Sampling of bandlimited functions on unions of shifted lattices

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Abstract

We consider Shannon sampling theory for sampling sets which are unions of shifted lattices. These sets are not necessarily periodic. A function f can be reconstructed from its samples provided the sampling set and the support of the Fourier transform of f satisfy certain compatibility conditions. An explicit reconstruction formula is given for sampling sets which are unions of two shifted lattices. While explicit formulas for unions of more than two lattices are possible, it is more convenient to use a recursive algorithm. The analysis is presented in the general framework of locally compact abelian groups, but several specific examples are given, including a numerical example implemented in MATLAB. Our methods also provide a new tool for designing sampling sets of minimal density.

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1 Introduction

The classical sampling theorem permits reconstruction of a bandlimited function from its values on a set of equidistant points on the real line $\mathbb{R}$ [17, 21, 25]. It has been extended in many directions; see the reviews [3, 11, 14] as well as the volumes [12, 13, 18, 19, 27]. Kluvánek's important generalization results from replacing $\mathbb{R}$ by an arbitrary locally compact abelian (LCA) group G [15]. The sampling set is then a coset of a closed subgroup of G. Periodic sampling, introduced by Kohlenberg [16], is well established and considers sampling sets which are unions of cosets of one subgroup; see, e.g., [5, 9, 20, 26, 28]. The present work investigates the case where the sampling set is a union of cosets of possibly different subgroups. Such sets are not necessarily periodic. Seminal results for this case have been derived by Walnut [22] and applied in [8, 23, 24]. These theorems were proved for $G = \mathbb{R}$ and extended to higher dimensions by means of tensor products. The approach taken here works for general LCA groups G, the sampling sets in case of $G = \mathbb{R}^n$ need not be tensor products of a one-dimensional set, and the support K of the Fourier transform of f need not be a hypercube. On the other hand, the sampling set and the set K need to satisfy certain compatibility conditions. If these conditions are satisfied, we obtain a recursive algorithm for reconstructing f from its samples on cosets of subgroups $H_1, \ldots, H_N$. Our results do lead in principle to explicit reconstruction formulas, and for the case of two lattices such a formula is given in Corollary 3.7. However, for more than two lattices the formulas tend to become very complicated, while the recursive algorithm is both convenient to state and easy to program. While the sampling sets we consider here could also be treated in many cases with the general methods for irregular sampling developed in the last decade, (see, e.g., [2, 6, 7]), our results make explicit use of the structure these sets possess.

The paper is organized as follows. We begin with a review of basic definitions and facts, leading up to Kluvánek's general version of the classical sampling theorem. For a more detailed introduction to sampling theory on LCA groups we refer to the recent article by Dodson and Beaty [4]. In section 3 the main results are developed and illustrated with examples. Theorem 3.5 provides a method to reduce the original problem to a simpler one, which can be used to obtain new sampling theorems from known ones. Explicit reconstruction formulas for sampling sets which are unions of two shifted lattices are given in Corollary 3.7. Several examples are given where our techniques yield sampling sets of minimal density. Applying Theorem 3.5 repeatedly yields the recursive reconstruction method of Theorem 3.9 for sampling on unions of N shifted lattices. The paper concludes with a numerical example implementing the algorithm in MATLAB for the group $G = \mathbb{Z}_L$.

2 Standard definitions and facts

Let $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ denote the integers, reals, and complex numbers, respectively. Let G denote a locally compact abelian (LCA) group written additively. The character group $\widehat{G}$ consists of the continuous homomorphisms of G into the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The value of the character $\xi \in \widehat{G}$ at the point $x \in G$ is written $\langle x, \xi \rangle$. $\widehat{G}$ has a natural addition and a natural topology relative to which it is also an LCA group. On every LCA group there exists a non-negative regular measure m_G, the so-called Haar measure of G, which is not identically zero and translation invariant. The Haar measure is uniquely determined up to multiplication by a constant. $L_p(G)$ denotes the space of all Borel functions on G such that $\| f \|_p = \left(\int_G |f(x)|^p \, dm_G(x)\right)^{1/p}$ is finite.

The Fourier transform of a function $f \in L_1(G)$ is the continuous function $\hat{f}$ on $\widehat{G}$ defined by

$$\hat{f}(\xi) = \int_G f(x) e^{-2\pi i \langle x, \xi \rangle} \, dm_G(x).$$

We will always normalize the Haar measure on $\widehat{G}$ such that the following holds.

Theorem 2.1 *(Fourier inversion formula) If $f \in L_1(G)$ is continuous and $\hat{f} \in L_1(\widehat{G})$, then*

$$f(x) = \int_{\widehat{G}} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} \, dm_{\widehat{G}}(\xi) = (\hat{f})^\wedge(-x). \tag{1}$$

The Fourier transform can be extended to a linear isomorphism of $L_2(G)$ onto $L_2(\widehat{G})$ by means of the Plancherel Theorem (cf. [10, Sec. 31.18]).

Let H be a closed subgroup of an LCA group G. The annihilator of H is the set $H^\perp \subset \widehat{G}$ given by $H^\perp = \{\eta \in \widehat{G} : \ \langle y, \eta \rangle = 0 \text{ for all } y \in H\}$. $H^\perp$ is a closed subgroup of $\widehat{G}$ and is isomorphically homeomorphic to the character group of G/H, i.e., $H^\perp = (G/H)^\wedge$. Furthermore we have that $(H^\perp)^\perp = H$, and $\widehat{H} = \widehat{G}/H^\perp$.

Definition 2.2 *A closed discrete subgroup H of G such that $H^\perp$ is also discrete is called a* lattice. *A measurable subset R of $\widehat{G}$ such that every $\xi \in \widehat{G}$ can be uniquely written as $\xi = \rho + \eta$, where $\rho \in R$ and $\eta \in H^\perp$ is called a* fundamental domain of $H^\perp$.

