Construction of Sampling Theorems for Unions of Shifted Lattices

Hamid Behmard
Mathematics Department, Western Oregon University,
Monmouth, Oregon 97361
behmarh@wou.edu

Adel Faridani
Department of Mathematics, Oregon State University,
Corvallis, Oregon 97331
faridani@math.oregonstate.edu

David Walnut
Department of Mathematical Sciences, George Mason University,
Fairfax, Virginia 22030-4444
dwalnut@gmu.edu

Abstract

The classical sampling theorem permits reconstruction of a bandlimited function f from its values on a shifted lattice. This work considers sampling sets which are unions of possibly different shifted lattices, using the following basic approach. Assume the Fourier transform of a function f vanishes outside a set K. Let K admit a disjoint decomposition $K = K_0 \cup K_1$ with a corresponding decomposition of f, $f(x) = f_0(x) + P(x)f_1(x)$, such that the Fourier transform of f_i vanishes outside K_i, $i = 0, 1$, and P is known. Let M_0, M_1 be sampling sets such that f_i can be reconstructed from its samples on M_i and P vanishes everywhere on M_0 but nowhere on M_1. Then f can be reconstructed from its values on $M_0 \cup M_1$. Two methods to construct such decompositions are given, subject to K satisfying certain compatibility conditions. It is demonstrated how the decompositions can be used to construct sampling theorems or recursive reconstruction algorithms. Several examples and a numerical implementation in two dimensions are presented.

Key words and phrases: Shannon sampling, multidimensional sampling, multichannel sampling, multirate sampling, nonuniform sampling, periodic sampling, non-equidistant sampling, non-periodic sampling, irregular sampling, locally compact abelian groups.

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1 Introduction

The classical sampling theorem permits reconstruction of a bandlimited function from its values on a set of equidistant points on the real line $\mathbb{R}$ [11, 14]. It is readily generalized to higher dimensions with the sampling set being a coset (shifted copy) of a lattice. Periodic sampling, introduced by Kohlenberg [9], is a further extension and considers sampling sets which are unions of cosets of a single lattice; see, e.g., [3, 5, 10, 15, 16]. Such sets are periodic in the sense of being invariant with regard to translations by an element of the lattice. In this paper we continue work begun in [1, 12, 13] and present an approach for finding sampling theorems for sampling sets which are unions of cosets of possibly different lattices. Such sampling sets are not necessarily periodic.

The basic ideas underlying the approach presented here are as follows. Assume that the Fourier transform $\hat{f}$ of a function f vanishes outside a certain set K. We call K the bandregion of f. Assume K has the form $K = K_0 \cup K_1$ with K_0, K_1 disjoint and that we have sampling sets M_0 and M_1 such that functions whose Fourier transform vanishes outside K_i can be reconstructed from their samples on M_i, $i = 0, 1$, respectively. We seek decompositions of the form

$$f(x) = f_0(x) + P(x) f_1(x) \tag{1}$$

where $P(x)$ is a known function vanishing everywhere on M_0 but vanishing nowhere on M_1, and $\widehat{f_0}$, $\widehat{f_1}$ are known to vanish outside K_0 and K_1, respectively. Such a decomposition can be used in the following way to reconstruct f from its samples on $M_0 \cup M_1$. Since $P(x)$ vanishes on M_0, $f(x) = f_0(x)$ on M_0, and so f_0 can be reconstructed from the samples of f on M_0. Since $P(x) \neq 0$ on M_1, we have $f_1(x) = (f(x) - f_0(x)) / P(x)$ for $x \in M_1$. Hence we can find the samples of f_1 on M_1 and use these to reconstruct f_1 everywhere. Then the function f itself can be found from (1).

In the present paper this approach will be applied in two different ways. The first arises from choosing $M_0 = x_0 + H$, with H a lattice, and $K_0 \subseteq R$, with R a fundamental domain of the reciprocal lattice $H^\perp$. In this case K_1 needs to satisfy the compatibility conditions $K_1 - \eta' \subseteq K = K_0 \cup K_1$, and $K_1 \subseteq \bigcup_{j=1}^{N-1} (j\eta' + R)$ for some non-zero element η' of $H^\perp$ and some positive integer N. Here $j\eta'$ is defined inductively by letting $0\,\eta' = 0$ and $j\eta' = (j-1)\eta' + \eta'$ for positive integers j. It will always be assumed that the $j\eta'$ are distinct elements of $H^\perp$ for $j = 0, \ldots, N-1$.

This generalizes the situation considered in [1] where more restrictive conditions were needed. It was shown in [1] how to apply this decomposition recursively to more complicated sets and we show that the recursive reconstruction algorithm carries over to the generalization presented here.

In the second application a decomposition for the case of a set K of the form $K = K_0 \cup K_1$ with $K_0 = \bigcup_{j=1}^{m-1} (R + \eta_j)$, and $K_1 \subseteq R$ is given, where $0 \neq \eta_j \in H^\perp$

and R is again a fundamental domain of $H^\perp$. Here the non-zero elements η_j need not be multiples of one element. The idea is to first use periodic sampling with M_0 being a union of m cosets of the lattice H to "strip off" the part K_0 and then to choose M_1 to be a coset of a possibly different lattice in order to deal with K_1.

The paper is organized as follows. In the next section we review some standard definitions and facts from Fourier analysis. We use a general notation which encompasses a wide range of settings. In §3 we prove the two basic decompositions and then show examples and applications in §4. Besides extending the range of applicability of the recursive reconstruction algorithm of [1], we present a two-dimensional example for finding sampling sets of minimum density when the bandregion K is a rectangle with two tabs (Example 4.4), and obtain a complete answer for sampling functions of one variable when K is an interval (Example 4.6 combined with Algorithm 4.8). The final section is devoted to a numerical implementation of Algorithm 4.8 in two dimensions using MATLAB.

2 Standard definitions and facts

The Fourier transform is defined in many different settings, and we will use a general notation which applies to a large number of these settings. Let $\mathbb{Z}, \mathbb{R}$ denote the integers and real numbers, respectively. Let G be the domain of the function f. For example, G could be $\mathbb{R}$ or $\mathbb{R}^n$. If f is a periodic function of n variables we can choose $G = [0,1)^n = \mathbb{T}^n$, where we use the interval $[0,1)$ with addition modulo 1 as a model for the circle group $\mathbb{T}$. If f is a function of a discrete variable, then $G = \mathbb{Z}$, or in the case of the discrete Fourier transform we have $G = \mathbb{Z}_L$, that is $G = \{0, \ldots, L-1\}$ with addition modulo L. In the general case G is a locally compact abelian group [2, 3, 8]. For each of these domains integration is defined using a translation invariant measure m_G, the so-called Haar measure on G, which is unique up to normalization by a multiplicative constant. $L_p(G)$ denotes the space of all functions on G such that $\| f \|_p = (\int_G |f(x)|^p \, dm_G(x))^{1/p}$ is finite.

The Fourier transform of a function $f \in L_1(G)$ is the continuous function $\hat{f}$ defined by

$$\hat{f}(\xi) = \int_G f(x) e^{-2\pi i \langle x, \xi \rangle} \, dm_G(x), \tag{2}$$

where ξ is an element of the corresponding Fourier space $\widehat{G}$. Some examples of the measures and meaning of $\langle x, \xi \rangle$ for different G are given in the following table, where dx denotes the Lebesgue measure on $\mathbb{R}^n$.

G	$\widehat{G}$	$\int_G f(x)\,dm_G(x)$	$\langle x,\xi\rangle,$	$x\in G,\ \xi\in\widehat{G}$
$\mathbb{R}$	$\mathbb{R}$	$\int_{\mathbb{R}} f(x)\,dx$	$x\xi,$	$x,\xi\in\mathbb{R}$
$\mathbb{R}^n$	$\mathbb{R}^n$	$\int_{\mathbb{R}^n} f(x)\,dx$	$\sum_{i=1}^n x_i\xi_i,$	$x,\xi\in\mathbb{R}^n$
$\mathbb{T}$	$\mathbb{Z}$	$\int_0^1 f(x)\,dx$	$xk,$	$x\in[0,1),\ k\in\mathbb{Z}$
$\mathbb{Z}$	$\mathbb{T}$	$\sum_{l\in\mathbb{Z}} f(l)$	$lt,$	$l\in\mathbb{Z},\ t\in[0,1)$

Throughout this paper we will assume that the measure m_G is given and then normalize the Haar measure on $\widehat{G}$ such that the Fourier inversion formula holds in the form (3) given below.

