arc-length parameter on Γ, and define $g : I \times I \to S^2$ by $g(s,t) = [\gamma(s) - \gamma(t)]/\|\gamma(s) - \gamma(t)\|, t \neq s$. Computing the differential of g one finds that Dg has rank zero precisely when the tangent line at $\gamma(s)$ coincides with the tangent line at $\gamma(t)$. Applying Lemma A.2. with $k = 2, \nu = 0, m = 2$ we see that if B is the set where the rank of Dg is zero, then $\mathcal{H}^1(g(B)) = 0$. But $g(B)$ is precisely the set of directions of bitangents. This proves the following lemma.

Lemma A.4. *If Γ is C^2 then for almost every tangent direction, any tangent line having that direction is tangent at only one point.*

Finally we show that if we think of Γ as 'lying around' Ω then Γ is a planar convex curve.

Lemma A.5. *Suppose that Γ satisfies conditions 1) and 2) and further that some chord of Γ meets Ω. Then Γ is a planar convex curve.*

Proof. By what we have already proved, $M = 1$ or $M = 2$. If $M = 1$ then Γ is a straight line segment exterior to Ω, so no chord meets Ω. Thus $M = 2$. If Γ is not planar then there is some third point so that the plane determined by the two endpoints of the chord and this third point is transverse to Γ at these three points. But then $M \geq 3$ contradicting $M = 2$. If Γ is not convex then one can make a similar argument with three collinear points.

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