Convention 2.3 *Throughout this paper we assume that m_G is given and normalize the Haar measure on $\widehat{G}$ such that the Fourier inversion formula (1) holds. For a lattice H and R a fundamental domain of $H^\perp$ we normalize the Haar measures on H, $H^\perp$, and $\widehat{G}/H^\perp$ such that*
(i) m_H equals the counting measure,
(ii) $m_{H^\perp}$ equals $m_{\widehat{G}}(R)$ times the counting measure, and
(iii) $m_{\widehat{G}/H^\perp}(\widehat{G}/H^\perp) = 1$.

We always have $0 < m_{\widehat{G}}(R) < \infty$ and the above normalizations imply that for every integrable function F on $\widehat{G}$

$$\int_{\widehat{G}} F(\xi) dm_{\widehat{G}}(\xi) \;=\; m_{\widehat{G}}(R) \int_{\widehat{G}/H^\perp} \sum_{\eta \in H^\perp} F(\xi + \eta) dm_{\widehat{G}/H^\perp}(\xi + H^\perp). \qquad (2)$$

Let g be a function on $\widehat{G}/H^\perp$ and let F in (2) be given by $F(\xi) = g(\xi + H^\perp)$ for $\xi \in R$, and $F(\xi) = 0$ otherwise. Then (2) allows to identify integration over $\widehat{G}/H^\perp$ with integration over R:

$$\int_{\widehat{G}/H^\perp} g(\xi + H^\perp) dm_{\widehat{G}/H^\perp}(\xi + H^\perp) \;=\; \frac{1}{m_{\widehat{G}}(R)} \int_R g(\xi + H^\perp) dm_{\widehat{G}}(\xi). \qquad (3)$$

3 Sampling theorems

We begin with Kluvánek's version of the classical sampling theorem. A key ingredient is the function φ_R defined in the following lemma, proven in Kluvánek's paper [15].

Lemma 3.1 *Let H be a lattice and R a fundamental domain of $H^\perp$. Then the function φ_R defined by*

$$\varphi_R(x) = \frac{1}{m_{\widehat{G}}(R)} \int_R e^{2\pi i \langle x, \xi \rangle} dm_{\widehat{G}}(\xi), \quad x \in G, \qquad (4)$$

is continuous on G and satisfies $\varphi_R(0) = 1$, $\varphi_R(y) = 0$ for $0 \neq y \in H$, $\| \varphi_R \|_2 = 1/\sqrt{m_{\widehat{G}}(R)}$, and

$$\int_G \varphi_R(x)\,\overline{\varphi_R(x-y)}\,dm_G(x) = 0 \;\; \text{for} \;\; 0 \neq y \in H.$$

Kluvánek's theorem reads as follows.

Theorem 3.2 *Let H be a lattice and R a fundamental domain of $H^\perp$. Suppose $f \in L_2(G)$ and $\hat{f}(\xi) = 0$ for almost all $\xi \notin R$. Then f is equal almost everywhere to a continuous function. If f itself is continuous, then*

$$f(x) = \sum_{y \in H} f(y)\varphi_R(x-y) \tag{5}$$

uniformly on G and in the sense of convergence in $L_2(G)$. Furthermore, the L_2-norm of f is given by

$$\| f \|_2^2 = \frac{1}{m_{\widehat{G}}(R)} \sum_{y \in H} |f(y)|^2.$$

The last equation shows that the restriction of f to the discrete subgroup H gives a function in $L_2(H)$. This property is needed for the reconstruction formula (5) to be well-defined. We would like to apply this formula also to functions whose Fourier transform is supported in a set K larger than R. The following corollary to Kluvánek's theorem deals with this case.

Corollary 3.3 *Let H be a lattice and R a fundamental domain of $H^\perp$. Let $f \in L_2(G)$ be continuous and $\hat{f}(\xi) = 0$ a.e. outside a measurable subset K of $\widehat{G}$. Assume that there is $P < \infty$ such that $K \subseteq \bigcup_{j=1}^{P} (\eta_j + R)$ with $\eta_1, \ldots, \eta_P$ distinct elements of $H^\perp$. Let $M = x_0 + H$ be a coset of H. Then the function $S_M f$ defined by*

$$S_M f(x) = \sum_{y \in H} f(x_0 + y)\,\varphi_R(x - x_0 - y) \tag{6}$$

is continuous and square integrable on G, and satisfies $S_M f(z) = f(z)$ for all $z \in M$.

Proof: We may decompose f as $f = \sum_{j=1}^{P} f_j$ with continuous functions f_j satisfying $\hat{f}_j(\xi) = \hat{f}(\xi)$ for $\xi \in \eta_j + R$, and $\hat{f}_j = 0$ a.e. outside $\eta_j + R$. Hence

$$S_M f(x) = \sum_{j=1}^{P} \sum_{y \in H} f_j(x_0 + y)\, \varphi_R(x - x_0 - y).$$

Now Theorem 3.2 can be applied to the functions $g_j(x) = f_j(x_0 + x)$ whose Fourier transform vanishes a.e. outside $R_j = \eta_j + R$, which is also a fundamental domain of $H^\perp$. This gives

$$f_j(x) = g_j(x - x_0) = \sum_{y \in H} f_j(x_0 + y)\varphi_{R_j}(x - x_0 - y),$$

where the right-hand side converges uniformly and defines a continuous function in $L_2(G)$. Because of $\varphi_{R_j}(x) = \varphi_R(x)e^{2\pi i\langle x, \eta_j\rangle}$ this implies that the sums $\sum_{y \in H} f_j(x_0 + y)\varphi_R(x - x_0 - y)$ also define continuous functions in $L_2(G)$. Hence $S_M f$ is continuous and square integrable. Since $\varphi_R(0) = 1$ and $\varphi_R(y) = 0$ for $0 \neq y \in H$ it follows immediately that $S_M f(z) = f(z)$ for $z \in M$. $\square$

Our point of departure for deriving nonperiodic sampling theorems is the following lemma, which is closely related to Lemma 3.1 in [22]. We consider the case where the support of the Fourier transform is no longer contained in a fundamental domain of $H^\perp$, but is contained in the union of a fundamental domain and one of its translates. A more general result has been obtained in [1], where K only needs to be contained in the union of finitely many translates of R. Since this result requires a more technical proof and is not needed here, we do not present it.