Theorem 2.1 *(Fourier inversion formula) If $f\in L_1(G)$ is continuous and $\hat{f}\in L_1(\widehat{G})$, then*

$$f(x)=\int_{\widehat{G}}\hat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}\,dm_{\widehat{G}}(\xi)=(\hat{f})^\wedge(-x). \tag{3}$$

The Fourier transform can be extended to a linear isomorphism of $L_2(G)$ onto $L_2(\widehat{G})$ by means of the Plancherel Theorem (cf. [6, Sec. 31.18]).

Definition 2.2 *1. Let H be a closed subgroup of G. The annihilator of H is the closed subgroup $H^\perp$ of $\widehat{G}$ given by*

$$H^\perp=\{\eta\in\widehat{G}:\ e^{2\pi i\langle y,\eta\rangle}=1 \text{ for all } y\in H\}.$$

2. *A closed discrete subgroup H of G such that $H^\perp$ is also discrete is called a* lattice. *$H^\perp$ is sometimes called the reciprocal lattice.*

3. *A measurable subset R of $\widehat{G}$ such that every $\xi\in\widehat{G}$ can be uniquely written as $\xi=\rho+\eta$, where $\rho\in R$ and $\eta\in H^\perp$ is called a* fundamental domain *of $H^\perp$.*

4. *For H a lattice and R a fundamental domain of $H^\perp$ we define a function $\varphi_R: G\to\mathbb{C}$ by*

$$\varphi_R(x)=\frac{1}{m_{\widehat{G}}(R)}\int_R e^{2\pi i\langle x,\xi\rangle}\,dm_{\widehat{G}}(\xi),\quad x\in G, \tag{4}$$

where

$$m_{\widehat{G}}(R)=\int_R dm_{\widehat{G}}(\xi).$$

Remark 2.3 *Let χ_V denote the indicator function of a set V, that is $\chi_V(x)=1$ if $x\in V$ and $\chi_V(x)=0$ otherwise. Note that (4) implies that*

$$\widehat{\varphi_R}(\xi)=\frac{1}{m_{\widehat{G}}(R)}\,\chi_R(\xi). \tag{5}$$

Furthermore it was shown in [8] that φ_R *is continuous on* G *and satisfies* $\varphi_R(0) = 1$, $\varphi_R(y) = 0$ *for* $0 \neq y \in H$, $\| \varphi_R \|_2 = 1/\sqrt{m_{\widehat{G}}(R)}$, *as well as the orthogonality relation* $\int_G \varphi_R(x)\, \overline{\varphi_R(x-y)}\, dm_G(x) = 0$ *for* $0 \neq y \in H$.

Note also that in the case of $G = \widehat{G}$ *the real line and* R *an interval we obtain the familiar sinus cardinalis: With* $G = \mathbb{R}$, $H = \frac{1}{2b}\mathbb{Z}$, $H^\perp = 2b\mathbb{Z}$, *and* $R = [-b, b)$, *we have* $m_{\widehat{G}}(R) = \int_{-b}^{b} dx = 2b$, *and* $\varphi_R(x) = \frac{1}{2b}\int_{-b}^{b} e^{2\pi i x\xi}\, d\xi = sinc(2bx)$, *where* $sinc(t) = \frac{sin(\pi t)}{\pi t}$.

The classical sampling theorem now reads as follows [8].

Theorem 2.4 *Let* H *be a lattice and* R *a fundamental domain of* $H^\perp$. *Suppose* $f \in L_2(G)$ *and* $\hat{f}(\xi) = 0$ *for almost all* $\xi \notin R$. *Then* f *is equal almost everywhere to a continuous function. If* f *itself is continuous, then*

$$f(x) = \sum_{y \in H} f(y)\varphi_R(x-y) \tag{6}$$

uniformly on G *and in the sense of convergence in* $L_2(G)$. *Furthermore, the* L_2*-norm of* f *is given by*

$$\| f \|_2^2 = \frac{1}{m_{\widehat{G}}(R)} \sum_{y \in H} |f(y)|^2.$$

We would like to apply the formula (6) to functions whose Fourier transform is supported in a set K larger than R. The following corollary to the classical sampling theorem deals with this case.

Corollary 2.5 *Let* H *be a lattice and* R *a fundamental domain of* $H^\perp$. *Let* $f \in L_2(G)$ *be continuous and* $\hat{f}(\xi) = 0$ *a.e. outside a measurable subset* K *of* $\widehat{G}$. *Assume that there is* $P < \infty$ *such that* $K \subseteq \bigcup_{j=1}^{P}(\eta_j + R)$ *with* $\eta_1, \ldots, \eta_P$ *distinct elements of* $H^\perp$. *Let* $M = x_0 + H$ *be a coset of* H. *Then the function* $S_M f$ *defined by*

$$S_M f(x) = \sum_{y \in H} f(x_0 + y)\, \varphi_R(x - x_0 - y) \tag{7}$$

is continuous and square integrable on G, *and satisfies* $S_M f(z) = f(z)$ *for all* $z \in M$.

For a proof of this corollary see [1].

3 Two decompositions

As stated in the introduction, our first decomposition applies to sets K of the form $K = K_0 \cup K_1$ with $K_0 \subseteq R$, $K_1 - \eta' \subset K$, and $K_1 \subseteq \bigcup_{j=1}^{N-1}(j\eta' + R)$ for some non-zero element η' of $H^\perp$ and some positive integer N. As before, H is a lattice, R a fundamental domain of $H^\perp$, and $j\eta'$ is defined inductively by letting $0\,\eta' = 0$ and $j\eta' = (j-1)\eta' + \eta'$ for positive integers j. We will always assume that the $j\eta'$ are distinct elements of $H^\perp$ for $j = 0, \ldots, N-1$. In particular these conditions imply that K_0 and K_1 are disjoint. To fix ideas let us consider a simple example.

Example 3.1 *Let $G = \mathbb{R} = \widehat{G}$, $H = \mathbb{Z} = H^\perp$, $R = [0, 1)$, and $K = [0, 3)$. We decompose K as $K = K_0 \cup K_1$ with $K_0 = [0, 1)$ and $K_1 = [1, 3)$. With $\eta' = 1$ we see that K_1 satisfies the conditions $K_1 - \eta' = K_1 - 1 = [0, 2) \subset K$, and $K_1 \subseteq [1, 2) \cup [2, 3) = \bigcup_{j=1}^{2}(j\eta' + R)$.*

We now give an equivalent statement of the structure of the set K, describing it in terms of the sets $\tilde{K}_j = K \cap (j\eta' + R)$. Consider the conditions

$$\begin{aligned} &K = \bigcup_{j=0}^{N-1} \tilde{K}_j \text{ with } \tilde{K}_j \text{ measurable such that} \\ &\tilde{K}_0 \subseteq R \text{ and } \tilde{K}_j \subseteq \tilde{K}_{j-1} + \eta' \text{ for } j = 1, \ldots, N-1, \\ &j\eta' \text{ distinct for } j = 0, \ldots, N-1. \end{aligned} \tag{8}$$

Letting $K_0 = \tilde{K}_0$ and $K_1 = \bigcup_{j=1}^{N-1} \tilde{K}_j$ one verifies that this is equivalent to the original conditions given in the first paragraph of this section. Note that the conditions (8) imply that $\tilde{K}_j \subseteq R + j\eta'$, and that the $\tilde{K}_j$ are mutually disjoint since the $j\eta'$ are distinct elements of $H^\perp$.

