RIGIDITY OF MINIMAL ISOMETRIC IMMERSIONS OF SPHERES INTO SPHERES

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Abstract.

We show two specific uniqueness properties of a fixed minimal isometric immersion from S^3 into S^6. This particular immersion has been extensively studied. We give the first uniqueness results within the full class of all minimal isometric immersion from $S^3(1)$ into $S^6(\frac{1}{4})$.

Introduction

Since the 1960's minimal isometric immersions of Riemannian manifolds into round spheres have been extensively studied ([C], [DW1], [DW2], [L], [T]). Naturally arising questions were: What are necessary and sufficient conditions for the existence of such immersions? Exactly which manifolds admit such immersions? Do these immersions have any kind of uniqueness properties? The first two questions both deal with existence only, for answers, see [E1], [E2], [DZ]. This paper addresses the last question, whose answer provides more insight into the structure of the moduli space of all minimal isometric immersions. In fact, we show two specific uniqueness properties of a fixed minimal isometric immersion from S^3 into S^6. This particular immersion has been extensively studied; the only known uniqueness results restrict the class of admissible immersions considerably. Our results are the first ones to address uniqueness properties within the full class of all minimal isometric immersion from S^3 into S^6.

A necessary and sufficient condition for the minimality of such immersions was given by by T. Takahashi [T] who observed that if $\Phi\colon M \to S^N(r) \subset \mathbb{R}^{N+1}$ is a minimal isometric immersion, then all components of Φ must be eigenfunctions of the Laplace operator on M with respect to the same eigenvalue. And conversely if Φ is an isometric immersion such that all the components are eigenfunctions of the Laplace operator for the same eigenvalue, then Φ is a *minimal* isometric immersion into a round sphere.

Takahashi also observed that if M is an isotropy irreducible Riemannian homogeneous space, i.e., if the isotropy group of a point acts irreducibly on the tangent space, then an orthonormal basis of each eigenspace automatically gives rise to

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a minimal isometric immersion into a round sphere. We call these the *standard* minimal immersions.

When $M = S^n(1)$, there is a sequence of standard minimal isometric immersions, one for each nonzero eigenvalue. For the first such eigenvalue one obtains the standard embedding into $\mathbb{R}^{n+1}$, for the second eigenvalue an immersion into $S^{\frac{n(n+3)}{2}-1}\left(\sqrt{\frac{n}{2(n+1)}}\right)$, which gives rise to the Veronese embedding of $\mathbb{R}P^n$. For odd-numbered eigenvalues the images are all embedded spheres and for even-numbered eigenvalues the images are all embedded real projective spaces.

The first uniqueness result was obtained by E. Calabi [C] who showed that *every* minimal isometric immersion of $S^2(1)$ into $S^N(r)$ is congruent to one of these standard eigenspace immersions. Here two immersions are called *congruent* if they differ by an isometry of the ambient space, in this case by an element of $O(N+1)$, the isometry group of $S^N(r)$. However, M. DoCarmo and N. Wallach [DW2] showed that in higher dimensions there are in general many minimal isometric immersions of $S^n(1)$ into $S^N(r)$ and that they are parametrized by a compact convex body in a finite dimensional vector space. A recent development is the finding of the exact dimension of this convex body by G. Toth [To1]. For further references on this matter, see [DW1], [DW2], [L], [T].

In this paper we will focus on a particular immersion which arose in the following way. In 1971 a question was posed by DoCarmo and Wallach [DW2, Remark 1.6]. For a given n, what is the smallest dimension N for which there exist minimal isometric immersions of $S^n(1)$ into $S^N(r)$ which are not totally geodesic? In this question one can also specify the radius r of the target sphere or equivalently fix the eigenvalue one wants to consider. A lower bound for N was given by J. D. Moore [Mr] who showed that no such immersions exist if $N \leq 2n-1$. In [DW2] it was suggested that the probable answer is $N = \frac{n(n+3)}{2} - 1$, which is achieved by the Veronese embedding. That this is false, at least for $n = 3$, was first observed by N. Ejiri [Ej] who showed that there exists a minimal isometric immersion of $S^3(1)$ into $S^6(\frac{1}{4})$ which is not totally geodesic.

