MICROLOCAL ANALYSIS OF THE X-RAY TRANSFORM WITH SOURCES ON A CURVE

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ABSTRACT. This paper surveys several settings where distributions associated to paired Lagrangians appear in inverse problems. We make a closer study of a particular case: the microlocal analysis of the X-ray transform with sources on a curve.

1. INTRODUCTION

As it was mentioned in the preface to this book microlocal analysis (MA) is very useful in inverse problems in determining singularities of the medium parameters. In this chapter we survey several such applications, including an elaboration of some applications of MA to tomography that were already mentioned in Faridani's chapter, section 5. We recall below the general setting.

While in two-dimensional tomography it is often possible to irradiate an unknown object from all directions, in three dimensions it is usually not practical to obtain this many data. Moreover, since the manifold of lines in $\mathbb{R}^3$ is four dimensional, while the object under investigation is a function of three variables, it should suffice to restrict the measurements to a three-dimensional submanifold of lines. Of course, in practice one has only finitely many measurements, but the considerations of the continuous case can be used to guide the design of algorithms and sampling geometries.

Reconstruction from line integral data is never local. That is, to reconstruct a function f at a point $x \in \mathbb{R}^n$ requires more than the data of the line integrals of f over all lines passing through a neighborhood of x. However, if the full line integral transform is composed with its adjoint, the resulting operator is an elliptic pseudo-differential operator, which preserves singular supports. Moreover, to compute this composition at a point x only requires line integrals for lines which pass through a neighborhood of x, and so useful information can be determined about the unknown function from just local data. In the planar setting, these observations have been elaborated upon and built into a useful tool for microtomography (see the discussion and references in section 6 of Faridani's chapter). In three dimensions, the observation is less useful since it still involves X-rays from all directions, and it is natural to wonder what might be done for local reconstruction of the singular support, or more refined information about the singularities of the object under consideration, when only those lines in a three-dimensional family which also pass through an arbitrarily small neighborhood of the point x are used. That is, what may be reconstructed when the data is local, in the sense of this paragraph, and

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restricted in the sense of the preceding paragraph? The first tools to handle the restricted X-ray transform were developed in [GrU1], where the setting was the more general geodesic transform on a Riemannian manifold. This will be further discussed below. The theory of paired Lagrangian distributions plays an important role in the analysis.

In this chapter we also describe other situations in inverse problems where the theory of paired Lagrangian distributions is important. These distributions, whose wave front set consists of two cleanly intersecting conic Lagrangian manifolds, were initially defined in [MU], and further developed in the papers [GuU], [AU], [GrU], [GrU2]. For a summary of some of the results see §2.

The forward fundamental solution for the wave equation $\Box^{-1}$ is an operator whose wave front set consists of two cleanly intersecting conic Lagrangian manifolds One is the diagonal and the other is the forward flowout by the Hamiltonian vector field H_p of the characteristic variety, $\{p = 0\}$, of the wave equation. In fact, as it is shown in [MU], this is valid for any operator of real-principal type, (see §3). The diagonal part preserves singularities while the flowout moves them.

This fact was used in [GrU] to show that from the singularities of the backscattering data one can determine the singularities of a conormal potential with some restrictions on the type of singularity but including bounded potentials. A conormal potential has wave front set contained in the normal bundle of a submanifold. A fundamental step in the proof of this result is the construction of geometrical optics solutions for the wave equation plus the conormal potential q with data a plane wave in the far past,

$$\begin{cases} \big(\Box + q(x)\big)u(x,t,\omega) = 0 \quad \text{on} \quad \mathbb{R}^{n-1} \times S^{n-1} \\ u(x,t,\omega) = \delta(t - x\cdot\omega), \ t << 0, \end{cases}$$

where $\Box = \frac{\partial^2}{\partial t^2} - \Delta_{\mathbb{R}^n}$ is the wave operator on $\mathbb{R}^{n+1}$ acting independently of ω.

The solution is constructed in the form

$$u = \delta(t - x\cdot\omega) + u_1 + u_2$$

where

$$u_1 = \Box^{-1}(q\delta(t - x\cdot\omega))$$

and u_2 is a smoother distribution.

In [GrU] a detailed study of the singularities of u_1 was made, using the fact that we know exactly how $\Box^{-1}$ propagates singularities. This is reviewed in §4.

Another inverse problem where the theory of paired Lagrangian distributions appears naturally is Calderón's problem [C]. The question is whether one can determine a conductivity $\gamma > 0$ by making voltage and current measurements at the boundary. This information is encoded in the so-called Dirichlet-to-Neumann map.

An important technique in the study of this inverse problem has been the construction of complex geometrical optics solutions for the conductivity equation $\operatorname{div} \gamma\nabla u = 0$. Let $\rho \in \mathbb{C}^n\backslash 0$ satisfy $\rho\cdot\rho = 0$. For $|\rho|$ large these solutions have the form

$$u_\rho(x) = e^{x\cdot\rho}\gamma^{-\frac{1}{2}}(1 + \psi_1(x,\rho) + \psi_2(x,\rho))$$

Furthermore ψ_1 decays in $|\rho|$ for $|\rho|$ large uniformly in compact sets and ψ_2 decays in $|\rho|$ faster than ψ_1. See [U] for a recent survey and further developments and references.

The term ψ_1 is constructed by solving

$$\Delta_\rho \psi = q$$

where $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$ and

$$\Delta_\rho u = (\Delta + 2\rho\nabla)u.$$

As we show in §5, by taking the inverse Fourier transform in ρ, the operator Δ_ρ can be viewed as an operator of complex principal type and its inverse is in the class of operators associated to two intersecting Lagrangian manifolds, one being the diagonal and the other a flowout of a codimension two characteristic variety. The use of this fact to study Calderón's problem for conormal bounded conductivities is investigated in [GrLU1]. The case of conormal potentials has been considered in [GrLU].

In §6 we review the microlocal approach of [GuS], [G] to invert a class of generalized Radon transforms $\mathcal{R}$. In particular under the Bolker condition $\mathcal{R}^t \circ \mathcal{R}$ is an elliptic pseudodifferential operator and preserves singularities. Here $\mathcal{R}^t$ denotes the transpose of $\mathcal{R}$. For an application of this to seismic imaging see the chapter by de Hoop in this volume. Guillemin and Sternberg showed that the range of a generalized Radon transform satisfying the Bolker condition can be characterized as the solution set of a system of pseudodifferential equations. A left parametrix for this system is another example of a paired Lagrangian distribution.

In §7 we describe in more detail the setting of restricted Radon transforms studied in [GrU1] that were already mentioned above. If $\mathcal{C}$ is a geodesic complex satisfying certain geometric conditions, and $\mathcal{R}_\mathcal{C}$ is the restricted Radon (geodesic) transform for $\mathcal{C}$, then $\mathcal{R}_\mathcal{C}$ is a Fourier integral operator and $\mathcal{R}_\mathcal{C}^t \circ \mathcal{R}_\mathcal{C}$, microlocalized away from certain bad points, falls in the class of operators associated to the diagonal and a flowout Lagrangian. The symbol of this operator is calculated on the diagonal and a relative left parametrix is constructed.

In §8, we specialize the discussion of §7 to the complex of lines meeting a space curve in $\mathbb{R}^3$: this is motivated by the tomographic scanner design wherein an X-ray source moves on a trajectory in space and for each source point measurements are made on a two-dimensional detector. One of the results of the analysis of the operators $\mathcal{R}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}^t$ in [GrU1] is the relation of the wave front set of a distribution μ and those of $\mathcal{R}_\mathcal{C}\mu$ and $\mathcal{R}_\mathcal{C}^t \circ \mathcal{R}_\mathcal{C}\mu$. (This was used by Quinto [Q1], who gave a more elementary presentation of the relation of the first two. His work is better known in the tomography community.) At about the same time, Louis and Maass proposed $\mathcal{R}_\mathcal{C}^t \circ \Delta \circ \mathcal{R}_\mathcal{C}$ as a local tomography operator and made some experimental reconstructions, [LoM]. (The operator Δ was the Laplacian on the sphere. Their weighting in the adjoint is different, but that is immaterial to the analysis.) They wrote down an integral which gave the symbol for their operator. Subsequently, A. Katsevich [Ka] studied the mapping properties of $\mathcal{R}_\mathcal{C}^t \circ \mathcal{R}_\mathcal{C}$ (with the adjoint weighting of Louis and Maass) on wave front sets, found an expression for the principal symbol, and computed some asymptotic expansions of $\mathcal{R}_\mathcal{C}^t \circ \mathcal{R}_\mathcal{C}\mu$ near the additional geometric singularities, in the case where μ is a piecewise smooth function. At the same time, and independently, the second author studied $\mathcal{R}_\mathcal{C}^t \circ \mathcal{R}_\mathcal{C}$ in his thesis, [La]. He computed the symbol of $\mathcal{R}_\mathcal{C}^t \circ \mathcal{R}_\mathcal{C}$ on the diagonal and on the flowout Lagrangian, and the symbol for $\mathcal{R}_{\mathcal{C}}^t \circ \mathcal{R}_{\mathcal{C}}\mu$, when μ is a conormal distribution (satisfying certain geometric hypotheses on its wave front set). These results, somewhat reworked, are presented here for the first time, along with a few subsequent developments. A microlocal analysis of the restricted Doppler transform has been done recently in [R].

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2. Spaces of Paired Lagrangian Distributions

In this section we recall the spaces of conormal distributions and distributions associated with either a single Lagrangian manifold or two cleanly intersecting Lagrangian manifolds.