Theorem 3.2 *Let H be a lattice, R a fundamental domain of $H^\perp$, $0 \neq \eta' \in H^\perp$ and $K \subset \widehat{G}$ such that the conditions (8) hold. Assume that $f \in L_2(G)$ is continuous, and that $\hat{f}$ vanishes a.e. outside K. Then*

$$f(x) = f_0(x) + \left(e^{-2\pi i\langle x, \eta'\rangle} - 1\right) f_1(x) \tag{9}$$

with $f_0, f_1 \in L_2(G)$ continuous, $\widehat{f_0}(\xi)$ vanishing outside $K_0 = \tilde{K}_0$, and $\widehat{f_1}(\xi)$ vanishing outside $K_1 = \bigcup_{j=1}^{N-1} \tilde{K}_j$.

Proof. If $N = 1$ then (9) holds trivially with $f_0 = f$ and f_1 vanishing everywhere. Assume in the following that $N > 1$. We define f_0 and f_1 on the Fourier transform side as follows.

$$\widehat{f_1}(\xi) = \begin{cases} -\sum_{k=0}^{N-1-j} \hat{f}(\xi + k\eta'), & \text{for } \xi \in \tilde{K}_j, \ \ j = 1, \ldots, N-1 \\ 0 & \text{for } \xi \notin K_1. \end{cases} \tag{10}$$

For $\xi \in \tilde{K}_0$ let

$$\widehat{f_0}(\xi) = \hat{f}(\xi) - \widehat{f_1}(\xi + \eta'),$$

and let $\widehat{f_0}(\xi) = 0$ otherwise.

We now establish (9) on the Fourier transform side by showing that

$$\hat{f}(\xi) = \widehat{f_0}(\xi) + \widehat{f_1}(\xi + \eta') - \widehat{f_1}(\xi). \tag{11}$$

First consider the case $\xi \notin K$. Then $\hat{f}(\xi) = \widehat{f_0}(\xi) = \widehat{f_1}(\xi) = 0$. Since $\xi + \eta' \notin K + \eta'$ and by (8)

$$K + \eta' = \bigcup_{j=0}^{N-1} \tilde{K}_j + \eta' \supseteq \bigcup_{j=0}^{N-2} \tilde{K}_j + \eta' \supseteq \bigcup_{j=0}^{N-2} \tilde{K}_{j+1} = K_1,$$

we have that $\xi + \eta' \notin K_1$ so that $\widehat{f_1}(\xi + \eta')$ also vanishes and (11) holds.

If $\xi \in \tilde{K}_0$, then $\widehat{f_1}(\xi) = 0$ and (11) holds by definition of $\widehat{f_0}$. If $\xi \in \tilde{K}_{N-1}$ then $\xi + \eta' \notin K_1$ since the $j\eta'$ are distinct elements. Hence $\widehat{f_1}(\xi + \eta') = 0$, $\widehat{f_0}(\xi) = 0$, and $\widehat{f_1}(\xi) = -\hat{f}(\xi)$ by (10), so that (11) holds. Finally, if $\xi \in \tilde{K}_j$ for $j = 1, \ldots, N-2$ then $\widehat{f_0}(\xi) = 0$ and (8) implies that either $\xi + \eta' \in \tilde{K}_{j+1}$ or $\xi + \eta' \notin K$. If $\xi + \eta' \in \tilde{K}_{j+1}$ then (10) gives

$$\widehat{f_1}(\xi + \eta') = -\sum_{k=0}^{N-1-(j+1)} \hat{f}(\xi + \eta' + k\eta') = -\sum_{k=1}^{N-1-j} \hat{f}(\xi + k\eta') = \widehat{f_1}(\xi) + \hat{f}(\xi),$$

which is what is needed. If $\xi + \eta' \notin K$, then it follows from the condition $\tilde{K}_j \subseteq \eta' + \tilde{K}_{j-1}$ that $\xi + k\eta' \notin K$ for $k = 1, \ldots, N-1-j$, so that $\widehat{f_1}(\xi) = -\hat{f}(\xi)$. Since in this case both $\widehat{f_0}(\xi)$ and $\widehat{f_1}(\xi + \eta')$ vanish, (11) holds. ∎

Our second decomposition applies to bandregions of the form $K = K_0 \cup K_1$ with $K_0 = \bigcup_{j=1}^{m-1}(R + \eta_j)$, $K_1 \subseteq R$, and η_j distinct non-zero elements of $H^\perp$. In particular, the η_j need not be multiples of one element as was the case with the $j\eta'$ in our first decomposition.

Theorem 3.3 *Let H be a lattice, R a fundamental domain of $H^\perp$, $K = K_0 \cup K_1$ with $K_0 = \bigcup_{j=1}^{m-1}(R + \eta_j)$, $K_1 \subseteq R$, $\eta_1, \ldots, \eta_{m-1}$ distinct non-zero elements of $H^\perp$, and*

$$P(x) = 1 + \sum_{j=1}^{m-1} c_j \, e^{2\pi i \langle x, \eta_j \rangle}. \tag{12}$$

If $f \in L_2(G)$ is continuous and $\hat{f}$ vanishes a.e. outside K then

$$f(x) = f_0(x) + P(x) f_1(x) \tag{13}$$

with $f_i \in L_2(G)$ continuous, and supp$(\widehat{f_i}) \subseteq K_i$, $i = 0, 1$.

Proof. Let f_1 be given on the Fourier transform side by $\widehat{f_1}(\xi) = \chi_{K_1}(\xi)\hat{f}(\xi)$. Recall that χ_{K_1} denotes the indicator function of K_1. Let $g(x) = f_0(x) + P(x)f_1(x)$. Then

$$\hat{g}(\xi) = \widehat{f_0}(\xi) + \widehat{f_1}(\xi) + \sum_{j=1}^{m-1} c_j \widehat{f_1}(\xi - \eta_j).$$

The third term is supported in $\bigcup_{j=1}^{m-1}(K_1+\eta_j) \subseteq \bigcup_{j=1}^{m-1}(R+\eta_j) = K_0$. Therefore, define

$$\widehat{f_0}(\xi) = \hat{f}(\xi) - \sum_{j=1}^{m-1} c_j \widehat{f_1}(\xi - \eta_j), \quad \xi \in \bigcup_{j=1}^{m-1}(R+\eta_j),$$

and $\widehat{f_0}(\xi) = 0$ otherwise. Then $\hat{g} = \hat{f}$ and f_0, f_1 have the desired properties. ∎

4 Examples and applications.

We begin with an application of our second decomposition. In order to apply Theorem 3.3 to the construction of sampling theorems, we let M_0 be a periodic sampling set, that is $M_0 = \bigcup_{n=1}^{m-1}(x_n+H)$ with the x_n chosen such that functions whose Fourier transform is supported in K_0 can be reconstructed from their samples on M_0 according to the method given in [3]. We further assume that the c_j in (12) can be chosen such that

$$P(x_n) = 0, \quad n = 1, \ldots, m-1. \tag{14}$$

Since it follows from (12) that P is constant on cosets of H, the equations (14) imply that $P(x)$ vanishes on M_0. Hence $f(x) = f_0(x)$ on M_0, so that f_0 can be found from samples of f on M_0. Let M_1 be a sampling set permitting reconstruction of functions whose Fourier transform vanishes outside K_1. If in addition $P(x) \neq 0$ on M_1, then for $x \in M_1$, $f_1(x) = (f(x) - f_0(x))/P(x)$. Hence we can find the samples of f_1 on M_1 and use these to reconstruct f_1 everywhere. Then the function f itself can be found from (13).