Lemma 3.4 *Let H be a lattice and R a fundamental domain of $H^\perp$. Let $K = R \cup (\eta' + K')$ with $K' \subset R$ measurable and $0 \neq \eta' \in H^\perp$. Assume that $f \in L_2(G)$ is continuous, vanishes on the coset $x_0 + H$ and that $\hat{f}$ vanishes a.e. outside K. Then*

$$f(x) = h(x)\left(1 - e^{2\pi i\langle x - x_0, \eta'\rangle}\right)$$

with $h \in L_2(G)$ continuous and $\hat{h}$ vanishing a.e. outside K'.

Proof: Consider the function $g(x) = f(x + x_0)$. Then g is continuous, vanishes on H, and $\hat{g}$ vanishes a.e. outside K. Hence $\hat{g} \in L_1(\widehat{G})$, and therefore the periodization $\sum_{\eta\in H^\perp}\hat{g}(\xi+\eta)$ is in $L_1(\widehat{G}/H^\perp)$. Let F in (2) be given by $F(\xi)=\hat{g}(\xi)e^{2\pi i\langle y,\xi\rangle}$ with y a fixed element of H. Because of $\langle y,\xi+\eta\rangle=\langle y,\xi\rangle$, the Fourier inversion formula and (2) now give

$$\begin{aligned} g(y) &= \int_{\widehat{G}}\hat{g}(\xi)e^{2\pi i\langle y,\xi\rangle}dm_{\widehat{G}}(\xi) \\ &= m_{\widehat{G}}(R)\int_{\widehat{G}/H^\perp}\sum_{\eta\in H^\perp}\hat{g}(\xi+\eta)e^{2\pi i\langle y,\xi\rangle}dm_{\widehat{G}/H^\perp}(\xi+H^\perp). \end{aligned}$$

Since g vanishes on H, this means that the Fourier transform (with respect to $\widehat{G}/H^\perp$) of $\sum_{\eta\in H^\perp}\hat{g}(\xi+\eta)$ vanishes identically. Hence

$$\sum_{\eta\in H^\perp}\hat{g}(\xi+\eta)=0 \text{ a.e.}. \tag{7}$$

We now decompose the set K into three disjoint subsets, i.e.,

$$K=K'\cup(\eta'+K')\cup(R\backslash K').$$

Since the translated sets $R+\eta$, $\eta\in H^\perp$ are disjoint, we have for $\xi\in R\backslash K'$ and $\eta\in H^\perp$ that $\xi+\eta\in K$ if and only if $\eta=0$. It now follows from (7) that $\hat{g}$ must vanish a.e. on $R\backslash K'$. Let $\tilde{h}\in L_2(G)$ be such that $\hat{\tilde{h}}(\xi)=\hat{g}(\xi)$ for $\xi\in K'$, and $\hat{\tilde{h}}(\xi)=0$ for a.e. $\xi\notin K'$. Since $m_{\widehat{G}}(K')\le m_{\widehat{G}}(R)<\infty$, $\tilde{h}$ can be chosen to be continuous. For $\xi\in\eta'+K'$ we have that $\xi+\eta\in K$ if and only if $\eta\in\{0,-\eta'\}$. Then (7) gives for a.e. $\xi\in\eta'+K'$ that

$$\hat{g}(\xi)=-\hat{g}(\xi-\eta')=-\hat{\tilde{h}}(\xi-\eta').$$

Since $\hat{\tilde{h}}(\xi-\eta')$ vanishes a.e. outside $\eta'+K'$ we have for a.e. $\xi\in\widehat{G}$

$$\hat{g}(\xi)=\hat{\tilde{h}}(\xi)-\hat{\tilde{h}}(\xi-\eta').$$

An inverse Fourier transform now gives

$$g(x)=\tilde{h}(x)\left(1-e^{2\pi i\langle x,\eta'\rangle}\right)=f(x_0+x).$$

The lemma now follows by letting $h(x)=\tilde{h}(x-x_0)$. □

The lemma can be used in the following general way to reduce the problem of reconstructing f to the problem of reconstructing h.

Theorem 3.5 *Let H be a lattice and R a fundamental domain of $H^{\perp}$. Let $K = R \cup (\eta' + K')$ with $K' \subset R$ measurable and $0 \neq \eta' \in H^{\perp}$. Assume that $f \in L_2(G)$ is continuous, and that $\hat{f}$ vanishes a.e. outside K. Let $M' \subset G$ be such that continuous functions $h \in L_2(G)$ whose Fourier transform vanishes a.e. outside K' can be reconstructed from their samples $h(z')$, $z' \in M'$. Let x_0 be such that*

$$\langle z' - x_0, \eta' \rangle \neq 0 \text{ for all } z' \in M'. \tag{8}$$

Then f can be reconstructed from its samples $f(z)$, $z \in M \cup M'$, where $M = x_0 + H$.

Proof: By Corollary 3.3 the function $g(x) = f(x) - S_M f(x)$ is continuous, square integrable and vanishes on M. It follows from (6) and (4) that $(S_M f)^\wedge(\xi)$ vanishes for a.e. ξ outside R. Hence Lemma 3.4 can be applied to g, yielding a continuous function $h(x) \in L_2(G)$ with $\hat{h}$ vanishing a.e. outside K' such that

$$f(x) = S_M f(x) + h(x)\left(1 - e^{2\pi i \langle x - x_0, \eta' \rangle}\right). \tag{9}$$

Since $\langle z' - x_0, \eta' \rangle \neq 0$ for $z' \in M'$, we can compute the sampled values

$$h(z') = \frac{f(z') - S_M f(z')}{1 - e^{2\pi i \langle z' - x_0, \eta' \rangle}}, \quad z' \in M'. \tag{10}$$

By hypothesis, $h(x)$, $x \in G$, can be computed from these samples. Then $f(x)$ is given by (9). □

The theorem provides a general method to generate new sampling theorems from known ones. If a sampling theorem for a set K' is known, we can obtain one for $K = R \cup (\eta' + K')$ by adding a coset of the subgroup H to the original sampling set M'. Aside from the condition (8) the primary limitation of this method is the requirement that H must be sufficiently dense so that $K' \subset R$.