Let X be an n-dimensional smooth manifold, and $\Lambda \subset T^*X\backslash 0$ a conic Lagrangian manifold. The Hörmander space $I^m(\Lambda)$ of Lagrangian distributions on X associated with Λ consists [H] of all locally finite sums of distributions of the form

$$u(x) = \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x,\theta) d\theta \ ,$$

where $\phi(x,\theta)$ is a nondegenerate phase function parametrizing Λ and

$$a \in S^{m+\frac{n}{4}-\frac{N}{2}}\big(X \times (\mathbb{R}^N\backslash 0)\big) = \{a \in C^\infty\big(X \times (\mathbb{R}^N\backslash 0)\big) \colon \\ |\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \le C_{\alpha\beta K} \langle\theta\rangle^{m+\frac{n}{4}-\frac{N}{2}-|\alpha|}, \forall \alpha \in \mathbb{Z}_+^N, \beta \in \mathbb{Z}_+^n, x \in K \Subset X\} \ .$$

(Here we use the standard notation $\langle\theta\rangle = (1+|\theta|^2)^{\frac{1}{2}}$.) For $u \in I^m(\Lambda)$, the wave front set $WF(u) \subset \Lambda$.

Let X and Y be smooth manifolds. The operators $F : C_0^\infty(X) \to \mathcal{D}'(Y)$ whose Schwartz kernel $K_F \in \mathcal{D}'(X \times Y)$ is a Lagrangian distribution associated to a conic Lagrangian manifold Γ (also called canonical relation) with respect to the twisted symplectic form $\omega_{T^*(X\times Y)} = \omega_{T^*(X)} - \omega_{T^*(Y)}$ are called Fourier integral operators. Here $\omega_{T^*(X)}, \omega_{T^*(Y)}$ denote the symplectic forms on $T^*(X), T^*(Y)$ respectively. We have that the twisted wave front set $WF'(K_F) \subset \Lambda$ where

$$WF'(K_F) = \{(x,y,\xi,\eta) \in T^*(X \times Y)\backslash 0 \colon (x,y,\xi,-\eta) \in WF(K_F)\}.$$

Now let $S \subset X$ be a smooth submanifold of codimension k. Then the conormal bundle of S,

$$N^*S = \{(x,\xi) \in T^*X\backslash 0 \colon x \in S, \xi \perp T_xS\} \ ,$$

is a Lagrangian submanifold of $T^*X\backslash 0$; the space of distributions on X conormal to S is by definition

$$I^\mu(S) = I^{\mu+\frac{k}{2}-\frac{n}{4}}(N^*S) \ .$$

If $h \in C^\infty(X, \mathbb{R}^k)$ is a defining function for S, with rank $(dh) = k$ at S, then $u(x) \in I^\mu(S) \Rightarrow$

$$u(x) = \int_{\mathbb{R}^k} e^{ih(x)\cdot\theta} a(x,\theta) d\theta, a \in S^\mu\big(X \times (\mathbb{R}^k\backslash 0)\big) \ .$$

For example, if δ_S is a smooth density on S, then $\delta_S \in I^0(S)$, while a distribution on $X\backslash S$ having a Heaviside-type singularity at S belongs to $I^{-k}(S)$. One easily sees that

$$I^\mu(S) \subset L^p_{loc}(X) \text{ if } \mu < -k\big(1 - \frac{1}{p}\big) \ .$$

Now, let $\Lambda_0,\ \Lambda_1 \subset T^*X\backslash 0$ be a cleanly intersecting pair of conic Lagrangians in the sense of [MU]. Thus, $\Sigma = \Lambda_0 \cap \Lambda_1$ is smooth and

$$T_{\lambda_0}\Sigma = T_{\lambda_0}\Lambda_0 \cap T_{\lambda_0}\Lambda_1, \quad \forall \lambda_0 \in \Sigma\ .$$

Associated to the pair (Λ_0, Λ_1) is a class of Lagrangian distributions, $I^{p,\ell}(\Lambda_0, \Lambda_1)$, indexed by $p, \ell \in \mathbb{R}$, which satisfy $WF(u) \subset \Lambda_0 \cup \Lambda_1$ [MU],[GuU]. Microlocally, away from Σ,

$$I^{p,\ell}(\Lambda_0, \Lambda_1) \subset I^{p+\ell}(\Lambda_0\backslash\Lambda_1) \text{ and } I^{p,\ell}(\Lambda_0, \Lambda_1) \subset I^{p}(\Lambda_1)\ . \tag{2.1}$$

We have that

$$\bigcap_\ell I^{p,\ell}(\Lambda_0, \Lambda_1) = I^p(X, \Lambda_1), \quad \bigcap_p I^{p,\ell}(\Lambda_0, \Lambda_1) = C^\infty(X).$$

The principal symbol of a paired Lagrangian distribution $I^{p,\ell}(\Lambda_0, \Lambda_1)$ consists of the pair of symbols $(\sigma^0_{p+\ell}, \sigma^1_\ell)$ of the Lagrangian distributions $I^{p+\ell}(\Lambda_0\backslash\Lambda_1)$ and $I^p(\Lambda_1\backslash\Lambda_0)$ away from the intersection Σ. For the definition of the symbol of a Lagrangian distribution see [H] and section 8 in this paper for more details. The symbols σ^0 and σ^1 each have a conormal singularity as they approach the intersection and the singularities satisfy a compatibility condition at the intersection (see [GuU] for more details).

The symbol calculus of [GuU] implies

Theorem 2.1. *Let $u \in I^{p,\ell}(\Lambda_0, \Lambda_1)$. If $\sigma_{p+\ell}(u) = 0$ on $\Lambda_0\backslash\Sigma$ then $u \in I^{p,\ell-1} + I^{p-1.\ell}(\Lambda_0, \Lambda_1)$.*

If $Y_2 \subset Y_1 \subset X$ are smooth submanifolds with $\operatorname{codim}_X(Y_1) = d_1, \operatorname{codim}_X(Y_2) = d_1 + d_2$, then N^*Y_1 and N^*Y_2 intersect cleanly in codimension d_2. The space of distributions on X conormal to the pair (Y_1, Y_2) of orders μ, μ' is

$$I^{\mu,\mu'}(Y_1, Y_2) = I^{\mu+\mu'+\frac{d_1+d_2}{2}-\frac{n}{4}, -\frac{d_2}{2}-\mu'}(N^*Y_1, N^*Y_2) \tag{2.2}$$

$$= I^{\mu+\frac{d_1}{2}-\frac{n}{4}, \mu'+\frac{d_2}{2}}(N^*Y_2, N^*Y_1). \tag{2.3}$$

If one introduces local coordinates $(x_1, \dots, x_n)$ on X such that

$$Y_1 = \{x_1 = \cdots = x_{d_1} = 0\} = \{x' = 0\}, \text{ and} \tag{2.4}$$

$$Y_2 = \{x_1 = \cdots = x_{d_1+d_2} = 0\} = \{x' = 0, x'' = 0\}\ , \tag{2.5}$$

then $u(x) \in I^{\mu,\mu'}(Y_1, Y_2)$ iff it can be written locally as

$$u(x) = \int_{\mathbb{R}^{d_1+d_2}} e^{i(x'\cdot\xi' + x''\cdot\xi'')} a(x;\xi';\xi'') d\xi' d\xi''$$

with $a(x;\xi';\xi'')$ belonging to the product-type symbol class

$$S^{\mu,\mu'}(X \times (\mathbb{R}^{d_1}\backslash 0) \times \mathbb{R}^{d_2}) = \{a \in C^\infty : \\ |\partial_x^\gamma \partial_{\xi''}^\beta \partial_{\xi'}^\alpha a(x,\xi)| \le C_{\alpha\beta\gamma K} \langle \xi', \xi''\rangle^{\mu - |\alpha|} \langle \xi''\rangle^{\mu' - |\beta|}\}\ .$$

Let X be a smooth manifold of dimension n. We denote the diagonal

$$D = \{(x,\xi,x,\xi);\ (x,\xi) \in T^*(X)\backslash 0\}. \tag{2.6}$$

The class of operators $F : C_0^\infty(X) \to \mathcal{D}'(X)$ whose twisted wave front set consists of two intersecting conic Lagrangian manifolds, one being the diagonal, is called the class of pseudodifferential operators with singular symbols. An important class of pseudodifferential operators with singular symbols are those whose other Lagrangian manifold, Λ_Σ, is a flowout. Let $\Sigma \subset T^*X\backslash 0$ be a smooth, codimension k conic submanifold, $1 \leq k < n$ which is involutive with respect to the symplectic form $\omega_{T^*(X)}$ (that is, the ideal of functions vanishing on Σ is closed under the Poisson bracket). Thus $T_{(x,\xi)}\Sigma^\omega \subset T_{(x,\xi)}\Sigma$ is a k plane for all $(x,\xi) \in \Sigma$, where $T_{(x,\xi)}\Sigma^\omega$ denotes the orthogonal complement of $T_{(x,\xi)}\Sigma$ with respect to the symplectic form. The distribution $\{T_{(x,\xi)}\Sigma^\omega\}$ is integral with integral submanifolds $\Xi_{(x,\xi)}$ called the bicharacteristic leaves of Σ. The flowout of Σ is the canonical relation $\Lambda_\Sigma \subset (T^*(X)\backslash 0) \times T^*(Y)\backslash 0)$ given by

$$\Lambda_\Sigma = \{(x,\xi,y,\eta) \in \Sigma \times \Sigma \colon (y,\eta) \in \Xi_{(x,\xi)}\}. \tag{2.7}$$

In [AU] a calculus a composition calculus was developed for pseudodifferential operators with singular symbols when the other Lagrangian Λ_Σ is a flow out. Notice that $D \circ D = D$, $D \circ \Lambda_\Sigma = \Lambda_\Sigma$, $\Lambda_\Sigma \circ D = \Lambda_\Sigma$ and $\Lambda_\Sigma \circ \Lambda_\Sigma = \Lambda_\Sigma$. Here $C_1 \circ C_2$ denotes the composition of the relations C_1 and C_2. Thus one can expect that the composition of pseudodifferential operators with singular symbols for which the second Lagrangian is a flowout is again in the same class. A theorem of [AU] shows that this indeed the case. More precisely we have

Theorem 2.2. *Let $A_i \in I^{p_i,\ell_i}(D,\Lambda_\Sigma), i = 1,2,$ with Λ_Σ a flowout as above. Then $A_1A_2 \in I^{p_1+p_2+k/2,l_1+l_2-k/2}(D,\Lambda_\Sigma)$. The principal symbol of A_1A_2 on D, away from the intersection, is given by $\sigma(A_1A_2)|_{D\backslash(D\cap\Lambda_\Sigma)} = (\sigma(A_1)\sigma(A_2))|_{D\backslash(D\cap\Lambda_\Sigma)}$.*

In [AU] the symbol of A_1A_2 is also computed on the flowout Lagrangian away from the intersection with the diagonal.