Example 4.1 *In Theorem 3.3 let $m = 3$ and $\eta_2 = -\eta_1$, so that the bandregion K has the form*

$$K = (R - \eta_1) \cup K_1 \cup (R + \eta_1), \quad K_1 \subseteq R, \quad \eta_1 \in H^\perp, \quad \eta_1 \neq -\eta_1.$$

According to (12) $P(x)$ has the form

$$P(x) = 1 + c_1\, e^{2\pi i\langle x,\eta_1\rangle} + c_2\, e^{-2\pi i\langle x,\eta_1\rangle},$$

and the equations (14) yield

$$P(x) = \left(1 + e^{2\pi i\langle x_1 - x_2,\eta_1\rangle}\right)^{-1} \left(1 - e^{-2\pi i\langle x - x_1,\eta_1\rangle}\right) \left(1 - e^{2\pi i\langle x - x_2,\eta_1\rangle}\right),$$

provided that x_1, x_2 are chosen such that

$$1 + e^{2\pi i\langle x_1 - x_2, \eta_1\rangle} \neq 0. \tag{15}$$

To be specific, let us consider $G = \mathbb{R}$, $0 < a \leq b$, $H = \frac{1}{2b}\mathbb{Z}$, $H^\perp = 2b\mathbb{Z}$, $R = [-b, b)$, $K_1 = [-a, a)$, and $\eta_1 = 2b$. Then $K = [-3b, -b) \cup [-a, a) \cup [b, 3b)$. If $a < b$ then K is the interval $[-3b, 3b)$ with two gaps at $[-b, -a)$ and $[a, b)$. The classical sampling theorem would require sampling with a set at least as dense as the lattice $\frac{1}{6b}\mathbb{Z}$, which is suboptimal. Our goal is to sample with a set $M_0 \cup M_1$ of minimal density, by letting $M_0 = \bigcup_{n=1}^{2}(x_n + H)$ and $M_1 = x_3 + \tilde{H}$, with $\tilde{H} = \frac{1}{2a}\mathbb{Z}$. The condition (15) now requires that $e^{2\pi i\langle x_1 - x_2, \eta_1\rangle} = e^{2\pi i(x_1 - x_2)2b} \neq -1$, i.e., $4b(x_1 - x_2)$ may not be an odd integer. However, the property that M_0 is a suitable sampling set for functions with bandregion $K_0 = [-3b, -b) \cup [b, 3b)$ requires according to [3, Theorem 3.5] the sharper condition $4b(x_1 - x_2) \notin \mathbb{Z}$. Once we have found f_0 in this way, we need to find x_3 such that $P(x) \neq 0$ for all $x \in x_3 + \tilde{H}$. This gives the additional requirement

$$2b(x_3 - x_i) + nb/a \notin \mathbb{Z}, \quad \text{for all } n \in \mathbb{Z},\ i = 1, 2.$$

We now show that our first decomposition, Theorem 3.2, leads to a larger range of validity of the theory and algorithms developed in [1]. The key is to use Corollary 4.2 below in place of [1, Lemma 2].

Corollary 4.2 *Let H be a lattice and R a fundamental domain of $H^\perp$. Let $K = K_0 \cup (\eta' + K')$ with K_0, K' measurable sets such that $K_0 \subseteq R$, $K' \subset K$, $K' \subseteq \bigcup_{j=0}^{N-2}(j\eta' + R)$, and $0 \neq \eta' \in H^\perp$ such that $j\eta'$ are distinct elements of $H^\perp$ for $j = 0, \ldots, N-1$. Assume that $f \in L_2(G)$ is continuous, vanishes on the coset $x_0 + H$, and that $\hat{f}$ vanishes a.e. outside K. Then*

$$f(x) = h(x)\left(1 - e^{2\pi i\langle x - x_0, \eta'\rangle}\right) \tag{16}$$

with $h \in L_2(G)$ continuous and $\hat{h}$ vanishing a.e. outside K'.

Proof: We first show that K satisfies the hypothesis of Theorem 3.2. It follows immediately from the hypothesis that

$$\begin{aligned} K &= K_0 \cup (\eta' + K') \\ &\subseteq R \cup \left(\eta' + \bigcup_{j=0}^{N-2}(j\eta' + R)\right) = \bigcup_{j=0}^{N-1}(j\eta' + R). \end{aligned}$$

Therefore $K = \bigcup_{j=0}^{N-1}\tilde{K}_j$ with $\tilde{K}_j = K \cap (j\eta' + R)$, $j = 0, \ldots, N-1$.

Since $K' \subset K$ we have $K' \cap (j\eta' + R) \subseteq \tilde{K}_j$ for $j = 0, \ldots, N-1$. For $j = 1, \ldots, N-1$ we have $K_0 \cap (j\eta' + R) = \emptyset$ so that

$$\tilde{K}_j = (\eta' + K') \cap (j\eta' + R) = \eta' + (K' \cap [(j-1)\eta' + R]) \subseteq \eta' + \tilde{K}_{j-1}.$$

Hence K has the structure (8) required in Theorem 3.2. Furthermore the relations

$$\begin{aligned} \eta' + K' &= \bigcup_{j=1}^{N-1} \tilde{K}_j && (17) \\ &\subseteq \bigcup_{j=1}^{N-1} (j\eta' + R) && (18) \end{aligned}$$

hold. Since the $j\eta'$, $j = 0, \ldots, N-1$ are distinct, (18) implies that $R \cap (\eta' + K') = \emptyset$, so that $\tilde{K}_0 = K \cap R = K_0$. Applying Theorem 3.2 to the function $g(x) = f(x + x_0)$ now yields

$$g(x) = g_0(x) + \left(e^{-2\pi i\langle x, \eta'\rangle} - 1\right) g_1(x) \tag{19}$$

with $\widehat{g_0}$ vanishing outside K_0 and $\widehat{g_1}$ vanishing outside $K_1 = \bigcup_{j=1}^{N-1} \tilde{K}_j = \eta' + K'$, cf. (17).

For $x \in H$ equation (19) gives that $g(x) = g_0(x)$. Since by hypothesis g vanishes on H, g_0 must vanish identically by the classical sampling theorem (Theorem 2.4). Now let $h(x) = e^{-2\pi i\langle x - x_0, \eta'\rangle} g_1(x - x_0)$. With $f(x) = g(x - x_0)$ it now follows that equation (16) holds and we see that $\hat{h}(\xi) = e^{-2\pi i\langle x_0, \xi\rangle}\, \widehat{g_1}(\xi + \eta')$ vanishes outside

$$K_1 - \eta' = (\bigcup_{j=1}^{N-1} \tilde{K}_j) - \eta' = (\eta' + K') - \eta' = K'.$$

■

Next, Corollary 4.2 can be used in the following way to reduce the problem of reconstructing f to the problem of reconstructing h.

Theorem 4.3 *Let H be a lattice and R a fundamental domain of $H^\perp$. Let $K = K_0 \cup (\eta' + K')$ with K_0, K' measurable sets such that $K_0 \subseteq R$, $K' \subset K$, $K' \subseteq \bigcup_{j=0}^{N-2} (j\eta' + R)$, and $0 \neq \eta' \in H^\perp$ such that $j\eta'$ are distinct elements of $H^\perp$ for $j = 0, \ldots, N-1$. Assume that $f \in L_2(G)$ is continuous, and that $\hat{f}$ vanishes a.e. outside K. Let $M' \subset G$ be such that continuous functions $h \in L_2(G)$ whose Fourier transform vanishes a.e. outside K' can be reconstructed from their samples $h(z')$, $z' \in M'$. Let x_0 be such that*

$$e^{2\pi i\langle z' - x_0, \eta'\rangle} \neq 1 \text{ for all } z' \in M'. \tag{20}$$

Then f can be reconstructed from its samples $f(z)$, $z \in M \cup M'$, where $M = x_0 + H$.

Proof: Note that $K = K_0 \cup (K' + \eta') \subseteq R \cup (\bigcup_{j=0}^{N-2} (j\eta' + R) + \eta') = \bigcup_{j=0}^{N-1} (j\eta' + R)$ where $j\eta'$ are distinct elements of $H^\perp$ for $j = 0, \ldots, N-1$. Hence, Corollary 2.5 applies to f. It follows that the function $g(x) = f(x) - S_M f(x)$ is continuous, square integrable and vanishes on M. It follows from (7) and (5) that the Fourier transform $\widehat{S_M f}(\xi)$ vanishes for a.e. ξ outside R. Hence Corollary 4.2 can be applied to g, yielding a continuous function $h(x) \in L_2(G)$ with $\hat{h}$ vanishing a.e. outside K' such that

$$f(x) = S_M f(x) + h(x) \left(1 - e^{2\pi i \langle x - x_0, \eta' \rangle}\right). \tag{21}$$

Since $e^{2\pi i \langle z' - x_0, \eta' \rangle} \neq 1$ for $z' \in M'$, we can compute the sampled values

$$h(z') = \frac{f(z') - S_M f(z')}{1 - e^{2\pi i \langle z' - x_0, \eta' \rangle}}, \quad z' \in M'. \tag{22}$$

By hypothesis, $h(x)$, $x \in G$, can be computed from these samples. Then $f(x)$ is given by (21). ■

As a first illustration we apply the theorem to sampling on $G = \mathbb{R}^2$ when the support of the Fourier transform is a rectangle with two unequal tabs attached.