For $n = 3$, the Ejiri immersion is optimal in that it maps to the smallest target sphere possible. Our goal is to prove specific uniqueness properties of this immersion within the full class of minimal isometric immersions. Previous work addresses uniqueness only by restricting the class of immersions. The following is a condensed survey of known results. Ejiri showed that the immersion is totally real with respect to the natural almost complex structure on S^6. His construction is not explicit, as it uses the fundamental theorem for isometric immersions to prove existence. In [Ma1] Mashimo constructed this immersion more explicitly as an $SU(2)$-equivariant immersion. In [Ma2] he shows that it is also an orbit of a subgroup of the exceptional Lie group G_2 acting on S^6 and proves that every totally real immersion of $S^3(1)$ into $S^6(\frac{1}{4})$ is congruent to this example. In [DVV] it was observed that the immersion is a 24-fold cover of its image. D. DeTurck and W. Ziller [DZ] were able to identify the image as the tetrahedral manifold S^3/T^*. Here T^* denotes the binary tetrahedral group, the double cover of the group of symmetries of the tetrahedron. They also described the immersion explicitly as follows.

$$\begin{aligned}
&\tilde{F} \,:\, S^3(1) \hookrightarrow S^6(\frac{1}{4})\\
&(z,w,\bar{z},\bar{w}) \mapsto [\frac{\sqrt{6}}{4}\,(\bar{z}\,\bar{w}^5 \,-\, \bar{z}^5\bar{w}\,), \frac{5}{4}\,z\,\bar{z}\,\bar{w}^4 \,-\, \frac{1}{4}\,w\,\bar{w}^5 \,+\, \frac{5}{4}\,w\,\bar{z}^4\,\bar{w} \,-\, \frac{1}{4}\,z\,\bar{z}^5\,,\\
&\qquad\qquad \frac{\sqrt{10}}{2}\,z^2\,\bar{z}\,\bar{w}^3 \,-\, \frac{\sqrt{10}}{4}\,z\,w\,\bar{w}^4 \,+\, \frac{\sqrt{10}}{4}\,z\,w\,\bar{z}^4 \,-\, \frac{\sqrt{10}}{2}\,w^2\,\bar{z}^3\,\bar{w}\,,\\
&\qquad\qquad -\,\frac{\sqrt{15}}{4}\,\imath\,(z^3\,\bar{z}\,\bar{w}^2 \,-\, z\,w^2\,\bar{z}^3) \,+\, \frac{\sqrt{15}}{4}\,\imath\,(z^2\,w\,\bar{w}^3 \,-\, w^3\,\bar{z}^2\,\bar{w})\,]\,.
\end{aligned}$$

Here $\imath$ stands for the standard square root of -1. One easily shows that this isometrically immerses $S^3(1)$ into $S^6(\frac{1}{4}) \subset \mathbb{C}^3 \oplus \mathbb{R} = \mathbb{R}^7$ and hence is a minimal isometric immersion by the above mentioned Takahashi result. The map is invariant under $\alpha(z,w) = (\imath\,z, -\imath\,w)$, $\beta(z,w) = (-w, z)$ and $\gamma(z,w) = (\frac{1}{2}\,(1{+}\imath)\,(z{-}w), \frac{1}{2}\,(1{-}\imath)\,(z{+}w))$. Then α, β, and γ generate a group of order 24 isomorphic to the binary tetrahedral group T^* and D. DeTurck and W. Ziller show that the immersion defines an embedding of S^3/T^* into $S^6(\frac{1}{4})$.

They also prove that this immersion is unique within the class of $SU(2)-$ equivariant minimal isometric immersions of $S^3(1)$ into $S^6(r)$. They conjectured that it is possible to remove the equivariance condition and to turn this into a global uniqueness result. The purpose of this paper is to show that this is in fact *impossible* in the case of infinitesimal uniqueness.