The proof above outlines the following reconstruction algorithm:

Algorithm 3.6 *1) Compute $S_M f(x)$, $x \in G$, from the samples $f(z)$, $z \in M$, according to (6).*

2) Compute the samples $h(z')$, $z' \in M'$, according to (10).

3) Reconstruct $h(x)$, $x \in G$, from these samples, which is possible by hypothesis.

4) Compute $f(x)$, $x \in G$, according to (9).

As a first illustration we apply the theorem to sampling on the real line when the support of the Fourier transform has a gap. For such a case, Theorem 3.5 provides a convenient way to generate sampling sets of minimal density. Let $G = \widehat{G} = \mathbb{R}$ and $K = [0, \alpha) \cup [\alpha + 1, \alpha + 2)$, $\alpha \geq 2$. Observe that with our normalization, m_G and $m_{\widehat{G}}$ are equal to the Lebesgue measure on $\mathbb{R}$. For any β satisfying $\frac{\alpha}{2} + 1 \leq \beta \leq \alpha$, K may be partitioned into two subsets so that the theorem applies. If $R = [0, \beta)$, $\eta' = \beta$, and $K' = [0, \alpha - \beta) \cup [\alpha - \beta + 1, \alpha - \beta + 2)$, then $K = R \cup (\eta' + K')$. Let $H = \frac{1}{\beta}\mathbb{Z}$, $M = x_0 + H$, and let $M' \subset \mathbb{R}$ be such that continuous functions with Fourier transform supported in K' can be reconstructed from their samples on M'. Then a function f with Fourier transform supported in K can be reconstructed from its samples on $M \cup M'$ if the condition (8) is satisfied. We consider two particular choices of β, i.e., $\beta = \alpha$ and $\beta = \alpha - 1$. First let $\beta = \alpha$, so that $R = [0, \alpha)$, $\eta' = \alpha$, $K' = [1, 2)$, and $H = \frac{1}{\alpha}\mathbb{Z}$. According to Kluvánek's theorem, functions with Fourier transform supported in K' can be reconstructed from their samples on a coset of $H' = \mathbb{Z}$. So suitable candidates for sampling sets would be sets of the form $(x_0 + \frac{1}{\alpha}\mathbb{Z}) \cup (x_1 + \mathbb{Z})$, subject to condition (8). This condition requires that $(x_1 - x_0 + l)\alpha \notin \mathbb{Z}$ for all $l \in \mathbb{Z}$, which may be written as $x_1 - x_0 \notin (\frac{1}{\alpha}\mathbb{Z} + \mathbb{Z})$. In this particular case this is equivalent to the intersection of $M = x_0 + \frac{1}{\alpha}\mathbb{Z}$ and $M' = x_1 + \mathbb{Z}$ being empty. However, in general the condition $M \cap M' = \emptyset$ is only necessary but not sufficient for (8) to hold.

If α is irrational, we obtain a genuinely non-periodic sampling set. When sampling on the group $G = \mathbb{R}^n$ one also has to consider stability. In the case of irrational α it is possible that the reconstruction is unstable, since there is no positive minimum distance between adjacent sampling points. Hence a numerical implementation would require regularization. We do not investigate stability considerations here, but refer the reader to [8, 24] for an investigation of stability and remedies for instability in a similar case.

If α is rational, let $\alpha^{-1} = p/q$ with p, q mutually prime. Then a sampling set of the form $(x_0 + \frac{1}{\alpha}\mathbb{Z}) \cup (x_1 + \mathbb{Z})$ is periodic, namely a union of $p + q$ cosets of the group $p\mathbb{Z}$. However, for $p + q$ large Theorem 3.5 may be more convenient to apply than the results for periodic sampling developed in [5] or [9] and the references cited there.

The set M' in Theorem 3.5 does not need to be a coset of one subgroup. E.g., let $\alpha > 4$ and $\beta = \alpha - 1$. Then we can choose $R = [0, \alpha - 1)$, $K' = [0, 1) \cup [2, 3)$, $\eta' = \alpha - 1$, and $H = (\alpha - 1)^{-1}\mathbb{Z}$. The function h whose Fourier transform is supported in K' can now be reconstructed from its samples on two cosets of $\mathbb{Z}$ according to Kohlenberg's [16] theory. In this case M' is a periodic set of the form $M' = (x_1 + \mathbb{Z}) \cup (x_2 + \mathbb{Z})$.

In the above example with $\beta = \alpha$, reconstruction of the function h was furnished by the classical sampling theorem. The following corollary gives an explicit reconstruction formula for this case.

Corollary 3.7 *Let H_1, H_2 be lattices, and $R_1 \subset R_2$ fundamental domains of $H_1^\perp$ and $H_2^\perp$, respectively. Let $f \in L_2(G)$ be continuous and such that $\hat{f}$ vanishes a.e. outside the set $K = R_2 \cup (\eta' + R_1)$, where $0 \neq \eta' \in H_2^\perp$. Let x_1, x_2 be such that*

$$\langle x_1 - x_2 + y, \eta' \rangle \neq 0 \quad \text{for all } y \in H_1. \tag{11}$$

Then

$$\begin{aligned} f(x) &= S_{M_2} f(x) + \left(1 - e^{2\pi i \langle x - x_2, \eta' \rangle}\right) \\ &\quad \times \sum_{y \in H_1} \frac{f(x_1 + y) - S_{M_2} f(x_1 + y)}{1 - e^{2\pi i \langle x_1 - x_2 + y, \eta' \rangle}} \varphi_{R_1}(x - x_1 - y) \end{aligned} \tag{12}$$

with $M_2 = x_2 + H_2$,

$$S_{M_2} f(x) = \sum_{v \in H_2} f(x_2 + v) \varphi_{R_2}(x - x_2 - v),$$

and φ_{R_j}, $j = 1, 2$ as defined in (4).