Example 4.4 *Let $0 < r_1 < r_2$ and let $\rho = r_1 + r_2$. Consider the set $K \subseteq \mathbb{R}^2$ defined by*

$$K = [-r_2, r_2]^2 \cup [-\rho, -\rho + r_1] \times [-r_1, r_1] \cup [\rho - r_1, \rho + 2r_2] \times [-r_1, r_1].$$

Thus, K is a square of side length $2r_2$ with two unequal "tabs" on each side; see Fig. 2. On the left, we have a rectangle with dimension $r_1 \times 2r_1$, and on the right, a rectangle with dimension $(2r_2 + r_1) \times 2r_1$. K may be partitioned into two subsets so that the theorem applies. Let $H = \frac{1}{2r_2}\mathbb{Z}^2$ so that $H^\perp = 2r_2\mathbb{Z}^2$ and let $\eta' = (2r_2, 0)$. Note that a natural choice for a fundamental domain of $H^\perp$ would be to choose $R_0 = [-r_2, r_2]^2$. However, we consider a different choice for R, which is obtained by cutting the rectangle $[r_2 - r_1, r_2] \times [-r_1, r_1]$ from the right side of R_0 and shifting it to the left by the amount $2r_2$, obtaining the rectangle $[-\rho, -r_2] \times [-r_1, r_1]$. This gives

$$R = [-\rho, -r_2] \times [-r_1, r_1] \cup \left([-r_2, r_2]^2 \setminus ([r_2 - r_1, r_2] \times [-r_1, r_1])\right); \tag{23}$$

see Fig. 1. In the notation of Theorem 4.3 this set would be K_0. Let

$$K' = [-\rho, \rho] \times [-r_1, r_1].$$

Then $K' \subseteq R \cup (\eta' + R)$ (see Fig. 1), and $K = K_0 \cup (\eta' + K')$; see Fig. 2. Note that while K thus satisfies the hypothesis of Corollary 4.2 with $N = 3$, it is not

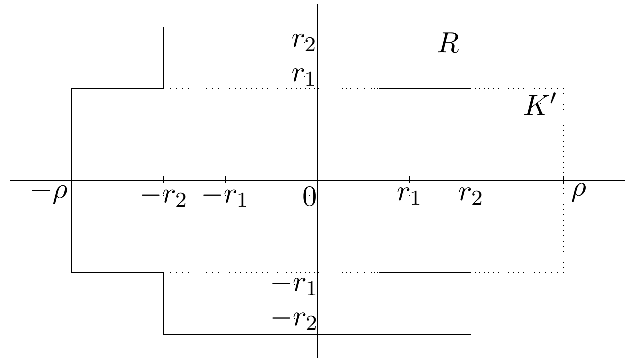

Figure 1: The fundamental domain R of (23) and the set $K' = [-\rho, \rho] \times [-r_1, r_1]$. With $\eta' = (2r_2, 0)$ one has $K' \subset R \cup (\eta' + R)$

an admissible subset of $\widehat{G}$ as defined in [1, Def. 2] because clearly $K' \not\subseteq R$. Now let $M = x_0 + H$ and $M' = x_1 + H'$, where

$$H' = \{(n/(2r_1 + 2r_2), m/(2r_1)) : (n, m) \in \mathbb{Z}^2\}.$$

Since K' is a fundamental domain of H', any continuous function with Fourier transform supported in K' can be reconstructed from its samples on M'. Then a function f with Fourier transform supported in K can be reconstructed from its samples on $M \cup M'$ provided the condition (20) is satisfied. If $x_0 = (x_{01}, x_{02})$ and $x_1 = (x_{11}, x_{12})$, then the condition (20) requires that $(x_{11} - x_{01} + \frac{n}{2(r_1+r_2)})2r_2 \notin \mathbb{Z}$ for all $n \in \mathbb{Z}$.

A recursive reconstruction algorithm can be defined if the set K has a certain structure. The appropriate modification to [1, Definition 2] is in condition *ii)*, which had been $K_j \subseteq R_{j+1}$.

Definition 4.5 *Let $H_1, \ldots, H_N$ be lattices with corresponding fundamental domains R_i of $H_i^\perp$. We call $K \subset \widehat{G}$ an admissible subset of $\widehat{G}$ with respect to $H_1, \ldots, H_N$ if there are subsets $K_1, \ldots, K_N$ of $\widehat{G}$ such that the following conditions hold:*

i) $K_1 = R_1$.

ii) $K_j \subset K_{j+1}$, and there is $P_j \in \mathbb{N}$ such that $K_j \subseteq \bigcup_{l=0}^{P_j-2}(l\eta_{j+1} + R_{j+1})$, with $0 \neq \eta_{j+1} \in H_{j+1}^\perp$, and $l\eta_{j+1}$ are distinct elements of $H_{j+1}^\perp$ for $l = 0, \ldots, P_j - 1$.

iii) $K_{j+1} = R_{j+1} \cup (\eta_{j+1} + K_j)$ with η_{j+1} as in ii).

iv) $K_N = K$.

Observe that because of conditions *ii)* and *iii)* each intermediate set K_{j+1} has the structure of the set K in Corollary 4.2 with $K = K_{j+1}$, $K_0 = R_{j+1}$, $K' = K_j$ and $\eta' = \eta_{j+1}$. The above conditions imply in particular that $K_1 \subset K_2 \subset \ldots \subset K_N$ but not necessarily $K_j \subset R_{j+1}$, $j = 1, \ldots, N-1$ as it was required in [1]. Instead the less restrictive condition that $K_j \subseteq \bigcup_{l=0}^{P_j-2}(l\eta_{j+1} + R_{j+1})$ is used.

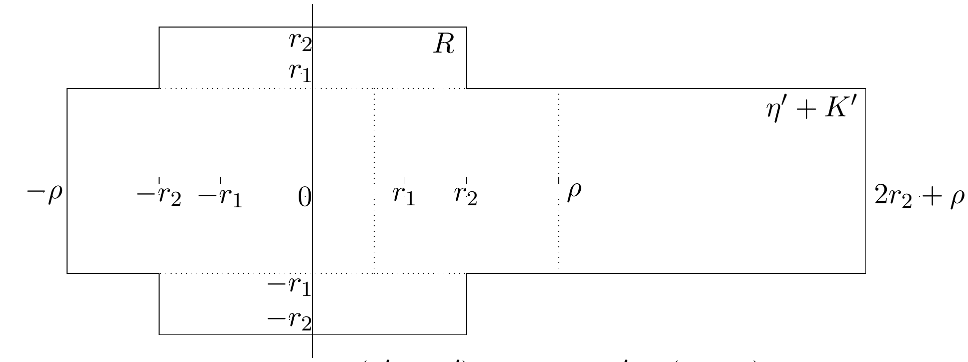

Figure 2: $K = K_0 \cup (\eta' + K')$, $K_0 = R$, $\eta' = (2r_2,\, 0)$.

In addition Definition 4.5 does encompass certain cases of periodic sampling where $H_1 = \ldots = H_N$. We will give examples of both periodic and nonperiodic sampling sets for the group $G = \mathbb{Z}_L \times \mathbb{Z}_L$ in the next section.

To illustrate the structure of the sets described in Definition 4.5, consider the following example.