Theorem 1. *The fixed minimal isometric immersion $\tilde{F} \,:\, S^3(1) \hookrightarrow S^6(\frac{1}{4})$ as described above is not infinitesimally rigid within the class of* **all** *degree* 6 *minimal isometric immersions.*

Here we use the same notation as in [To2] which will be explained in the following section. We also prove a weaker uniqueness property of this immersion.

Theorem 2. *The fixed minimal isometric immersion $\tilde{F} \,:\, S^3(1) \hookrightarrow S^6(\frac{1}{4})$ as described above is linearly rigid within the class of* **all** *degree* 6 *minimal isometric immersions.*

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1. Geometric Preliminaries

Let M be an n-dimensional compact Riemannian manifold and $S^N(r)$ a sphere of dimension N and radius r.

A fundamental result of T. Takahashi [T] is the following theorem:

Theorem (Takahashi). *Let M be an n-dimensional compact Riemannian manifold and $f : M \to \mathbb{R}^N$ an isometric immersion. Then f is a minimal isometric immersion into a round sphere if and only if all components of f are eigenfunctions of the Laplace operator on M with respect to the same eigenvalue.*

Therefore the main idea in constructing minimal isometric immersions of a manifold M into a sphere is to find eigenvalues of the Laplacian on M of sufficiently high multiplicity in order to provide the coordinate functions of the immersions.

Another result of Takahashi [T] is that all isotropy irreducible homogeneous Riemannian manifolds, i.e. manifolds $M = G/H$ whose isotropy group H acts irreducibly on the tangent space, do admit such immersions. To see this we consider the eigenspace E_λ to a fixed eigenvalue $\lambda \neq 0$. On E_λ we have the inner product induced by that of $L^2(M)$ and the group G acts on E_λ by isometries. Let $\{\phi_1, \ldots, \phi_N\}$ be an orthonormal basis of E_λ and let $\phi = (\phi_1, \ldots, \phi_N) : M \to \mathbb{R}^N$. Then $\sum d\phi_i^2$ on the one hand can be regarded as the inner product on E_λ and on the other hand as the metric on M which is the pull back of the standard metric on $\mathbb{R}^N$ under ϕ. In the first interpretation it is clear that $\sum d\phi_i^2$ must be invariant under the action of G and hence the metric $\sum d\phi_i^2$ on M must be also. But then $\sum d\phi_i^2$ must be a multiple of the given metric on M as both are invariant under the irreducible action of the isotropy group H. This multiple cannot be zero as the functions ϕ_i are not constant. Therefore, after multiplying the metric on M by a constant, $\phi : M \to \mathbb{R}^N$ is an isometric immersion, which by our first stated theorem by Takahashi [T] gives rise to a minimal isometric immersion into a sphere. This immersion is called the *standard minimal immersion* of degree g if λ is the gth nonzero eigenvalue. We call two immersions *congruent* if they differ by an isometry of the ambient space. Note that a different choice of orthonormal basis for E_λ gives rise to a congruent immersion.

An example of such a homogeneous Riemannian manifold is the n-dimensional sphere, realized as the homogeneous space $SO(n+1)/SO(n)$. The eigenfunctions of $S^n(1)$ are simply the restrictions of harmonic homogeneous polynomials on $\mathbb{R}^{n+1}$ to $S^n(1)$. All the harmonic homogeneous polynomials of degree g restrict to eigenfunctions on S^n with the same eigenvalue $\lambda_g = g\,(g+n-1)$ and the dimension of this eigenspace is equal to $N_g = (2g+n-1)(g+n-2)!/\,(g!(n-1)!)$.