Using this calculus one can prove the following estimate:

Theorem 2.3. *Let $A \in I^{p,\ell}(X,D,\Lambda_\Sigma)$ with Λ_Σ a flowout as above. Then*

$$A : H^s_{comp}(X) \to H^{s+s_0}_{loc}(X), \forall s \in \mathbb{R}$$

if

$$\max(p + \frac{k}{2}, p + l) \leq -s_0.$$

3. Parametrices for Principal Type Operators

The simplest example of an operator of principal type on $\mathbb{R}^n, n \geq 2$ is the operator $D_{x_1} = \frac{1}{i}\frac{\partial}{\partial x_1}$. The forward fundamental solution is given by

$$E_+f(x) = i\int_{-\infty}^{x_1} f(s,x')ds \tag{3.1}$$

where we are using coordinates $x = (x_1, x')$. Let

$$\Lambda_+ = \{(x_1,x',\xi_1,\xi',y_1,x',\xi_1,\xi') \in T^*(\mathbb{R}^n) \times T^*(\mathbb{R}^n) \colon y_1 \geq x_1\}.$$

It is readily seen that Λ_+ is the forward flowout from $D \cap \{\xi_1 = 0\}$ by H_{ξ_1} where H_p denotes the Hamiltonian vector field of p. We have that

$$WF'E_+ = D \cup \Lambda_+,$$

and in fact $E_+ \in I^{-\frac{1}{2},-\frac{1}{2}}(D,\Lambda_+)$.

Another example of the class of operators whose Schwartz kernel has wave front set in two conic Lagrangian manifolds which intersect cleanly is the forward fundamental solution of the wave operator $\square = \partial_t^2 - \sum_{i=1}^n \partial_{x_i}^2$. The forward fundamental solution is given by

$$\square^{-1} f(t,x) = \int_0^t \int \frac{((t-s)^2 - |x-y|^2)_+^{-\frac{(n-1)}{2}}}{\Gamma\left(\frac{(-n+3)}{2}\right)} f(s,y)dyds. \tag{3.2}$$

(The distribution $\frac{x_+^s}{\Gamma(s+1)}$, is defined by analytic continuation (see [H1, section 3.2]). Notice that for n odd, $n \geq 3$, $\frac{x_+^{-\frac{(n-1)}{2}}}{\Gamma(\frac{(-n+3)}{2})} = \delta^{\frac{(n-3)}{2}}$. We have that $\square^{-1} \in I^{-\frac{3}{2},-\frac{1}{2}}(D,\Lambda)$ where Λ is the forward flowout from $\Delta \cap \{p=0\}$ by the Hamiltonian vector field H_p. Here p denotes the principal symbol of the wave operator: $p(t,x,\tau,\xi,t',x',\tau',\xi') = \tau^2 - |\xi|^2$.

The paper [MU] contains a symbolic construction of the forward parametrices ($PE = I + R$, with R smoothing) for pseudodifferential operators of real principal type. These parametrices were first studied in [DH]. We recall,

Definition 3.1. *Let $P(x,D)$ be an m^{th} order classical pseudodifferential operator, with real homogeneous principal symbol $p_m(x,\xi)$. We say that P is of real principal type if a) $dp_m \neq 0$ at char$(P) = \{(x,\xi) \in T^*X\backslash 0 \colon p_m(x,\xi) = 0\}$ so that char(P) is smooth, and b) char(P) has no characteristics trapped over a compact set of X.*

For $(x,\xi) \in \text{char}(P)$, let $\Xi_{(x,\xi)}$ be the bicharacteristic of $P(x,D)$ (i.e., integral curve of H_{p_m}) through (x,ξ). Then the flowout canonical relation generated by char(P),

$$\Lambda_P = \{(x,\xi;y,\eta) \colon (x,\xi) \in \text{char}(P), (y,\eta) \in \Xi_{(x,\xi)}\} \ , \tag{3.3}$$

intersects the diagonal D cleanly in codimension 1. In [MU], it was shown that $P(x,D)$ has a parametrix $Q \in I^{\frac{1}{2}-m,-\frac{1}{2}}(D,\Lambda_P)$.

We now review the mapping properties of a parametrix for a pseudodifferential operator of real principal type, acting on the spaces of distributions associated with one and two Lagrangians described in §2 (see [GrU]).

Proposition 3.1. *Suppose $\Lambda_0 \subset T^*X\backslash 0$ is a conic Lagrangian intersecting char(P) transversally and such that each bicharacteristic of P intersects Λ_0 a finite number of times. Then, if $T \in I^{p,\ell}(D,\Lambda_P)$*

$$T \colon I^r(\Lambda_0) \to I^{r+p,\ell}(\Lambda_0,\Lambda_1) \ ,$$

where $\Lambda_1 = \Lambda_P \circ \Lambda_0$ is the flowout from Λ_0 on char(P). Furthermore, for $(x,\xi) \in \Lambda_1\backslash\Lambda_0$,

$$\sigma(Tu)(x,\xi) = \sum_j \sigma(T)(x,\xi;y_j,\eta_j)\sigma(u)(y_j,\eta_j) \ ,$$

where $\{(y_j,\eta_j)\} = \Lambda_0 \cap \Xi_{(x,\xi)}$.

The action of $I^{p,\ell}(D,\Lambda_p)$ on the class $I^{p',\ell'}(\Lambda_0,\Lambda_1)$ is described in the next proposition.

Proposition 3.2. *Under the same assumptions as Proposition 3.1*

$$T : I^{p',\ell'}(\Lambda_0,\Lambda_1) \to I^{p+p'+\frac{1}{2},\ell+\ell'-\frac{1}{2}}(\Lambda_0,\Lambda_1) \ .$$

Thus, if Q is a parametrix for $P(x,D)$,

$$Q : I^{p',\ell'}(\Lambda_0,\Lambda_1) \to I^{p'+1-m,\ell'-1}(\Lambda_0,\Lambda_1)\ .$$

The following result is also useful.

Proposition 3.3. *Suppose $\Lambda_1 \subset T^*X\backslash 0$ is a conic Lagrangian which is characteristic for P: $\Lambda_1 \subset char(P)$. Then, if $T \in I^{p,\ell}(D,\Lambda_P)$,*

$$T : I^r(\Lambda_1) \to I^{r+p+\frac{1}{2}}(\Lambda_1)$$

and thus

$$Q : I^r(\Lambda_1) \to I^{r+1-m}(\Lambda_1)\ .$$

4. The Inverse Backscattering Problem for a Conormal Potential

In the wave equation approach to the inverse backscattering problem in the framework of the Lax-Phillips theory of scattering, the continuation problem of solving the wave equation plus a potential with data a plane wave in the far past is fundamental [GrU].

$$\begin{cases} \big(\Box + q(x)\big)u(x,t,\omega) = 0 \quad \text{on} \quad \mathbb{R}^{n-1}\times S^{n-1} \\ u(x,t,\omega) = \delta(t - x\cdot\omega),\ t << 0, \end{cases} \tag{4.1}$$

where $\Box = \frac{\partial^2}{\partial t^2} - \Delta_{\mathbb{R}^n}$ is the wave operator on $\mathbb{R}^{n+1}$ acting independently of ω. For the inverse scattering problem one needs to understand the behavior of the solution $u(x,t,\omega)$ for t large.

In the case that q is a compactly supported smooth function we can write a solution of (4.1) in the form

$$u = \delta(t - x\cdot\omega) + a(t,x,\omega)H(t - x\cdot\omega). \tag{4.2}$$

where $H(x)$ denotes the Heaviside function and a is a smooth function of all variables. We have then that the wave front set of the solution satisfies

$$WFu \subset N^*\{t = x\cdot\omega)\} =: \Lambda_+.$$

Thus singularities propagate forward as time increases.

We now sketch the construction of the solution of (4.1) under the assumption that the potential $q(x)$ is conormal to a smooth codimension k submanifold. Let S be given by a defining function,

$$S = \{x \in \mathbb{R}^n : h(x) = 0\}\ ,$$

where $h \in C^\infty(\mathbb{R}^n,\mathbb{R}^k)$ satisfies $\operatorname{rank}(dh(x)) = k$ for $x \in S$; in addition we assume S has compact closure. Let

$$q(x) \in I^\mu(S), \begin{cases} \mu < -\max\big((1-\frac{2}{n})k, k-1\big), & n \geq 5 \\ \mu < -\max(\frac{k}{2}, k-1), & n = 3 \text{ or } 4 \end{cases}$$

be compactly supported and real-valued. We have that $q \in L^p(\mathbb{R}^n)$ for $p = \frac{n}{2}$ when $n \geq 5$ and $p > 2$, when $n = 3$ or 4.