Example 4.6 *Let $G = \widehat{G} = \mathbb{R}$, and $H_i = \frac{1}{2r_i}\mathbb{Z}$ for $i = 1, 2, 3$ where r_1, r_2, and r_3 are positive real numbers. Let $\rho = r_1 + r_2 + r_3$, define $K = [-\rho, \rho)$, and let fundamental domains R_i of $H_i^{\perp}$ be given by $R_i = [-\rho, -\rho + 2\,r_i)$, $i = 1, 2, 3$. The R_i thus form three nested intervals with the common left boundary $-\rho$ and length $2r_i$ as shown in Figure 3.*

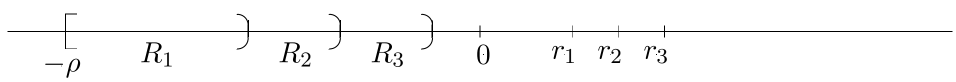

Figure 3: $R_i = [-\rho, -\rho + 2\,r_i)$

Let $\eta_i \in H_i^{\perp}$ be given by $\eta_i = 2\,r_i$ for $i = 2, 3$. Then $K_1 = R_1$ and $K_2 = R_2 \cup (\eta_2 + K_1) = [-\rho, -\rho + 2\,r_1 + 2\,r_2)$, and $K = K_3 = R_3 \cup (\eta_3 + K_2) = R_3 \cup (\eta_3 + R_2) \cup (\eta_3 + \eta_2 + K_1)$. Hence

$$\begin{aligned} K &= R_3 \cup (\eta_3 + K_2) \\ &= [-\rho, -\rho + 2\,r_3) \cup (2\,r_3 + [-\rho, -\rho + 2\,r_1 + 2\,r_2)) \\ &= [-\rho, -\rho + 2\,r_1 + 2\,r_2 + 2\,r_3)) \\ &= [-\rho, \rho) \end{aligned}$$

as shown in Figure 4.

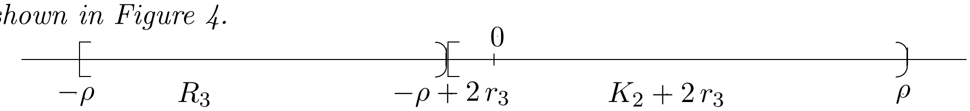

Figure 4: $K = R_3 \cup (\eta_3 + K_2)$

Observe that this example falls under the theory developed in [1] only if $r_1 \leq r_2$ and $r_1 + r_2 \leq r_3$. Definition 4.5 does not require these restrictions. As the example shows, there are in fact no restrictions in case of sampling on the real line and K being an interval. Indeed, it follows from Definition 4.5 and the theory developed below that for $K = [-\rho, \rho)$ a sampling set can be obtained from suitably shifted copies of $H_i = \frac{1}{2r_i}\mathbb{Z}$, $i = 1, \ldots, N$ with $r_1, \ldots, r_N$ such that $r_i > 0$ and $\sum_{i=1}^{N} r_i \geq \rho$.

Theorem 4.7 *Suppose that K is an admissible subset of $\widehat{G}$ with respect to the lattices $H_1, \ldots, H_N$, with R_j, K_j, η_j as in Definition 4.5. Let $M_j = x_j + H_j$, $j = 1, \ldots, N$ be such that if $N > 1$*

$$e^{2\pi i\langle z - x_j, \eta_j\rangle} \neq 1 \quad \text{for} \quad z \in \bigcup_{k=1}^{j-1} M_k, \quad j = 2, \ldots, N. \tag{24}$$

Let $f \in L_2(G)$ be continuous and such that $\hat{f}$ vanishes a.e. outside K. Then there are continuous functions $f_j \in L_2(G)$ such that $\hat{f}_j$ vanishes outside K_j, and for all $x \in G$:

$$\begin{aligned} f_1(x) &= S_{M_1} f_1(x), \\ f_j(x) - S_{M_j} f_j(x) &= f_{j-1}(x)\left(1 - e^{2\pi i\langle x - x_j, \eta_j\rangle}\right), \quad j = 2, \ldots, N, \\ f_N(x) &= f(x). \end{aligned}$$

Using this recursion, the function f can be reconstructed from sampled values $f(z)$, $z \in \bigcup_{k=1}^{N} M_k$.

Proof: The proof is by induction on N. If $N = 1$, then $K = K_1 = R_1$ and $f = S_{M_1} f$ by the classical sampling theorem. Hence f can be reconstructed from its samples on M_1. Now assume $N > 1$ and that the theorem holds with N replaced by $N-1$. Let $f_N = f$ and consider the function $g(x) = f_N(x) - S_{M_N} f_N(x)$. Since $K \subseteq \bigcup_{l=0}^{P_{N-1}-1}(l\eta_N + R_N)$, by Corollary 2.5 g is continuous, square-integrable, and vanishes on M_N. Since $\widehat{S_{M_N} f}$ vanishes outside $R_N \subseteq K$, $\hat{g}$ vanishes a.e. outside K. Since $K = R_N \cup (\eta_N + K_{N-1})$ and $K_{N-1} \subseteq \bigcup_{l=0}^{P_{N-1}-2}(l\eta_N + R_N)$, we can apply Corollary 4.2 to g, with K_0, K', x_0 and η' replaced by R_N, K_{N-1}, x_N, and η_N, respectively. Hence there is a continuous, square-integrable function f_{N-1} such that

$$g(x) = f_N(x) - S_{M_N} f_N(x) = f_{N-1}(x)\left(1 - e^{2\pi i\langle x - x_N, \eta_N\rangle}\right),$$

and $\widehat{f_{N-1}}$ vanishes a.e. outside K_{N-1}. Because of (24) the values

$$f_{N-1}(z) = \frac{f(z) - S_{M_N} f(z)}{1 - e^{2\pi i\langle z - x_N, \eta_N\rangle}}, \quad z \in \bigcup_{k=1}^{N-1} M_k,$$

can be computed. Now the hypothesis of the theorem is satisfied if f, K, and N are replaced by f_{N-1}, K_{N-1}, and $N-1$, respectively. By induction hypothesis the theorem holds in this case, yielding the functions f_j, $j = 1, \dots, N-2$, and the reconstructed function $f_{N-1}(x)$ for all $x \in G$. Now f is reconstructed via

$$f(x) = \left(1 - e^{2\pi i \langle x - x_N, \eta_N \rangle}\right) f_{N-1}(x) + S_{M_N} f(x), \quad x \in G.$$

■

The theorem establishes the following recursive algorithm for reconstruction of f from sampled values $f(z)$, $z \in \bigcup_{k=1}^{N} M_k$:

Algorithm 4.8 *:*

IF $N = 1$ *THEN* $f(x) = S_{M_1} f(x)$.

ELSE

Compute

$$g(z) = \frac{f(z) - S_{M_N} f(z)}{1 - e^{2\pi i \langle z - x_N, \eta_N \rangle}}, \quad z \in \bigcup_{k=1}^{N-1} M_k.$$

Invoke the algorithm to compute $g(x)$, $x \in G$ *from the computed values* $g(z)$, $z \in \bigcup_{k=1}^{N-1} M_k$.

$f(x) = g(x) \left(1 - e^{2\pi i \langle x - x_N, \eta_N \rangle}\right) + S_{M_N} f(x), \quad x \in G.$

END

5 A Two-dimensional Numerical Implementation

In this section we implement Algorithm 4.8 for the group $G = \mathbb{Z}_L \times \mathbb{Z}_L$ using MATLAB.

Let $G = \mathbb{Z}_L \times \mathbb{Z}_L$, i.e., $G = \{0, \dots, L-1\} \times \{0, \dots, L-1\}$ with addition modulo L. Then $\widehat{G} = \{\nu/L, \ \nu = 0, \dots, L-1\} \times \{\nu/L, \ \nu = 0, \dots, L-1\}$ with addition modulo 1. Let m_G be the counting measure. Then $m_{\widehat{G}}$ equals $1/L^2$ times the counting measure, where according to our convention the normalization constant $1/L^2$ is determined by the Fourier inversion formula (3). We consider lattices H that are tensor products of two subgroups of $\mathbb{Z}_L$, that is

$$H = H(h_1, h_2) = \{h_1 m, \ m = 0, \dots, L/h_1 - 1\} \times \{h_2 n, \ n = 0, \dots, L/h_2 - 1\}$$

where $0 < h_1, h_2 \le L$ and h_1, h_2 divide L. The reciprocal lattice can be written as

$$H(h_1, h_2)^\perp = \{\mu/h_1, \ \mu = 0, \dots, h_1 - 1\} \times \{\nu/h_2, \ \nu = 0, \dots, h_2 - 1\}.$$

A fundamental domain of $H(h_1,h_2)^\perp$ is given by

$$R = R(h_1,h_2) = \{(\mu/L, \nu/L):\ \mu = 0,\ldots,L/h_1-1,\ \nu = 0,\ldots,L/h_2-1\}, \quad (25)$$

with $m_{\widehat{G}}(R) = 1/(h_1h_2)$.