Proof: Apply Theorem 3.5 and Algorithm 3.6 with $H = H_2$, $R = R_2$, $K' = R_1$, $M = M_2 = x_2 + H_2$, and $M' = x_1 + H_1$. Then (8) becomes (11), and (10) gives the samples of h for $z' \in x_1 + H_1$, i.e.,

$$h(x_1 + y) = \frac{f(x_1 + y) - S_{M_2} f(x_1 + y)}{1 - e^{2\pi i \langle x_1 - x_2 + y, \eta' \rangle}}.$$

Since $\hat{h}$ is supported in the fundamental domain R_1 of $H_1^\perp$, we can apply Theorem 3.2 to the function $\tilde{h}(x) = h(x_1 + x)$, whose samples are available on H_1. This gives

$$h(x) = \sum_{y \in H_1} h(x_1 + y) \varphi_{R_1}(x - x_1 - y).$$

Now (9) yields (12). $\square$

We note that neither the requirement that $K = R_2 \cup (\eta' + R_1)$ nor its generalization in Definition 3.8 below exclude the case that K consists of a single interval or hypercube. For example, let $G = \mathbb{R}$ and $K = [-W, W)$. For any W_1 such that $0 \le W_1 < W$ we have $K = R_2 \cup (\eta' + R_1)$ with $R_2 = [-W, W_1)$, $\eta' = W + W_1$, and $R_1 = [-W, -W_1)$.

As another example, let us consider a bandpass signal whose Fourier transform vanishes outside the set $K = [-W_1 - W, -W_1) \cup [W_1, W_1 + W)$. Kohlenberg solved this case using periodic sampling sets [16]. Now assume that $0 < W_1 < W$ and let $H_2 = (W + W_1)^{-1}\mathbb{Z}$ with fundamental domain $R_2 = [-W_1 - W, -W_1) \cup [W, W_1 + W)$. (Note that a fundamental domain need not be a single interval.) With $R_1 = [-W, -W_1) \subset R_2$ and $\eta' = W + W_1$ we have $K = R_2 \cup (\eta' + R_1)$ and Corollary 3.7 applies with $H_1 = (W - W_1)^{-1}\mathbb{Z}$. Depending on whether the ratio $(W + W_1)/(W - W_1)$ is rational or irrational, one obtains a periodic or nonperiodic sampling set, respectively.

Finally, let K consist of the three intervals $K = [-W_1 - W, -W_1) \cup [-W_0, W_0) \cup [W_1, W_1 + W)$ such that $0 < W_0 < W_1 < W_0 + W$. We find a sampling set of minimal density by choosing $H_2 = (W + W_0 + W_1)^{-1}\mathbb{Z}$ with fundamental domain $R_2 = [-W_1 - W, -W_1) \cup [-W_0, W_0) \cup [W_1, W_1 + W)$, and $H_1 = (W + W_0 - W_1)^{-1}\mathbb{Z}$ with fundamental domain $R_1 = [-W_0 - W, -W_1)$. With $\eta_2 = W + W_0 + W_1$ we obtain $K = R_2 \cup (\eta_2 + R_1)$. Depending on the values of W, W_0 and W_1 an optimal periodic sampling set may not exist or the most efficient periodic sampling set, found, e.g., by the methods of [5] or [9] may be given as a union of a large number of cosets of a sparse subgroup. As mentioned before, this may be inconvenient to process. For this example our method always yields a representation using only two shifted lattices. The examples given here, as well as the one discussed in §4 below show that the methods presented here may improve on the methods for periodic sampling in some situations. However, they do by no means replace these methods in general, since our conditions on the set K are more restrictive.

In order to obtain further results we apply Theorem 3.5 repeatedly. This gives a recursive algorithm to reconstruct f from its samples on cosets of groups $H_1, \ldots, H_N$, provided that the subgroups H_j and the set K satisfy certain compatibility conditions. These conditions are given in the following definition which presents the structure of the sets K we consider as support of the Fourier transform $\hat{f}$. This structure is a generalization of the structure of the set K in Lemma 3.4.

Definition 3.8 *Let $H_1, \ldots, H_N$ be lattices with corresponding fundamental domains R_i of $H_i^\perp$. We call $K \subset \widehat{G}$ an admissible subset of $\widehat{G}$ with respect to $H_1, \ldots, H_N$ if there are subsets $K_1, \ldots, K_N$ of $\widehat{G}$ such that the following conditions hold:*

i) $K_1 = R_1$,

ii) $K_j \subset R_{j+1}$, $j = 1, \ldots, N-1$,

iii) $K_{j+1} = R_{j+1} \cup (\eta_{j+1} + K_j)$ *with* $0 \neq \eta_{j+1} \in H_{j+1}^\perp$, $j = 1, \ldots, N-1$

iv) $K_N = K$.

Observe that because of conditions *ii)* and *iii)* each intermediate set K_{j+1} has the structure of the set K in Lemma 3.4 with $R = R_{j+1}$, $K' = K_j$ and $\eta' = \eta_{j+1}$. The above conditions imply in particular that $R_1 \subset R_2 \subset \ldots \subset R_N$, so that the subgroups H_j are ordered by increasing density. In addition it follows that $H_j \neq H_k$ for $j \neq k$, with the exception that H_1 may be equal to H_2. Hence the theory developed here does not include periodic sampling, where $H_1 = \ldots = H_N$ as a special case, although some of the sampling sets we consider are indeed periodic.

As an example, let $G = \mathbb{R}^2$, and H_1, H_2 and H_3 be lattices of the form $H_i = W_i\mathbb{Z}^2$, with matrices

$$W_i = \begin{pmatrix} r_i & 0 \\ 0 & d_i \end{pmatrix}$$

such that r_i, $d_i \in \mathbb{R}^+$. Furthermore, assume that $r_3 < r_2 < r_1$ and $d_3 < d_2 < d_1$ such that $\frac{1}{d_1} + \frac{1}{d_2} \leq \frac{1}{d_3}$. Let fundamental domains R_i of $H_i^\perp$ be given by

$$R_i = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : 0 \leq \xi_1 < 1/r_i, \quad 0 \leq \xi_2 < 1/d_i\right\}, \quad i = 1, 2, 3,$$

as illustrated in the left part of Figure 1.