Now define

$$S_1 = \{(x,t,\omega) \in \mathbb{R}^{n-1}\times S^{n-1} : x \in S\}\ ; \tag{4.3}$$

regarding $q(x)$ as a distribution on $\mathbb{R}^{n-1} \times S^{n-1}$ independent of t and ω, one has

$$q \in I^{\mu}(S_1) \ .$$

We wish to find an approximate solution to (4.1). We look for an approximation

$$u \sim u_0 + u_1 + \cdots + u_j + \cdots$$

where $u_0(x,t,\omega) = \delta(t - x \cdot \omega)$ and such that the series on the right is (formally) telescoping when $\Box + q$ is applied. The terms in the series are increasingly smooth. This type of solution is called a *geometrical optics* solution. Thus, $u_{j+1} = -\Box^{-1}\big(q(x)u_j(x,t,\omega)\big)$, where $\Box^{-1}$ is the forward fundamental solution of $\Box$. We only consider the first two more singular terms, the other terms are smoother as shown in [GrU]. We have that

$$u_0 + u_1 = \delta(t - x \cdot \omega) - \Box^{-1}\big(q(x)\delta(t - x \cdot \omega)\big) \ . \tag{4.4}$$

Now, the most singular term in the expansion is

$$u_0(x,t,\omega) = \delta(t - x \cdot \omega) \in I^0(S_+) \ , \tag{4.5}$$

where

$$S_+ = \{(x,t,\omega) \in \mathbb{R}^{n+1} \times S^{n-1} \colon t - x \cdot \omega = 0\} \ . \tag{4.6}$$

The submanifolds S_+ and S_1 intersect transversally; let $S_2 = S_+ \cap S_1$ be the resulting codimension $k+1$ submanifold of $\mathbb{R}^{n+1} \times S^{n-1}$. Let $\Lambda_1 = N^*S_1, \Lambda_+ = N^*S_+$ and $\Lambda_2 = N^*S_2$ be the respective conormal bundles, which are conic Lagrangian submanifolds of $T^*(\mathbb{R}^{n+1} \times S^{n-1})\backslash 0$. The geometry of how these submanifolds intersect is summarized in the following

Proposition 4.1. *(1) $WF(q) \subset \Lambda_1$ and $WF(u_0) \subset \Lambda_+$.*
(2) Λ_1 and Λ_+ are disjoint.
(3) Λ_2 intersects Λ_1 and Λ_+ cleanly in codimensions 1 and k, respectively, so that (Λ_1, Λ_2) and (Λ_+, Λ_2) are intersecting pairs.

The second term in (4.4) is

$$u_1 = -\Box^{-1}\big(q(x,t)\delta(t - x \cdot \omega)\big) \ , \tag{4.7}$$

where $\Box^{-1}$ acts only in the (x,t) variables. We have that

$$q(x,t) \cdot \delta(t - x \cdot \omega) \in I^{0,\mu}(S_+, S_2) \ ,$$

so that

$$WF(q \cdot \delta) \subset \Lambda_+ \cup \Lambda_2 \ .$$

To obtain $WF(u_1)$, recall that

$$WF(\Box^{-1}v) \subset (D \cup \Lambda_\Box) \circ WF(v) \quad , \ \forall v \in \mathcal{E}'(\mathbb{R}^{n+1} \times S^{n-1}) \ , \tag{4.8}$$

where D is the diagonal of $T^*(\mathbb{R}^{n+1} \times S^{n-1})\backslash 0$ and $\Lambda_\Box$ is the flowout of the characteristic variety

$$\mathrm{char}(\Box) = \{(x,t,\omega;\xi,\tau,\Omega) \colon |\tau|^2 = |\xi|^2\}$$

of $\Box$ (acting on $\mathbb{R}^{n+1} \times S^{n-1}$). In (4.8), $D \cup \Lambda_\Box$ acts as a relation between subsets of $T^*(\mathbb{R}^{n+1} \times S^{n-1})\backslash 0$; of course, D acts as the identity. Also, $\Lambda_\Box \circ \Lambda_+ = \Lambda_+$ since Λ_+ is characteristic for $\Box$. Thus

$$WF(u_1) \subset \Lambda_+ \cup \Lambda_2 \cup \Lambda_\Box \circ \Lambda_2 \ .$$

Compared with the case of a smooth potential, u_1 has the additional singularity $\Lambda_2 \cup \Lambda_\Box \circ \Lambda_2$.

An analysis of this contribution (see [GrU]) gives

Proposition 4.2. $u_1 \in I^{-(\frac{n+1}{2})}(\Lambda_+ \backslash L) + I^{\mu+\frac{k-2-n}{2}}(\Lambda_- \backslash L), \quad t \gg 0.$

We note that when q is smooth, $u_1 \in I^{-(\frac{n+1}{2})}(\Lambda_+)$. We describe below what Λ_- and L are. We have

$$L = \Lambda_\Box \circ \Theta$$

where Θ is a conic neighborhood of $\Sigma_3 = \Lambda_2|_{S_3}$ with S_3 the set of points where the incoming plane wave and the surface S are tangent. We denote by $\Sigma = \Lambda_2 \cap \mathrm{char}(\Box)$. Now

$$\Sigma = \Sigma_+ \cup \Sigma_-$$

with $\Sigma_+ = \Lambda_+ \cap \Lambda_2$. The "new" Lagrangian Λ_- is the flowout of $\Sigma_- \backslash \Sigma_3$ by H_p.

Using this additional singularity it is shown in [GrU] that we can recover the symbol of q from the singularities of the backscattering kernel; that is, the location and strength of the singularities of the q is determined by the singularities of the backscattering kernel. The crucial element in the proof, of importance in its own right, is the construction of geometrical optics solutions of (4.1) for q conormal as above.

For the case that q is a general potential, the operator

$$U : \mathcal{E}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n \times \mathbb{R} \times S^{n-1})$$

defined by

$$Uq(x,t,\omega) = \Box^{-1}(q(x) \cdot \delta(t - x \cdot \omega)) \tag{4.9}$$

was studied in [GrU3], where the two following results are proved.

Theorem 4.1.

$$U \in I^{-(\frac{n+4}{4}),-\frac{1}{2}}(\Lambda_1, \Lambda_2),$$

where

$$\Lambda_1 = N^*\{(x,t,\omega,y;)x = y, t = x \cdot \omega\}, \quad \Lambda_2 = N^*\{|t - y \cdot \omega|^2 = |x-y|^2\}.$$

Theorem 4.2.

$$U : H^s_{comp}(\mathbb{R}^n) \longrightarrow H^{s+1}_{loc}(\mathbb{R}^n \times \mathbb{R} \times S^{n-1}), \quad \forall s < -\frac{1}{2},$$

with the endpoint result

$$U : H^{-\frac{1}{2}}_{comp}(\mathbb{R}^n) \longrightarrow B^{\frac{1}{2}}_{2,\infty,comp}(\mathbb{R}^n \times \mathbb{R} \times S^{n-1}).$$

Here $B^s_{p,\infty}$ denotes the standard Besov spaces of distributions with s derivatives having Littlewood-Paley components associated with large frequencies uniformly in L^p.

5. Operators of Complex Principal Type and Calderón's problem

We first recall the inverse conductivity problem, also known as Calderón's problem. Let $\gamma \in C^2(\overline{\Omega})$ be a strictly positive function on $\overline{\Omega}$. The equation for the potential in the interior, with conductivity γ under the assumption of no sinks or sources of current in Ω, is

$$\mathrm{div}(\gamma \nabla u) = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = f. \tag{5.1}$$

The Dirichlet-to-Neumann map is defined in this case as follows:

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega}.$$

Let $\rho \in \mathbb{C}^n \backslash 0$ satisfy $\rho \cdot \rho = 0$. A key step in the proof of unique determination of the conductivity γ from Λ_γ, and in the study of several other inverse problems, is the construction of *complex geometrical optics* solutions to the conductivity equation found in [SyU, SyU1]. (See also [U] and the references there for the applications of complex geometrical optics solutions to Calderón's problem and to other inverse problems.) For sufficiently large $|\rho|$ one can construct solutions to $\operatorname{div}(\gamma\nabla u) = 0$ in $\mathbb{R}^n$ (extending γ to be 1 outside a large ball) of the form

$$u_\rho(x) = e^{x\cdot\rho}\gamma^{-\frac{1}{2}}(1 + \psi(x, \rho)) \tag{5.2}$$

with

$$\|\psi(x, \rho)\|_{L^2(K)} \le \frac{C}{|\rho|}$$

for every compact set K.

The term ψ is constructed by solving the equation

$$\Delta_\rho \psi = q(1 + \psi) \tag{5.3}$$

where $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$ and

$$\Delta_\rho u = (\Delta + 2\rho\nabla)u. \tag{5.4}$$

The solution ψ is written in the form

$$\psi = \sum_{j=0}^{\infty} \psi_j$$

where

$$\psi_{j+1} = \Delta_\rho^{-1}(q\psi_j), \quad \psi_0 = 1.$$

The fundamental property of Δ_ρ^{-1} is that satisfies the estimate, for $0 < \delta < -1$ [SyU], [SyU1],

$$\|\Delta_\rho^{-1} f\|_{L^2_\delta(\mathbb{R}^n)} \le C \frac{\|f|_{L^2_{\delta+1}(\mathbb{R}^n)}}{|\rho|}$$

where $\|f\|^2_{L^2_\alpha(\mathbb{R}^n)} = \int |f(x)|^2(1 + |x|^2)^\alpha dx$. Here we show that Δ_ρ can be viewed as an operator of complex principal type in the sense of Duistermaat-Hörmander [DH] and Δ_ρ^{-1} is a pseudodifferential operator with a singular symbol.