The MATLAB code given at the end of this section implements Algorithm 4.8 for this setting. The parameters are specified and explained in the driver routine bf2d.m. This routine generates the function to be reconstructed by randomly specifying its non-zero Fourier coefficients, cf. [4]. The function M-file spect.m computes the support K of the Fourier transform of the function f according to the formula

$$K = R_N \cup (\eta_N + R_{N-1}) \cup ((\eta_{N-1}+\eta_N) + R_{N-2}) \cup \ldots \cup ((\eta_2 + \ldots + \eta_N) + R_1)$$

which follows from Definition 4.5 provided the compatibility conditions are met. This computation requires specification of the R_j and η_j by the user. In order to keep the code simple the assumption was made that all fundamental domains are of the form given in (25), so that specifying h_1 and h_2 determines both the lattice H and the fundamental domain R of $H^\perp$. Note however that the price for this convenience is a loss of generality in the code as our theory would permit other choices for R, a feature that is sometimes advantageous, such as in Example 4.4 above. Note also that depending on the values of the R_j and η_j, the set K may or may not be a hypercube, may be connected or have more than one connected component. The recursive algorithm is implemented in the function M-file bfmethod.m. The function M-file SM.m computes S_Mf by first computing the Fourier transform $\widehat{S_M f}$ on R and then finding S_Mf by an inverse Fourier transform. This computation is based on the following considerations. It follows from (7) and (5) that

$$\widehat{S_M f}(\xi) \;=\; \frac{\chi_R(\xi)}{m_{\widehat{G}}(R)}\, e^{-2\pi i\langle x_0,\xi\rangle} \sum_{y\in H} f(x_0+y)\, e^{-2\pi i\langle y,\xi\rangle}. \quad (26)$$

Every $y \in H(h_1,h_2)$ can be written as $y = y_{mn} = (h_1m, h_2n)$ for some $m \in \{0,\ldots,L/h_1-1\}$, $n \in \{0,\ldots,L/h_2-1\}$. Let $f_{mn} = f(x_0+y_{mn})$. Then, for $\xi = \xi_{\mu\nu} = (\mu/L, \nu/L) \in R$, R as in (25), we have

$$\begin{aligned} F_{\mu\nu} &= \sum_{y\in H} f(x_0+y)\, e^{-2\pi i\langle y,\xi_{\mu\nu}\rangle} \\ &= \sum_{m=0}^{L/h_1-1} \sum_{n=0}^{L/h_2-1} f_{mn}\, e^{-2\pi i(h_1m\mu+h_2n\nu)/L} \end{aligned}$$

which is just the two-dimensional Discrete Fourier Transform (DFT) of the array f_{mn} with size $L/h_1 \times L/h_2$. On the other hand we see from (26) that

$$F_{\mu\nu} = m_{\widehat{G}}(R)\, e^{2\pi i\langle x_0,\xi_{\mu\nu}\rangle}\, \widehat{S_M f}(\xi_{\mu\nu}).$$

Hence extending the array $F_{\mu\nu}$ by zero-padding to size $L \times L$, taking an Inverse DFT of size $L \times L$, and dividing by $m_{\widehat{G}}(R)$ gives $S_M f(x_0 + x)$, for $x \in G$. Reversing the shift by x_0 completes the task.

Example 5.1 *Our first numerical experiment involves three different subgroups of $G = \mathbb{Z}_L \times \mathbb{Z}_L$ with $L = 512$: $H_1 = H(8,8)$, $H_2 = H(4,8)$, and $H_3 = H(4,4)$. According to (25) we have*

$$\begin{aligned} R_1 &= \{\mu/512,\, \mu = 0, \dots, 63\}^2, \\ R_2 &= \{\mu/512 :\, \mu = 0, \dots, 127\} \times \{\nu/512 : \nu = 0, \dots, 63\} \\ R_3 &= \{\mu/512 : \mu = 0, \dots, 127\}^2 \end{aligned}$$

Let $\eta_2 = (0, 64)/512$ and $\eta_3 = (384, 0)/512$. Then

$$K = R_3 \cup (\eta_3 + R_2) \cup (\eta_2 + \eta_3 + R_1)$$

is the union of the two contiguous sets R_3 and $(\eta_3 + R_2) \cup (\eta_2 + \eta_3 + R_1)$ depicted in Figure 5. The shifts $x_j = (x_{j1}, x_{j2})$ need to satisfy the conditions (24) with $N = 3$. For the present example these conditions read as follows. For $j = 2$ we obtain the condition

$$e^{2\pi i \langle z - x_2, \eta_2 \rangle} \neq 1 \textit{ for } z \in M_1 = x_1 + H_1.$$

Since $H_1 = H(8,8)$, we have $z = z_{mn} = (x_{11} + 8m, x_{12} + 8n)$. With $\eta_2 = (0, 64)/512$ we obtain the condition

$$64\,(x_{12} - x_{22} + 8n) \notin 512\,\mathbb{Z} \quad \textit{for} \quad n = 0, \dots, 63,$$

which is equivalent to $(x_{12} - x_{22})$ not being a multiple of 8. For $j = 3$ we have the conditions

$$e^{2\pi i \langle z - x_3, \eta_3 \rangle} \neq 1 \textit{ for } z \in M_1 \cup M_2 = (x_1 + H_1) \cup (x_2 + H_2), \quad \eta_3 = (384,\, 0)/512.$$

For $z \in M_1$ this yields

$$384\,(x_{11} - x_{31} + 8m) \notin 512\,\mathbb{Z} \quad \textit{for } m = 0, \dots, 63,$$

which is equivalent to $x_{11} - x_{31}$ not being a multiple of 4. For $z \in M_2$ one obtains

$$384\,(x_{21} - x_{31} + 4m) \notin 512\,\mathbb{Z},; \quad \textit{for } m = 0, \dots, 127,$$

which yields that $x_{21} - x_{31}$ should not be a multiple of 4. Hence we see that the shifts can for example be chosen as $x_1 = (1,1)$, $x_2 = (1,0)$, and $x_3 = (0,1)$. The driver program bf2d.m given below contains the parameters for this example. Running the program demonstrates that the function f is recovered accurately. The relative errors in our numerical tests varied with the random signal, but stayed below $3 \cdot 10^{-13}$.

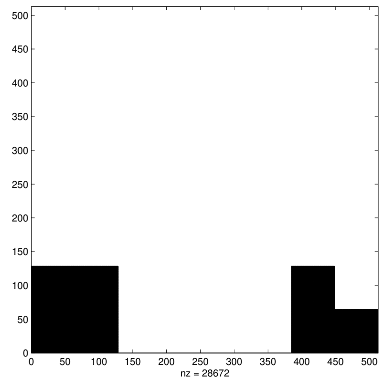

Figure 5: $K = R_3 \cup (\eta_3 + R_2) \cup ((\eta_2 + \eta_3) + R_1)$. $L = 512, R_1 = R(8,8)$, $R_2 = R(4,8)$, $R_3 = R(4,4)$, cf. (25). $\eta_2 = (0,64)/L$, $\eta_3 = (384,0)/L$. K contains 28,672 elements.

In our final experiment we consider a case of periodic sampling covered by Theorem 4.7, namely when K has the form $\bigcup_{j=0}^{N-1}(j\eta + R)$.