Let $\eta_2 \in H_2^\perp$ and $\eta_3 \in H_3^\perp$ be given by

$$\eta_2 = \begin{pmatrix} 0 \\ 1/d_2 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 1/r_3 \\ 0 \end{pmatrix}.$$

The set $K_2 = R_2 \cup (\eta_2 + R_1) \subset R_3$ is shown in the right part of Figure 1. The complete set $K = K_3$ is given by

$$\begin{aligned} K &= R_3 \cup (\eta_3 + K_2) \\ &= R_3 \cup (\eta_3 + R_2) \cup (\eta_3 + \eta_2 + R_1) \\ &= R_3 \cup \left(\begin{pmatrix} 1/r_3 \\ 0 \end{pmatrix} + R_2\right) \cup \left(\begin{pmatrix} 1/r_3 \\ 1/d_2 \end{pmatrix} + R_1\right) \end{aligned}$$

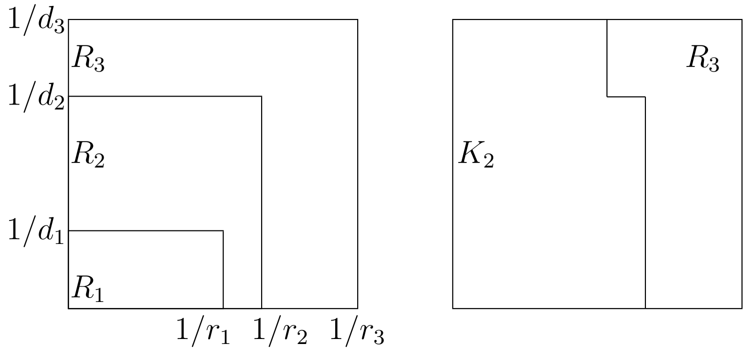

Figure 1: $R_1 \subset R_2 \subset R_3$ with $K_1 = R_1$ and $K_2 \subset R_3$

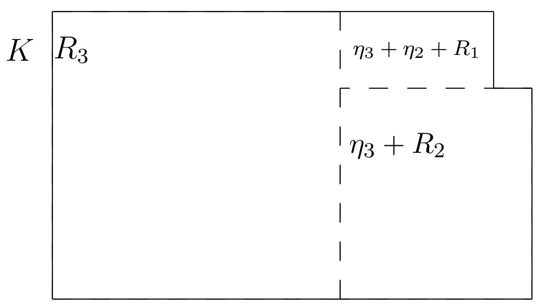

Figure 2: $K = R_3 \cup (\eta_3 + R_2) \cup ((\eta_3 + \eta_2) + R_1)$

and shown in Figure 2.

The following theorem is our main result:

Theorem 3.9 *Suppose that K is an admissible subset of G with respect to the lattices $H_1, \ldots, H_N$, with R_j, K_j, η_j, $j = 1, \ldots, N$ as in Definition 3.8. Let $M_j = x_j + H_j$, $j = 1, \ldots, N$ be such that if $N > 1$*

$$\langle z - x_j, \eta_j \rangle \neq 0 \quad \text{for} \quad z \in \bigcup_{k=1}^{j-1} M_k, \quad j = 2, \ldots, N. \tag{13}$$

Let $f \in L_2(G)$ be continuous and such that $\hat{f}$ vanishes a.e. outside K. Then there are continuous functions $f_j \in L_2(G)$ such that $\hat{f}_j$ vanishes outside K_j, and for all $x \in G$:

$$\begin{aligned} f_1(x) &= S_{M_1} f_1(x), \\ f_j(x) - S_{M_j} f_j(x) &= f_{j-1}(x) \left(1 - e^{2\pi i \langle x - x_j, \eta_j \rangle}\right), \quad j = 2, \ldots, N, \\ f_N(x) &= f(x). \end{aligned}$$

Using this recursion, the function f can be reconstructed from sampled values $f(z)$, $z \in \bigcup_{k=1}^{N} M_k$.

Proof: The proof is by induction on N. If $N = 1$, then $K = K_1 = R_1$ and $f = S_{M_1} f$ by Kluvánek's theorem. Hence f can be reconstructed from its samples on M_1. Now assume $N > 1$ and that the theorem holds with N replaced by $N-1$. Let $f_N = f$ and consider the function $g(x) = f_N(x) - S_{M_N} f_N(x)$. By Corollary 3.3 g is continuous, square-integrable, and vanishes on M_N. Since $\widehat{S_{M_N}} f$ vanishes outside $R_N \subseteq K$, $\hat{g}$ vanishes a.e. outside K. Since $K = R_N \cup (\eta_N + K_{N-1})$ and $K_{N-1} \subseteq R_N$, we can apply Lemma 3.4 to g, with R, K', x_0 and η' replaced by R_N, K_{N-1}, x_N, and η_N, respectively. Hence there is a continuous, square-integrable function f_{N-1} such that

$$g(x) = f_N(x) - S_{M_N} f_N(x) = f_{N-1}(x) \left(1 - e^{2\pi i \langle x - x_N, \eta_N \rangle}\right),$$

and $\widehat{f_{N-1}}$ vanishes a.e. outside K_{N-1}. Because of (13) the values

$$f_{N-1}(z) = \frac{f(z) - S_{M_N} f(z)}{1 - e^{2\pi i \langle z - x_N, \eta_N \rangle}}, \quad z \in \bigcup_{k=1}^{N-1} M_k,$$

can be computed. Now the hypothesis of the theorem is satisfied if f, K, and N are replaced by f_{N-1}, K_{N-1}, and $N-1$, respectively. By induction hypothesis the theorem holds in this case, yielding the functions f_j, $j = 1, \ldots, N-2$, and the reconstructed function $f_{N-1}(x)$ for all $x \in G$. Now f is reconstructed via

$$f(x) = \left(1 - e^{2\pi i \langle x - x_N, \eta_N \rangle}\right) f_{N-1}(x) + S_{M_N} f(x), \quad x \in G. \quad \square$$

The theorem establishes the following recursive algorithm for reconstruction of f from sampled values $f(z)$, $z \in \bigcup_{k=1}^{N} M_k$:

Algorithm 3.10 *:*

IF $N = 1$ *THEN* $f(x) = S_{M_1} f(x)$.