We take the Fourier transform of u in the $|\rho|$ variable.

$$v(x, r, \omega) = \int_{\mathbb{R}} e^{-ir\lambda} u(x, \lambda\omega) d\lambda. \tag{5.5}$$

Then the operator Δ_ρ is transformed into the operator

$$\Box_* = \Delta_x + 2i(\omega \cdot \nabla_x) \cdot \frac{\partial}{\partial r} \tag{5.6}$$

If $\rho \cdot \rho = 0$, we can write $\rho = \frac{|\rho|}{\sqrt{2}}(\omega_R + i\omega_I)$ with $\omega_r, \omega_I \in S^{n-1}$ and $\omega_R \cdot \omega_I = 0$. Let

$$\mathcal{V} = \{\omega_R + i\omega_I \in S^{n-1} + iS^{n-1} : \omega_R \cdot \omega_I = 0\}.$$

Considered as an operator acting only in the (x,r) variables $\Box_*$ is of complex principal type in the sense of Duistermaat and Hörmander: $\Box_*$ has symbol

$$\sigma(\Box_*)(\xi,\tau;\omega) = -(|\xi|^2 - 2i(\omega\cdot\xi)\tau)$$
$$= -[(|\xi|^2 - 2(\omega_I\cdot\xi)\tau) + i(\omega_R\cdot\xi)\tau] = -[p_R + ip_I].$$

The functions p_R and p_I have linearly independent gradients and Poisson commute ($\{p_R,p_I\}=0$) and so the characteristic variety Σ is codimension two and involutive. Furthermore, the two-dimensional bicharacteristics are not trapped over a compact set, so $\Box_*$ is locally solvable. Now, $\Box_*$ actually acts on $\mathcal{D}'(\mathbb{R}^{n+1}_{x,r}\times\mathcal{V})$, with coefficients that depend on ω but without differentiation in the ω directions, and the above facts remain true as long as we work away from $0_{T^*\mathbb{R}^{n+1}}\times T^*\mathcal{V}$, which will always be the case below. Away from there, $\Box_*$ possesses a parametrix $\Box_*^{-1}\in I^{-2,0}(D,C_\Sigma)$ as we describe below.

We can write the (complex) Hamiltonian vector field of $-\frac{1}{2}\sigma(\Box_*)$ as H_R+iH_I, where

$$H_R = (\xi-\tau\omega_I)\cdot\frac{\partial}{\partial x} - (\omega_I\cdot\xi)\frac{\partial}{\partial r} + \tau i^*_{\omega_I}(\xi)\cdot\frac{\partial}{\partial\Omega_I}$$

and

$$H_I = \tau\omega_R\cdot\frac{\partial}{\partial x} + (\omega_R\cdot\xi)\frac{\partial}{\partial r} - \tau i^*_{\omega_R}(\xi)\cdot\frac{\partial}{\partial\Omega_R}.$$

Here, $\Omega=(\Omega_R,\Omega_I)\in T^*_\omega\mathcal{V}$ and $i_{\omega_A}: T_{\omega_A}S^{n-1}\hookrightarrow T_{\omega_A}\mathbb{R}^n$ is the natural inclusion for $A=R,I$. H_R and H_I span the annihilator $T\Sigma^\perp$ of $T\Sigma$ with respect to the canonical symplectic form on $T^*(\mathbb{R}^{n+1}\times\mathcal{V})$, and Σ is nonradial since the radial vector field

$$\xi\cdot\frac{\partial}{\partial\xi} + \tau\frac{\partial}{\partial\tau} + \Omega_R\cdot\frac{\partial}{\partial\Omega_R} + \omega_I\cdot\frac{\partial}{\partial\Omega_I} \notin T\Sigma^\perp.$$

(Recall (*cf.,* [DH,7.2.4]) that Σ *nonradial* means that the (two-dimensional) annihilator of $T\Sigma$ with respect to the symplectic form σ on T^*X does not contain the radial vector field $\sum_{i=1}^N \xi_i\frac{\partial}{\partial\xi_i}$ at any point.) The family of two dimensional subspaces $T\Sigma^\perp$ forms an integrable distribution in the sense of Frobenius, and its integral surfaces are the bicharacteristic leaves of Σ. It is easy to verify that no bicharacteristic leaf is trapped over a compact set and that the bicharacteristic foliation is regular (see [DH,§7]). The flowout of Σ is then the canonical relation $C_\Sigma\subset\Big(T^*(\mathbb{R}^{n+1}\times\mathcal{V})\setminus 0)\times(T^*(\mathbb{R}^{n+1}\times\mathcal{V})\setminus 0)\Big)$ defined by

$$(5.7)\quad C_\Sigma = \Big\{(x,r,\omega,\xi,\tau,\Omega;x',t',\omega',\xi',\tau',\Omega') : (x,r,\omega,\xi,\tau,\Omega)\in\Sigma,$$
$$(x',r',\omega',\xi',\tau',\Omega') = exp(sH_R+tH_I)(x,r,\omega,\xi,\tau,\Omega) \text{ for some } (s,t)\in\mathbb{R}^2\Big\}.$$

By the results of [DH], $\Box_*$ is locally solvable, and admits a right-parametrix which we will denote by $\Box_*^{-1}$, so that $\Box_*\Box_*^{-1}=I+E$ with E a smoothing operator. Although not stated in this way, since [DH] predates [MU] and [GuU], the parametrix of [DH] has a Schwartz kernel belonging to $I^{-2,0}(D',C'_\Sigma)$. Here, D is the diagonal as before and the prime denotes the twisting

$$(x,r,\omega,\xi,\tau,\Omega;\tilde{x},\tilde{r},\tilde{\omega},\tilde{\xi},\tilde{\tau},\tilde{\Omega}) \to (x,r,\omega,\xi,\tau,\Omega;\tilde{x},\tilde{r},\tilde{\omega},-\tilde{\xi},-\tilde{\tau},-\tilde{\Omega}).$$

Greenleaf, Lassas and Uhlmann [GLU] are using this microlocal approach to consider Calderón's problem when the conductivity γ has conormal singularities.

6. Microlocal Characterization of the Range of Radon Transforms

By a well known theorem of Fritz John the range of the X-ray transform in $\mathbb{R}^3$ is characterized as a solution of an ultrahyperbolic equation. Guillemin and Sternberg in [GuS1] characterized microlocally the range of a very general class of Radon transforms. It was shown in [GuU] that the projection onto the range is an operator in the class of intersecting Lagrangians. In order to state these results we first describe the microlocal approach to the double fibration of Gelfand and Helgason.

Let X and Y be smooth manifolds with $\dim X = n$ and $\dim X \leq \dim Y$. Let Z be an embedded submanifold of $X \times Y$ of codimension $k < n$. We consider the double fibration diagram

(6.1)
$$\begin{array}{ccc} & Z & \\ {}^{\rho}\swarrow & & \searrow^{\pi} \\ Y & & X \end{array}$$

where π and ρ are the natural projections onto X and Y, respectively. We also assume that π is proper.

We denote by G_x the fibers of the projection $\pi : Z \to X$, considered as submanifolds of Y and H_y the fibers of $\rho : Z \to Y$, considered as submanifolds of X. If μ is a smooth, nonvanishing measure on Z, then μ induces measures $d\mu_x$ on G_x and $d\mu_y$ on H_y. This gives rise to the generalized Radon transform, defined for $f \in C_0^\infty(X)$ by

$$\mathcal{R}f(y) = \int_{H_y} f(x) d\mu_y(x), \ y \in Y. \tag{6.2}$$

The formal adjoint of $\mathcal{R}$ is given by

$$\mathcal{R}^t g(x) = \int_{G_x} g(y) d\bar{\mu}_x(y), \ x \in X. \tag{6.3}$$

By standard duality arguments, $\mathcal{R}$ and $\mathcal{R}^t$ extend to act on distributions, $\mathcal{R} : \mathscr{E}'(X) \to \mathscr{D}'(Y)$ and $\mathcal{R}^t : \mathscr{D}'(Y) \to \mathscr{D}'(X)$.

It is immediate from (6.2) that the Schwartz kernel of $\mathcal{R}$ is δ_Z, the delta function supported on Z defined by μ. Guillemin and Sternberg, see ([Gu], [GuS]), first introduced microlocal techniques to the study of generalized Radon transforms noting that δ_Z is a Fourier integral distribution, and then studying the microlocal analogue of the double fibration (6.1). It follows from Hörmander's theory [H] that $\mathcal{R}$ is a Fourier integral operator of order $(\dim Y - \dim Z)/2$ associated with the canonical relation $\Gamma = N^*Z'$. Similarly, $\mathcal{R}^t$ is a Fourier integral operator associated with the canonical relation $\Gamma^t \subset T^*X \times T^*Y$, which is simply Γ with (x, ξ) and (y, η) interchanged.

Now consider the microlocal diagram

(6.4)
$$\begin{array}{ccc} & \Gamma & \\ {}^{\rho}\swarrow & & \searrow^{\pi} \\ T^*Y & & T^*X \end{array}$$

where π and ρ again denote the natural projections, this time onto T^*X and T^*Y, respectively. We analyze the normal operator $\mathcal{R}^t \circ \mathcal{R}$. Concerning the wave front sets we have, by a theorem of Hörmander and Sato (see [H]), that for $f \in \mathscr{E}'(X)$,

$$WF((\mathcal{R}^t \circ \mathcal{R})f) \subset (\Gamma^t \circ \Gamma)(WF(f)).$$

In general $\Gamma^t \circ \Gamma$ can be a quite complicated object, but under certain assumptions one can prove that it is a canonical relation, in fact the diagonal D.

For example, if Γ is a canonical graph (i.e. the graph of a canonical transformation $\chi : T^*X \to T^*Y$), then this is the case and on the operator level, Hörmander's composition calculus applies to yield that $\mathcal{R}^t \circ \mathcal{R}$ is a pseudodifferential operator on X. This happens if π and ρ are local diffeomorphisms and ρ is 1-1. This is the case for the generalized Radon transforms considered in [B] and [Q].