Example 5.2 *Let* $L = 512$, $N = 4$, $H_1 = H_2 = H_3 = H_4 = H(32,8)$, $R = R(32,8)$ *as in (25), and* $\eta_j = (j-1)\eta$, $j = 2,3,4$ *with* $\eta = (16,64)/512 \in H(32,8)^{\perp}$. *Hence* $K = \bigcup_{j=0}^{3}(j\eta + R)$ *is the union of the four contiguous sets depicted in Figure 6.*

Again the shifts $x_j = (x_{j1}, x_{j2})$ *have to be chosen such that the sampling conditions (24) are satisfied. Since the sampling set consists of four cosets of a single lattice* H, *and* $\eta = \eta_4 = \eta_3 = \eta_2 = (16/L, 64/L)$, *the sampling conditions (24) are reduced to*

$$e^{2\pi i\langle x_k - x_j, \eta\rangle} \neq 1 \quad \textit{for} \quad j = 2,3,4 \ \textit{ and } \ k = 1, \cdots, j-1. \tag{27}$$

Conditions (27) are equivalent to $(x_{k1} - x_{j1}) + 4(x_{k2} - x_{j2})$ *not being a multiple of 32 for* $j = 2,3,4$, *and* $k = 1, \cdots, j-1$. *For example, one could chose* $x_1 = (0,0)$, $x_2 = (1,1)$, $x_3 = (2,2)$, *and* $x_4 = (3,3)$. *In order to run this example with the code given below, the parameters in the routine bf2d.m should be set as follows.*

```
L=512; h1=[32 32 32 32]; h2=[8 8 8 8]; x1=[0 1 2 3]; x2=[0 1 2 3];
eta1=[0 16 16 16]/L;  eta2=[0 64 64 64]/L;
```

Finally, note that our sampling theory does of course permit the bandregion K to be a hypercube. E.g., if in the example above we choose $\eta_4 = \eta_3 = \eta_2 = (16/L, 0/L)$, we obtain a square region of size 64×64 for K.

We conclude this section by giving the MATLAB source code for this two-dimensional implementation on $G = \mathbb{Z}_l \times \mathbb{Z}_L$.

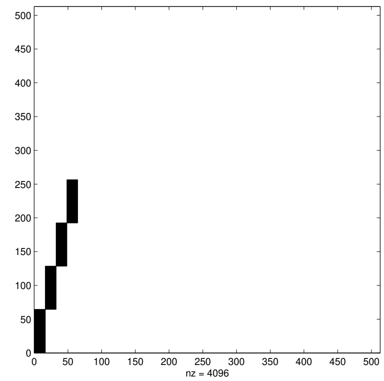

Figure 6: $K = R \cup (\eta_4 + R) \cup ((\eta_3 + \eta_4) + R) \cup ((\eta_2 + \eta_3 + \eta_4) + R)$. $R = R(32, 8)$ as in (25), $\eta_j = \eta = (16,\ 64)/512$. K contains 4,096 elements.

```
% bf2d.m : Driver for nonperiodic/periodic sampling
% on the group G = Z_L X Z_L = {0, ..., L-1} X {0, ..., L-1}
% with addition modulo L.
% Explanation of variables:
% L = number elements in Z_L
% The k-th lattice H_k is H(h1(k),h2(k))
% with h1 and h2 as described below
% h1: vector where h1(k) is the divisor of L which generates
% the points in the horizontal direction with the form <h1(k)>,
% i.e., <h1(k)> = {0,h1(k),2h1(k),...,L-h1(k)}
% h2: vector where h2(k) is the divisor of L which generates the
% points in the vertical direction with the form <h2(k)>, i.e.,
% <h2(k)> = {0,h2(k),2h2(k),...,L-h2(k)}
% x1: vector with shifts in the horizontal direction.
% x2: vector with shifts in the vertical direction.
% eta1: eta1(k) = first component of eta_k. eta1(k) = \eta_{k1}
% eta2: eta2(k) = second component of eta_k



%Input parameters:

%Experiment 1
L=512;            % Length of Z_L
h1=[8 4 4];       % Specify horizontal direction of the subgroups
h2=[8 8 4];       % Specify vertical direction of the subgroups
```

```
x1=[1 1 0];        % Specify first component of shifts.
                   % Need x1(k) in {0,...,h1(k)-1}
x2=[1 0 1];        % Specify second component of shifts;
                   % Need x2(k) in {0,...,h2(k)-1}
eta1=[0 0 384]/L; % Specify first components of eta_k.
eta2=[0 64 0]/L;  % Specify second components of eta_k.
                   % NOTE: The values of eta1(1) and eta2(1) are
                   % not used by the code but must be specified.
% End of input section


N = max(size(h1));                   % Number of subgroups

% Randomly generate signal to be sampled and reconstructed
fhat=complex(rand(L,L),rand(L,L));   % Random Fourier coefficients
filt = spect(L,h1,h2,N,eta1,eta2);   % The bandregion K
spy(filt.','k'); axis xy;            % Plot bandregion K
fhat=fhat.*filt;                     % Set frequencies
                                     % outside of K to zero
fexact=ifft2(fhat);
fexact=fexact/norm(fexact);          % Normalize signal
% Compute sampled values
f = zeros(L,L);
for k=1:N
 Hx1=x1(k)+[0:h1(k):L-h1(k)];
 % Hx1 = First components of points in coset M_k = x_k + H_k.
 Hx2=x2(k)+[0:h2(k):L-h2(k)];
 % Hx2 = Second components of points in coset M_k = x_k + H_k.
 f(1+Hx1,1+Hx2) = fexact(1+Hx1,1+Hx2);% Sampled values on M_k
end
% Reconstruct signal
F = bfmethod(f,L,h1,h2,eta1,eta2,x1,x2);

%Compute the l2 relative reconstruction error
relerr = norm(fexact - F) %Note that norm(fexact)=1.

%----------------------------------------------------------------

function filt = spect(L,h1,h2,N,eta1,eta2)
% Computes the spectrum according to Definition 4.3.
% Parameters need to satisfy the conditions of Definition 4.3
M = max(size(eta1));
```

```
ETA1=0;
ETA2=0;
filt=zeros(L,L);
for m=N:-1:1
    v=mod((L*ETA1)+(0:L/h1(m)-1),L);
    w=mod((L*ETA2)+(0:L/h2(m)-1),L);
    filt(v+1,w+1)=1;

    ETA1 = ETA1 + eta1(m);
    ETA2 = ETA2 + eta2(m);
end

%------------------------------------------------------------------

function F=bfmethod(f,L,h1,h2,eta1,eta2,x1,x2)

N = max(size(h1));
Hx1 = x1(N)+[0:h1(N):L-h1(N)];  % Coset M_N = Hx1 x Hx2
Hx2 = x2(N)+[0:h2(N):L-h2(N)];
fH = f(1+Hx1,1+Hx2);            % Sampled values on coset M_N
SMf = SM(fH,L,h1(N),h2(N),x1(N),x2(N));
V=([0:L-1]-x1(N));
W=([0:L-1]-x2(N));
TMP1 = zeros(L,L);
TMP = zeros(L,L);
if N==1
  F = SMf;
else
    for k = 1:L
        for m = 1:L
            tmp = 1-exp(2*pi*i*[V(k)*eta1(N)+W(m)*eta2(N)]);
            tmp1 = tmp;
                          % Avoid zero divisions
            tmp1(find(abs(tmp1 < 1.e-14)))=1;
            TMP(k,m) = tmp;
            TMP1(k,m) = tmp1;
        end
    end
f1 = (f - SMf)./TMP1;
fN1 = bfmethod(f1,L,h1(1:N-1),h2(1:N-1),eta1(1:N-1), ...
               eta2(1:N-1), x1(1:N-1),x2(1:N-1));
F = fN1.*TMP + SMf;
```

```
end

%-----------------------------------------------------------------

function S = SM(f,L,h1,h2,x1,x2)
%Computes S_Mf(z) for z in G
%G = {0,1,...,L-1} X {0,1,...,L-1} with addition mod L
%H = {0,h1,2h1,...,L-h1} X {0,h2,2h2,...,L-h2}
%f = row vector of length L/h1 X L/h2, with sampled
% values on x+H where x=(x1,x2).
% x = shift. Need x1 in {0,...,h1-1} and x2 in {0,...,h2-1}

chi = zeros(L,L);
chi(1:L/h1,1:L/h2) = fft2(f);
S = h1*h2*ifft2(chi);
if x1 > 0
  tmp = S(L-x1+1:L,1:L);
  S(x1+1:L,1:L)=S(1:L-x1,1:L);
  S(1:x1,1:L) = tmp;
end
if x2 > 0
    tmp = S(:,L-x2+1:L);
    S(:,x2+1:L) = S(:,1:L-x2);
    S(:,1:x2) = tmp;
end
```

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