ELSE

Compute

$$g(z) = \frac{f(z) - S_{M_N} f(z)}{1 - e^{2\pi i \langle z - x_N, \eta_N \rangle}}, \quad z \in \bigcup_{k=1}^{N-1} M_k.$$

Invoke the algorithm to compute $g(x)$, $x \in G$ *from the computed values* $g(z)$, $z \in \bigcup_{k=1}^{N-1} M_k$.

$$f(x) = g(x)\left(1 - e^{2\pi i \langle x - x_N, \eta_N \rangle}\right) \; + \; S_{M_N} f(x), \quad x \in G.$$

END

Clearly, Theorem 3.9 also gives rise to explicit formulas generalizing the case $N = 2$ treated in Corollary 3.7, but as N increases these formulas seem to become too complicated to be useful. On the other hand, Algorithm 3.10 is very easy to program if the progamming language allows for recursive function calls; see, e.g., the MATLAB M-file bfmethod.m in the next section.

4 A numerical example

In this section we illustrate Theorem 3.9 and Algorithm 3.10 with an example implemented in MATLAB.

Let $G = \mathbb{Z}/(L\mathbb{Z}) = \{0, \ldots, L-1\}$ with addition modulo L. Then $\widehat{G} = \{\nu/L, \ \nu = 0, \ldots, L-1\}$ with addition modulo 1. Let m_G be the counting measure. According to Convention 2.3 $m_{\widehat{G}}$ equals $1/L$ times the counting measure. We characterize subgroups H of G by specifying an element $h \in G$ such that h divides L and generates H. Hence $H = \{hl, \ l = 0, \ldots, L/h - 1\}$. We will use the notation $H = \langle h \rangle$ indicating that H is generated by h. The annihilator $H^{\perp}$ equals $H^{\perp} = \{\nu/h, \ \nu = 0, \ldots, h-1\}$, and a fundamental domain is given by $R = \{\nu/L, \ \nu = 0, \ldots, L/h - 1\}$. Hence $m_{\widehat{G}}(R) = (L/h)/L = 1/h$.

The MATLAB code given below implements Algorithm 3.10 for this setting. The parameters are specified and explained in the driver routine bfdriver.m. This routine also generates the function to be reconstructed by randomly specifying its non-zero Fourier coefficients, cf. [6]. The recursive algorithm is implemented in the function M-file bfmethod.m. The function M-file SM.m computes $S_M f$. In order to keep the code readable the simplifying assumption was made that all fundamental domains R_j are of the form given above, i.e., $R_j = \{\nu/L, \ \nu = 0, \ldots, L/h_j - 1\}$. More general code is available from the authors.

In the code below the parameters to be specified by the user are set as follows: We specified $L = 2520$ which leads to a rich collection of subgroups. As the set K we consider the union of the two contiguous sets $\{0, \ldots, 71\}/L$

and $\{1224, \ldots, 1274\}/L$, giving a total of 123 points. Now choose $H_1 = \langle 280 \rangle$, $H_2 = \langle 60 \rangle$, and $H_3 = \langle 35 \rangle$, with $R_1 = \{0, \ldots, 8\}/L$, $R_2 = \{0, \ldots, 41\}/L$, and $R_3 = \{0, \ldots, 71\}/L$. With $\eta_2 = 42/L$ and $\eta_3 = 1224/L$ we have $K = R_3 \cup (\eta_3 + R_2) \cup ((\eta_2 + \eta_3) + R_1)$, and the sets $K_1 = R_1$, $K_2 = R_2 \cup (\eta_2 + R_1)$, and $K_3 = K = R_3 \cup (\eta_3 + K_2)$ satisfy the conditions of Definition 3.8. The shifts x_j have to be chosen such that the sampling conditions (13) are satisfied. Let $\langle \eta \rangle$ denote the subgroup of $H^\perp$ generated by $\eta \in H^\perp$. Then the annihilator $\langle \eta \rangle^\perp$ is a subgroup of G containing H. If $\langle \eta_j \rangle^\perp = H_j$, then the condition

$$\langle z - x_j, \eta_j \rangle \neq 0 \quad \text{for} \quad z \in \bigcup_{k=1}^{j-1} M_k,$$

reduces to the requirement that the coset $M_j = x_j + H_j$ does not intersect the union of the cosets M_k, $k = 1, \ldots, j-1$. Since in the present example we do have that $\langle \eta_j \rangle^\perp = H_j$, $j = 2, 3$, the sampling condition (13) is equivalent to the cosets M_1, M_2, M_3 being mutually disjoint. Two cosets $x_i + \langle h_i \rangle$ and $x_j + \langle h_j \rangle$ will intersect if and only if the difference $x_i - x_j$ is an integer multiple of the greatest common divisor of h_i and h_j. Hence the conditions (13) require in this particular example that $x_1 - x_2$ should not be a multiple of 20, $x_1 - x_3$ should not be a multiple of 35, and $x_2 - x_3$ should not be a multiple of 5. An admissible choice is, e.g., $x_1 = 3$, $x_2 = 1$, and $x_3 = 0$. The relative errors in our numerical tests varied with the random signal, but stayed below $1.e - 12$. In order to assess the stability of the algorithm we computed as a comparison the relative error resulting from taking the FFT of the signal f and then reconstructing by an inverse FFT. The relative error resulting from this very stable procedure was about $2.e - 13$, indicating that our algorithm is stable in this case. This indication was confirmed by tests where noise was added to the signal.

Each sampled value yields a linear equation for the unknown Fourier coefficients of f. The condition number of this linear system is an indication if the problem of reconstructing f from its samples is well conditioned or not, independent of the algorithm used to perform the reconstruction. In this case the condition number is about 40, so the problem is fairly well conditioned.