If $\dim X < \dim Y$, however, Γ cannot be a canonical graph. This is the case for the X-ray transforms and geodesic X-ray transforms in dimensions ≥ 3. Guillemin[G], motivated by work of Bolker on the discrete Radon transform, introduced a condition that guarantees that $\Gamma^t \circ \Gamma$ is still the diagonal and allows the "clean intersection" composition calculus of Duistermaat and Guillemin [DG] to be applied. The Bolker condition is that

the map ρ *in* (6.4) *is an embedding.*

Guillemin then proved

Theorem 6.1. *If the Bolker condition is satisfied, then* $\mathcal{R}^t \circ \mathcal{R}$ *is an elliptic pseudodifferential operator on* X *of order dim* Y *− dim* Z*. Hence,* $\mathscr{R}$ *is locally invertible. Moreover,*

$$\mathcal{R} : H^s_{comp}(X) \to H^{s+\frac{\dim Z - \dim Y}{2}}_{loc}(Y).$$

Other examples of generalized Radon transforms to which Theorem 6.3 applies include X-ray transforms and, more generally a class of geodesic X-ray transforms (see [GrU1], section 2 for the precise class of geodesic X-ray transforms). A consequence of the fact that the X-ray transform P is a Fourier integral operator is a precise description of the wave front set of Pf in terms of the wave front set of f, and also microlocal description of the Sobolev singularities. An elementary account is given in Theorem 3.1 of [Q1].

If the Bolker condition is satisfied then $\rho(\Gamma) =: \Sigma$ is a co-isotropic submanifold of $T^*(Y)\backslash 0$ of codimension $k = \dim Y - \dim X$. Locally the submanifold Σ is defined by $p_1(y,\eta) = ... = p_k(y,\eta) = 0$ such that the Poisson brackets of all the $p_i's$ vanish, i.e. $\{p_i, p_j\} = 0, i, j = 1, ..., k$ on Σ. The problem of showing that the Radon transform has for its range the solution set of a system of pseudodifferential equations is reduced in [GS1] to the construction of left parametrices for pseudodifferential equations of the form

$$P = P_1^2 + ... + P_k^2 + \sum_{i=1}^{n} A_i P_i + B,$$

where the principal symbol of P_i is p_i, $A_i's$ and B are pseudodifferential operators of order zero.

The left parametrix $E \in I^{p,\ell}(D, \Lambda)$ where Λ denotes the joint flow out from $D \cap \Sigma$ by the $H_{p_j}, j = 1, ...n$. D and Λ intersect cleanly of codimension k on Σ.

We remark that the principal symbol of E on $D - \Sigma$ is $\frac{1}{p_1^2+\ldots+p_k^2}$.

7. Restricted X-ray transforms

If $W \subset Y$ is a submanifold, the restricted generalized Radon transform $\mathcal{R}_W f = \mathcal{R}f|_W$ will typically not satisfy the Bolker condition even if $\mathcal{R}$ does. It is then of interest to study what injectivity properties and estimates $\mathcal{R}_W$ satisfies (as compared with $\mathcal{R}$) and the operator theory associated with $\mathcal{R}_W$. This was done for the geodesic X-ray transform in [GrU1]. We denote by $X = (M, g)$ a complete, n dimensional simply connected Riemannian manifold. We assume, as in §2 of [GrU1], that the space of geodesics $Y =: \mathscr{M}$ is a smooth manifold of dimension $2n - 2$.

We now describe the structure of the microlocal diagram (6.4) for $\mathscr{C} \subset \mathscr{M}$, a geodesic complex satisfying an analogue of Gelfand's cone condition for the case of an admissible line complex [GGr]. Let

$$\mathscr{C}_x = \bigcup\{\gamma \in \mathscr{C}; x \in \gamma\}$$

which generates a cone with vertex at x

$$\Sigma_x = \bigcup\{\dot\gamma; \gamma \in \mathscr{C}_x\}.$$

Let $\gamma \in \mathscr{C}_x$ and $y \in \gamma$. The cone condition states that the tangent planes of Σ_x and Σ_y along $\gamma \in \mathscr{C}$ are parallel translates of each other.

We now describe the projections $\pi : \Gamma \rightarrow T^*M\backslash 0$ and $\rho : \Gamma \rightarrow T^*\mathscr{C}\backslash 0$ in the language of singularity theory. (Note that (6.4) is a diagram of smooth maps between manifolds, all of dimension $2n$.) First, one makes (see[GrU1,p.215) a curvature assumption on the cones Σ_x which guarantees that π has a Whitney fold (see[GoGu]), at least away from a codimension 3 submanifold of Γ (automatically empty if $n = 3$); furthermore, one microlocalizes away from the critical points of the complex. We denote by L the fold hypersurface of π, so that $\pi(L) \subset T^*M\backslash 0$ is an immersed hypersurface. Microlocally, the image $\pi(\Gamma)$ is a half-space in $T^*M\backslash 0$ with boundary $\pi(L)$. In fact, $\pi(\Gamma)$ is the support of the Crofton symbol $Cr_{\mathscr{C}}(x, \xi)$ of $\mathscr{C}$, defined by Gelfand and Gindikin[GGi]:

$$Cr_{\mathscr{C}}(x, \xi) = \#\{\gamma \in \mathscr{C}_x : \dot\gamma \perp \xi\}$$

if finite and 0 otherwise. $Cr_{\mathscr{C}}$ is piecewise constant and jumps by 2 across $\pi(L)$. So far, we have only used the curvature assumption, not the cone condition. The projection $\rho : \Gamma \rightarrow T^*\mathscr{C}\backslash 0$ is necessarily singular at L, since π is (this is a general fact about canonical relations), but for an arbitrary geodesic complex, little can be said about the structure of ρ. However, using Jacobi fields, one can show [GrU1,p.225] that, assuming that the curvature operator can be smoothly diagonalized, the cone condition forces ρ to be a $blow - down$ at L; that is, ρ has the singularity type of polar coordinates in R^2 at the origin (crossed with a diffeomorphism in the remaining $2n - 2$ variables). Thus, ρ is 1-1 away from L, $\rho|_L$ has 1-dimensional fibers, and $\rho(L)$, which is thus of codimension 2, is symplectic (noninvolutive) in the sense that $\omega_{T^*\mathscr{C}}|_{\rho(L)}$ is nondegenerate. Furthermore, the fibers of ρ are the lifts by π of the bicharacteristic curves of the hypersurface $\pi(L) \subset T^*M\backslash 0$.

Some canonical relations having the singular structure described above were independently considered by Guillemin[Gu1], for reasons arising in Lorentzian integral geometry.

We denote by $\mathcal{R}_{\mathcal{C}}, \mathcal{R}_{\mathcal{C}}^t$ the geodesic transform restricted to $\mathscr{C}$ and its transpose. In [GrU1] it is proven that

Theorem 7.1.

$$\mathcal{R}_{\mathcal{C}}^t \circ \mathcal{R}_{\mathcal{C}} \in I^{-1,0}(D, \Lambda_{\Pi(L)})$$

where $\Lambda_{\Pi(L)}$ denotes the flowout of $\Pi(L)$.

The symbol on D away from the intersection is computed in [GrU1]. By using Theorem 1.1 and the functional calculus of [AU] a relative left parametrix is constructed for $\mathcal{R}_{\mathcal{C}}$.

We remark that this result has as corollary Theorem 5.3 in Faridani's chapter, which was explicitly stated in Theorem 4.1 in [Q1].

More details are given in the next sections on the computation of the symbols in both Lagrangians for the case that the complex of curves are straight lines going through a curve satisfying some additional conditions.

8. The complex of lines through a curve in $\mathbb{R}^3$

In this section, we will study in more detail a specific case of a restricted X-ray transform, that of the complex of line passing through a curve in $\mathbb{R}^3$. Our goal is to compute the principal symbol on the diagonal, and the symbol on the flowout Lagrangian. In view of applications to limited data problems in computed tomography, we suppose that the restricted transform acts on functions (distributions) with support contained in a given set, and that the curve lies outside this set. Specifically, we suppose that Ω is a bounded open set in $\mathbb{R}^3$, that the curve C_v lies outside the closed convex hull of Ω, and that the tangent to C_v never points into Ω. Taking $a(t)$ to be an arc length parametrization, we parametrize the family of lines passing though C_v by $C_v \times S^2$, where the pair (a, θ) is associated to the line $a + \mathbb{R}\theta$ through a in direction θ. Notice that there is some redundancy, since the same line is also associated to $(a, -\theta)$, and any line which meets the curve C_v more than once is counted multiple times. Rewriting (6.2) and (6.3) for this specific restricted transform we have

$$\mathcal{R}_{\mathcal{C}} f(a, \theta) = \int_{\mathbb{R}} f(a + s\theta) ds \tag{8.1}$$

where s is an arc length parameter on $\mathbb{R}$. The formal adjoint $\mathcal{R}_{\mathcal{C}}^t$, which here maps $C^\infty(C_v \times S^2)$ to $C^\infty(\Omega)$, is defined for $x \in \Omega$ by

$$\mathcal{R}_{\mathcal{C}}^t g(x) = \int_{C_v} g\left(a(t), \frac{x - a(t)}{|x - a(t)|}\right) \frac{1}{|x - a(t)|^2} dt. \tag{8.2}$$

The operators $\mathcal{R}_{\mathcal{C}}$ and $\mathcal{R}_{\mathcal{C}}^t$ satisfy

$$\int_{C_v \times S^2} \mathcal{R}_{\mathcal{C}} f(a, \theta) g(a, \theta)\, dt\, d\theta = \int_{\Omega} f(x) \mathcal{R}_{\mathcal{C}}^t g(x) dx \tag{8.3}$$

for smooth f and g, with f compactly supported in Ω. This relation is used to extend $\mathcal{R}_{\mathcal{C}}$, by duality, to compactly supported distributions in Ω.

In the situation of the complex $\mathcal{C}$ described above, it is possible to study the geometry of $\Gamma^t \circ \Gamma$ directly to find the intersecting Lagrangians. (The definition of Γ is slightly modified, since it is only an immersed submanifold.) It is found that that $\Gamma^t \circ \Gamma$ is the union of a subset of the diagonal relation consisting of all $(x, \xi, x, \xi) \in T^*(\Omega) \backslash 0 \times T^*(\Omega) \backslash 0$ such that the plane through x with normal ξ intersects the curve C_v, and another set consisting of all $(x,\xi,y,\eta) \in T^*(\Omega)\backslash 0 \times T^*(\Omega)\backslash 0$ subject to the condition that x and y lie in a line through C_v, ξ and η are normal to the line and to the tangent to the curve at the point of intersection, and $s_2\xi = s_1\eta$, where s_1 (resp. s_2) is the distance from x (resp. y) to the point of intersection. This is parametrized by $(t,s_1,s_2,\theta,u) \to (a(t)+s_1\theta, us_2\beta, a(t)+s_2\theta, us_1\beta)$ where β is a conormal vector at $a(t)$ annihilating both the tangent vector to the curve and the tangent vector to the line in direction θ. Computations in local coordinates show that this map is an immersion when $s_1 \neq s_2$ and also when $s_1 = s_2$ provided that $a''(t)$, $a'(t)$ and θ are linearly independent. Moreover, in the second case, this is found to be precisely the condition for clean intersection between the image and the diagonal relation (see also the discussion prior to (3.20) in [GrU2]). We let $\overline{\Lambda}'$ be the full set, and Λ' be the image of the relatively open subset where $a''(t), a'(t)$, and θ are linearly independent. For (x_0,ξ_0) such that (x_0,ξ_0,x_0,ξ_0) lies in Λ', choose one (if there be more than one) t_0 such that $x_0 = a(t_0)+s\theta$ and ξ_0 is normal to θ and $a'(t_0)$, and then $a'(t_0)\cdot\xi_0 = 0$ while $a''(t_0)\cdot\xi_0 \neq 0$. By the implicit function theorem, there is a conic neighborhood of ξ_0 and a smooth function $t(\xi)$, homogeneous of degree 0, such that $a'(t(\xi))\cdot\xi = 0$. Defining $p(x,\xi) = (x - a(t(\xi))\cdot\xi$ it is then the case that the sheet of Λ parametrized using $t(\xi)$ is the H_p flowout of $p = 0$.

Theorem 8.1. *The symbol of $\mathcal{R}^t_{\mathcal{C}} \circ \mathcal{R}_{\mathcal{C}}$ on $D\backslash\Lambda'$ is given by*

$$\sigma^0(x,\xi) = \sum_{\{t:a(t)\in(x+\xi^\perp)\cap C_v\}} \frac{2\pi}{|\xi\cdot a'(t)||x-a(t)|^{n-2}}\,\sigma^0_{Id}(x,\xi), \tag{8.4}$$

where σ^0_{Id} is the symbol of the identity operator considered as a reference section of $L\otimes\Omega^{\frac{1}{2}}$, and where it is also assumed that the sum is finite. The symbol of $\mathcal{R}^t_{\mathcal{C}} \circ \mathcal{R}_{\mathcal{C}}$ on $\overline{\Lambda}'\backslash D$ is given by

$$\sigma^1(x,\xi,y,\eta) = c\frac{1}{\sqrt{|y-a|-|x-a|}}|d\nu|^{\frac{1}{2}}, \tag{8.5}$$

where $|d\nu|^{\frac{1}{2}}$ is the half-density on Λ' induced from the parametrization above, a is the point where the line through x and y meets the curve, and c incorporates some powers of 2π and of i.

The hypothesis that there are only finitely many intersections between any plane $x+\xi^\perp$ and C_v is true generically. Lan has proved also

Theorem 8.2. *Let C be a compact smooth space curve. If C has non-vanishing torsion, then the set of intersection numbers of C with planes is bounded above.*

We will outline the proof of the symbol result, but the calculations on the flowout are too lengthy to be presented in detail here. Most of them can be found in [La], and will also be reported in another work in preparation. The principal symbol (i.e. on the diagonal) can be calculated by several methods. The easiest is to find a Fourier representation of $\mathcal{R}^t_{\mathcal{C}} \circ \mathcal{R}_{\mathcal{C}} f$ by carrying through the calculations of formula (3.6) in [GrU1]. As this can be done expeditiously, we include it here. Moreover, for this part of the calculation, the dimension n may be greater than three as well.

Since the line integral of f through $a(t)$ in the direction $\xi = \frac{x-a}{|x-a|}$ is equal to the line integral of f through x in the same direction, we have

$$\begin{aligned}
\mathcal{R}_{\mathcal{C}}^t \circ \mathcal{R}_{\mathcal{C}} f(x) &= \int_{C_v} \int_R f(a(t)+s\xi)|x-a(t)|^{1-n} ds\, dt \\
&= \int_{C_v} \int_R f(x+s\xi)|x-a(t)|^{1-n}\, ds\, dt \\
&= (2\pi)^{1-n} \int_{C_v} \int_{\xi^\perp} e^{ix\cdot\eta} \hat{f}(\eta) dv_{\xi^\perp}(\eta)|x-a(t)|^{1-n} dt \\
&= (2\pi)^{-n} \int_{R^n} e^{ix\cdot\eta} b(x,\eta)\hat{f}(\eta) d\eta
\end{aligned}$$

where $b(x,\eta)$ is the pushforward of $2\pi|x-a(t)|^{1-n} dv_{(x-a)^\perp}(\eta) dt$ under the map $(\eta,t) \to \eta$, with $dv_{(x-a)^\perp}(\eta)$ the Lebesgue measure on $(x-a)^\perp$ and dt the arc length measure on C_v. In writing this, we have presumed that the pushforward measure has a density with respect to Lebesgue measure on $\mathbb{R}^n$. This will hold provided the set of critical points has measure zero [GuS, p. 304]. Let $\{u_1(t),\dots,u_{n-1}(t)\}$ be a smoothly varying orthonormal basis for $(x-a(t))^\perp$ and let $\zeta = (\zeta_1,\dots,\zeta_{n-1})$ be the coordinates of $\eta \in (x-a(t))^\perp$ with respect to this basis. Then the map $(\eta,t) \to \eta$ is given by $G(\zeta,t) = \sum \zeta_i u_i(t)$. The differential is given by

$$dG = [u_1,\dots,u_{n-1},\sum \zeta_i u_i'],$$

and since the $\{u_i\}_{i=1}^{n-1}$ span $(x-a)^\perp$, the absolute value of the determinant is just $|\sum \zeta_i u_i' \cdot \frac{x-a}{|x-a|}|$. Now since $u_i \cdot \frac{x-a}{|x-a|} = 0$, $u_i' \cdot \frac{x-a}{|x-a|} = -u_i \cdot \frac{d}{dt}\frac{x-a(t)}{|x-a(t)|}$ and thus

$$|\det(dG)| = \left|\sum \zeta_i u_i' \cdot \frac{x-a}{|x-a|}\right| = \left|\eta \cdot \frac{d}{dt}\frac{x-a}{|x-a|}\right|.$$

Simplifying, we obtain,

$$|\det(dG)| = |\eta\cdot a'||x-a|^{-1}.$$

From this is evident that the set of critical points has measure zero. Moreover, the density of the pushforward at a regular value is given by the sum over the preimages of (x,η) of the value of density $|x-a|^{1-n}$ times the reciprocal of the Jacobian, and thus

$$b(x,\eta) = 2\pi \sum_{\{t: a(t)\in(x+\eta^\perp)\cap C_v\}} \frac{1}{|\eta\cdot a'(t)||x-a(t)|^{n-2}}. \tag{8.6}$$

We note that for a given x, η is a regular value if and only if the plane $x+\eta^\perp$ has only transversal intersections with the curve C_v which holds precisely when $(x,\eta,x,\eta) \in D\backslash\Lambda'$.

The symbol of a Fourier integral operator is usually expressed in terms of an amplitude and phase function, when the Schwartz kernel of the operator is given explicitly as an oscillatory integral. We do not have an explicit phase function parametrizing the Lagrangian Λ, so we must approach the problem differently. Here we use an intrinsic characterization based on the asymptotics of testing the Schwartz kernel against localized oscillatory functions. A development can be found in sections 3.2 and 3.3 of [H], and in the specific form which we use it, in section 4.1 of [D].

Definition 8.1. *The principal symbol of order m of a Fourier integral distribution K of order m associated to the conic Lagrangian manifold Λ is the element in*

$$S^{m+\frac{n}{4}}(\Lambda,\,\Omega_{\frac{1}{2}}\otimes L)/S^{m+\frac{n}{4}-1}(\Lambda,\,\Omega_{\frac{1}{2}}\otimes L) \tag{8.7}$$

given by

$$\alpha \longrightarrow e^{i\psi(\pi(\alpha),\,\alpha)}\left\langle ue^{-i\psi(x,\,\alpha)},\,K\right\rangle. \tag{8.8}$$

Here $S^{\mu}\left(\Lambda,\,\Omega_{\frac{1}{2}}\otimes L\right)$ denotes the symbol space of sections of the complex line bundle $\Omega_{\frac{1}{2}}\otimes L$ over Λ, of growth order μ; $u\in C_0^{\infty}\left(X,\,\Omega_{\frac{1}{2}}\right)$; $\psi(x,\,\alpha)\in C^{\infty}(X\times\Lambda)$ is homogeneous of degree 1 in α and the graph of $x\mapsto d_x\psi(x,\,\alpha)$ intersects Λ transversally at α.