The sampling set above has minimal density in the sense that there are as many sampling points as there are points in the set K, i.e., 123. We may define the Nyquist distance as the ratio between the size L of the group G and the size of the spectrum K. The average spacing of our sampling set is equal to the Nyquist distance, which equals $2520/123 \simeq 20.49$. This can be compared to the optimal regular sampling distance, i.e., the spacing of the smallest subgroup H such that the set K is a subset of a fundamental domain of $H^\perp$ and Theorem 3.2 can be applied. For this example the smallest feasible subgroup has 280 elements and a spacing of 9.

We also used the standard frame method for irregular sampling as described by Feichtinger and Gröchenig [6] to solve the present example. Even with a near optimal relaxation parameter convergence was very slow, requiring thousands of iterations to obtain the same accuracy which our method achieved quickly. The reason is that iterative methods for general irregular sampling sets may not work well if the sampling set has gaps larger than the Nyquist distance, see [6]. In this example the largest gap is equal to the spacing of the largest subgroup, i.e., 35, hence significantly larger than the Nyquist distance. Our method can deal with such gaps because it makes explicit use of the specific structure of the sampling set.

The sampling set of our example is periodic with period 840 and can be obtained as a union of 41 cosets of the group $H = \langle 840 \rangle$. Hence the algorithm described in [5] could be used. However, this algorithm is most efficient and convenient when the number of cosets is much smaller than the number of elements in H, while the opposite is true in this example, since $H = \langle 840 \rangle$ has only 3 elements. We can construct a non-periodic example by replacing H_1 above by $H_1 = \langle\ 360\ \rangle$ and reducing the set K by two points by replacing $\{1224, \dots, 1274\}/L$ with $\{1224, \dots, 1272\}/L$. The fundamental domain R_1 now equals $\{0, \dots, 6\}/L$. Again $x_1 = 3$, $x_2 = 1$, and $x_3 = 0$ is an admissible choice of shifts. The resulting sampling set and spectrum K now both contain 121 points. Since 121 and $L = 2520$ are mutually prime the sampling set has no period smaller than L. The relative errors in our tests where larger than in the previous example and came out between 1.e-12 and 1.e-11. This is explained by the fact that the condition number of the linear system for the unknown Fourier coefficients is now about 487, i.e., the problem itself is more ill-conditioned than before.

One can also reconstruct f in this setting by solving the linear system for the unkown Fourier coefficients directly, and then find f by an inverse FFT. When compared with a direct solution of the linear system by means of a generic solver (MATLAB's \ command), our algorithm achieves the same accuracy but is usually faster. As implemented here, its complexity is dominated by the $2N$ Fast Fourier Transforms of length up to L, hence of order $O(NL \log L)$. This compares to $O(P^3) + O(L \log L)$ for the direct solution of the linear system followed by an inverse FFT. Here P is the number of points in the spectrum K. In the examples above our algorithm is only about three times faster, since P is small compared to L. For larger P this advantage becomes more pronounced.

```
% bfdriver.m : Driver for nonperiodic sampling on the group
%               G = Z_L = {0,...,L-1} with addition modulo L.
% Explanation of input parameters:
% L:    number elements in G
% h:    h(k) is the divisor of L which generates the subgroup
%       H_k, i.e., H_k = {0,h(k),2h(k),...,L-h(k)}
% x:    vector with shifts. M_k = x(k) + H_k
% eta:  eta(k) corresponds to eta_{k+1} in Theorem 3.9.
% filt: characteristic function of spectrum. filt(k)=1 if
%       the point (k-1)/L lies in the spectrum. Otherwise
%       filt(k)=0.

% Input parameters:
L=2520;                                  % Length of group G.
filt = zeros(1,L);                       % DO NOT CHANGE!
filt(1:72) = 1; filt(1225:1275)=1;       % Set filt(k)=1 if the
                                         % point (k-1)/L lies
                                         % in the spectrum

h=[280,60,35];                           % Specify subgroups
x=[3,1,0];                               % Specify shifts
eta = [42,1224]/L;                       % Specify eta_{k+1}
% End of input section

% Compute signal to be sampled and reconstructed
N = max(size(h));                        % Number of subgroups
fhat = complex(rand(1,L),rand(1,L));     % Generate random spectrum
fhat = fhat.*filt;                       % Set frequencies outside
                                         % of spectrum to zero
fexact = ifft(fhat);
fexact = fexact/norm(fexact);            % Normalize signal
```

```
% Compute sampled values
f = zeros(1,L);
for k=1:N
 Mk = x(k)+[0:h(k):L-h(k)];           % Coset M_k = x(k) + H_k
 f(1+Mk) = fexact(1+Mk);              % Sampled values on M_k
end

% Reconstruct signal
F = bfmethod(f,L,h,eta,x);

% Compute the l2 relative reconstruction error
relerr = norm(fexact-F)
%----------------------------------------


function F=bfmethod(f,L,h,eta,x)

N = max(size(h));
MN = x(N) + [0:h(N):L-h(N)];          % Coset M_N = x(N) + H_N
fH = f(1+MN);                         % Sampled values on M_N
SMf = SM(fH,L,h(N),x(N));
if N==1
  F = SMf;
else
  tmp = 1-exp(2*pi*i*([0:L-1]-x(N))*eta(N-1));
  tmp1=tmp;
  tmp1(find(abs(tmp < 1.e-14)))=1;   % Avoid zero divisions
  f1 = (f-SMf)./tmp1;
  fN1 = bfmethod(f1,L,h(1:N-1),eta(1:N-2),x(1:N-1));
  F = fN1.*tmp + SMf;
end
%----------------------------------------

function S = SM(f,L,h,x)
% Computes S_Mf(z) for z in G (cf. equation (6))
% G = {0,1,...,L-1} with addition mod L, M = x + H
```

```
% H = {0,h,2h,...,L-h}, R = {0,...,L/h-1}/L
% f = row vector of length L/h containing sampled values on M.
% x = shift. Need x in {0,...,h-1}

chi = zeros(1,L);
chi(1:L/h) = fft(f);
S = h*ifft(chi);
if x > 0
  tmp = S(L-x+1:L);
  S(x+1:L)=S(1:L-x);
  S(1:x) = tmp;
end
%---------------------------------------
```

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