To apply this, we need to find candidate functions ψ for which the graph of the differential is transverse to the Lagrangian where we wish to evaluate the symbol. If we can find such a ψ, which moreover has the form $\psi(x,y,w)=\psi_1(x,w)+\psi_2(y,w)$, where w is the point in the Lagrangian where the symbol is to be calculated, then we can evaluate the pairing when K is the Schwartz kernel of $\mathcal{R}_{\mathcal{C}}^t\circ\mathcal{R}_{\mathcal{C}}$ by

$$\begin{aligned}\langle K, e^{i(\psi_1+\psi_2)}\rho_1(x)\otimes\rho_2(y)\rangle &= \langle \mathcal{R}_{\mathcal{C}}^t\circ\mathcal{R}_{\mathcal{C}}e^{i\psi_2}\rho_2, e^{i\psi_1}\rho_1\rangle\\ &= \langle \mathcal{R}_{\mathcal{C}}e^{i\psi_2}\rho_2, \mathcal{R}_{\mathcal{C}}e^{i\psi_1}\rho_1\rangle.\end{aligned} \tag{8.9}$$

This last pairing is an ordinary five dimensional integral over the product of $\mathcal{C}$ with two copies of $\mathbb{R}$ (for the line integrals). We will evaluate its asymptotics using the method of stationary phase.

Initially, we will assume that a point $w=(x_0,y_0,\xi_0,-\eta_0)\in\Lambda\backslash\Sigma$ is given, and that it lies in the flowout of the clean intersection subset of Σ, and define

$$\begin{aligned}\psi(x,\,y,\,w) &= \langle x-x_0,\,\xi_0\rangle+\langle y-y_0,\,-\eta_0\rangle\\ &\quad+\frac{1}{2}\langle y-y_0,\,y-y_0\rangle k(\xi_0,\,-\eta_0)+\frac{1}{2}\langle x-x_0,\,x-x_0\rangle h(\xi_0,\,-\eta_0)\end{aligned} \tag{8.10}$$

where k and h are homogeneous of degree one in ξ_0 and η_0. When h is non-vanishing and k is identically zero, or the reverse, the graph of $d\psi$ is transverse to Λ at w. This is proved by showing that the 12×12 matrix whose first six columns represent the differential of the parametrization of Λ and whose last six columns represent the differential of the graph mapping has full rank. (The hypotheses of vanishing and non-vanishing of k and h are only to simplify the rank calculation.)

Next we substitute ψ for $w=(x_0,\tilde{\xi},y_0,\tilde{\eta})\in\Lambda\backslash\Sigma$ into the pairing (8.9) to obtain

$$\langle K, e^{i\psi}\rho_2(y)\otimes\rho_1(x)\rangle=\int\rho_1\left(a(t)+s_1\theta\right)\rho_2\left(a(t)+s_2\theta\right)e^{-i\tilde{\psi}}\,ds_1ds_2d\theta dt, \tag{8.11}$$

where $\tilde{\psi}$ is ψ evaluated at $(a(t)+s_1\theta, a(t)+s_2\theta)$. It is checked that $\tilde{\psi}$ has only the critical point corresponding to w if ρ_1 and ρ_2 have small enough support, then an application of stationary phase as $\tau\to\infty$ for $\tilde{\xi}=\tau\xi_0$, $\tilde{\eta}=\tau\eta_0$ gives the asymptotic expansion of the pairing. It is found, when $h=0,k\neq 0$ or $k=0,h\neq 0$ that the leading term of the asymptotic expansion is given by

$$\frac{(2\pi)^{\frac{5}{2}} e^{\frac{\pi i \sigma}{4}} \rho_1\rho_2(x_0, y_0)}{k\left(\tilde{\xi}, \tilde{\eta}\right) \left|\tilde{\xi}\right| \, ||y_0 - a| - |x_0 - a||^{\frac{1}{2}} \left|a(t(\tilde{\xi})) - y_0\right|^{\frac{1}{2}} \left|a'' \cdot \tilde{\xi}\right|^{\frac{1}{2}}} \qquad \text{if } h = 0, k \neq 0$$

$$\frac{(2\pi)^{\frac{5}{2}} e^{\frac{\pi i \sigma}{4}} \rho_1\rho_2(x_0, y_0)}{h\left(\tilde{\xi}, \tilde{\eta}\right) |\tilde{\eta}| \, ||y_0 - a| - |x_0 - a||^{\frac{1}{2}} \left|a(t(\tilde{\xi})) - x_0\right|^{\frac{1}{2}} |a'' \cdot \tilde{\eta}|^{\frac{1}{2}}} \qquad \text{if } k = 0, h \neq 0$$

where the signature factor σ is given by

$$\sigma = 2 + \operatorname{sgn}((a''(t) \cdot \tilde{\xi})(|\tilde{\xi}| - |\tilde{\eta}|)).$$

We note that since we have assumed that w lies in the flowout of the clean intersection subset we have $a''(t) \cdot \tilde{\xi} \neq 0$. Now we must divide by the value of $\rho_1\rho_2$ and account for the dependence of the asymptotics on the transverse Lagrangian, graph $d\psi$. Following the analysis in [H] or [D], it can be seen that the invariant expression of the symbol will be obtained by multiplying this asymptotic expression by $|P_L^*\omega|^{\frac{1}{2}}$, where P_L is the linear projection of the tangent space to the Lagrangian at w onto the tangent space to the fiber of $T^*(\Omega \times \Omega)$ along the tangent space to the graph of $d\psi$ at w, and ω is the volume induced in the fiber as the quotient of the volume from the symplectic form and the volume on the base. (The projection is non-singular by the hypothesis of transversal intersection.) These may be evaluated when $h = 0, k \neq 0$ and $h \neq 0, k = 0$ using the same coordinates as were used in the preceding calculations. Multiplying the leading terms above by these half-density factors, it is found that both expressions produce

$$\frac{(2\pi)^{\frac{5}{2}} e^{\frac{i\pi\sigma}{4}}}{||y_0 - a| - |x_0 - a||^{\frac{1}{2}}}. \tag{8.12}$$

Taking account that the signature factor changes by $\pm i$ along any line in the flowout when passing through the diagonal, we may incorporate this in the denominator, to obtain the square root of the difference of $|y_0 - a|$ and $|x_0 - a|$. This analysis breaks down when w corresponds to a point in $\overline{\Lambda}\backslash\Lambda$. However, one can also approach the analysis of the Schwartz kernel of $\mathcal{R}_{\mathcal{C}}^t \circ \mathcal{R}_{\mathcal{C}}$ by another method. It can also be expressed as the pushforward under the natural projection from $C_v \times \Omega \times \Omega$ to $\Omega \times \Omega$ of the pullback by a submersion of a conormal distribution on the product of two two-spheres. One can then check that the tranversality condition of [GuS] is satisfied above $\overline{\Lambda}$ away from the diagonal, so that the Schwartz kernel is a Lagrangian distribution (in fact, conormal) on the flowout of the nonclean intersection subset as well. Since the symbol must be smooth, we can extend by continuity the formula obtained above.

The method used above for computing the symbol σ^1 using (8.9) can also be used to obtain the specific form of the principal symbol σ^0; the details are worked out in [La]. This was the version used by Ramaseshan in [R] where he needed to compute the principal symbol of the Doppler transform restricted to the complex of straight line through a space curve.

Finally we would like to point out that Propositions 3.1 and 3.3 have some interesting consequences in tomography. It is sometimes taken as a useful approximation to represent the object to be reconstructed as a superposition of products of a smooth function with the characteristic function of a set with smooth boundary, which places it in the category of conormal distributions considered in section §2.

The tomographer is interested in reconstructing the discontinuities (singularities) of the object, but Propositions 3.1 and 3.3 say that a local method (applying a differential or pseudodifferential operator to $\mathcal{R}_\mathcal{C}^t \circ \mathcal{R}_\mathcal{C}$) will always produce artifacts due to the flowout Lagrangian. More specifically, suppose μ is a conormal distribution associated to a surface S and that $(x_0, \xi_0, x_0, \xi_0) \in \Sigma$. Let $p(x, \xi) = (x - a(t(\xi))) \cdot \xi$, for ξ in a conic neighborhood of ξ_0 be as described prior to Theorem 8.1, let $P(x, D)$ be a pseudodifferential operator with symbol $p(x, \xi)$, so that microlocally $\Lambda' = \Lambda_P$ with Λ_P as in (3.3). One can then prove that $\mathrm{char}(P)$ intersects N^*S transversally at (x_0, ξ_0) provided that $x_0 - a(t(\xi_0))$ is not an asymptotic vector to the surface at x_0. (Of course, this holds automatically if the surface has positive Gaussian curvature at x_0.) Using $\Lambda_0 = N^*S$ and $\Lambda_1 = \Lambda_P \circ \Lambda_0$ as in Proposition 3.1, we have from Theorem 7.1, Proposition 3.1, and Theorem 8.1 that if $\mu \in I^r(N^*S)$ has non-zero symbol at (x_0, ξ_0) then $\mathcal{R}_\mathcal{C}^t \circ \mathcal{R}_\mathcal{C} \mu \in I^{r-1}(\Lambda_0 \backslash \Lambda_1)$ and $\mathcal{R}_\mathcal{C}^t \circ \mathcal{R}_\mathcal{C} \mu \in I^{r-1}(\Lambda_1 \backslash \Lambda_0)$, and the symbol of the latter is non-zero. This means that the propagated singularities have the same strength as the singularities which were to be recovered. (A similar observation, in a specific case, was also made by Katsevich in [Ka].) However, the structure of Λ_1 in this case, being the conormal bundle of a ruled surface, may provide evidence that it could be an artifact. Furthermore, applying $P(x, D)$ to $\mathcal{R}_\mathcal{C}^t \circ \mathcal{R}_\mathcal{C} \mu$ would decrease the order of the singularities in the flowout, though at the expense of changing the symbol on Λ_0 as well.

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