In [DW2] the space of all minimal isometric immersions of $S^n(1)$ into $S^N(r)$ was examined in detail, and it was shown that for $n > 2$ there are many minimal isometric immersions besides the standard ones. If we fix $r = \sqrt{n/\lambda_g}$, or equivalently fix the degree g of the harmonic homogeneous polynomials, then these minimal isometric immersions (up to congruence of the ambient space) are parametrized by a compact convex body in a finite dimensional vector space, which we will now describe. Let $\phi_0 : S^n(1) \longrightarrow S^{N_g - 1}(\sqrt{n/\lambda_g})$ be the standard minimal isometric immersion of degree g. Then any other isometric immersion ϕ of degree g is given by $A \circ \phi_0$ where A is an $N_g \times N_g$ matrix. We can write $A = R \circ P$ where R is orthogonal and P symmetric and positive semi-definite. Hence $A \circ \phi_0$ is congruent to $P \circ \phi_0$ and one can show that $P \circ \phi_0$ is an isometric immersion if and only if $P^2 - \mathrm{Id}$ is orthogonal to $Sym^2((\phi_0)_*(TS^n)) \subset Sym^2\mathbb{R}^{N_g}$. Here TS^n denotes the tangent bundle of S^n. If we let W_g be the vector space of all symmetric matrices orthogonal to $Sym^2((\phi_0)_*(TS^n))$ and $B_g = \{S \in W_g | S + \mathrm{Id} \geq 0\}$, then in fact $P \circ \phi_0$ is an isometric immersion if and only if $P^2 - \mathrm{Id} \in B_g$. One can show that $S \in W_g$ implies $\mathrm{tr}(S) = 0$. It follows that the eigenvalues of elements in B_g are bounded and hence B_g is a compact convex body which parametrizes all congruence classes of minimal isometric immersions of degree g. An explicit parametrization is given by $S \in B_g \longrightarrow \sqrt{S + \mathrm{Id}} \circ \phi_0$.

In [DW2] it is shown that for $n = 2$ and any g and for $g = 2, 3$ and any n the space B_g is a point, i.e. any such minimal isometric immersion is congruent to the standard one, ϕ_0. For any other value of n and g they show that $\dim(B_g) \geq 18$ and that the dimension of B_g grows very quickly with n or g. Only recently Toth [To1] determined the exact dimension of B_g by showing that the lower estimate given by DoCarmo and Wallach was sharp.

In this work we will be concerned with the case of $M = S^3$, whose eigenfunctions are harmonic homogeneous polynomials in four real variables. The isometry condition translates into the following system of partial differential equations :

$$\sum_{k=1}^{N} \frac{\partial f_k}{\partial x_i} \frac{\partial f_k}{\partial x_j} = \delta_{ij} \, (\sum_{l=1}^{4} x_l^2)^{g-1} + C \, x_i \, x_j \, (\sum_{l=1}^{4} x_l^2)^{g-2} \tag{1.0}$$

Here the f_k denote the components of $f : S^3(1) \hookrightarrow S^{N-1}(r)$, g stands for the degree of the polynomials used and C is the constant $C = \dfrac{3g^2}{g\,(g+2)} - 1$.

A slightly different system of differential equations was used by N. Wallach [Wa]. He assumed the target sphere to have radius 1. For an explanation of equations (1.0), see [E1].

In this setting it is more natural to translate the system (1.0) of four real variables into a system of two complex variables. To do so we will set $z = x_1 + \imath\, x_2$, $w = x_3 + \imath\, x_4$ and

$$\begin{aligned}
&\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar z}\,; \ \frac{\partial f}{\partial x_2} = \frac{1}{\imath}\,(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar z}) \\
&\frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial w} + \frac{\partial f}{\partial \bar w}\,; \ \frac{\partial f}{\partial x_4} = \frac{1}{\imath}\,(\frac{\partial f}{\partial w} - \frac{\partial f}{\partial \bar w}) \\
&\triangle\, f = 4\,(\frac{\partial^2 f}{\partial z \partial \bar z} + \frac{\partial^2 f}{\partial w \partial \bar w})
\end{aligned}$$

Using these rules we obtain the following six equations:

$$\sum_{k=1}^{N+1} (\frac{\partial f_k}{\partial z})^2 \; = \; \frac{C}{4}\,(z\,\bar z + w\,\bar w)^{g-2}\,\bar z^2 \tag{1.1}$$

$$\sum_{k=1}^{N+1} (\frac{\partial f_k}{\partial w})^2 = \frac{C}{4}\,(z\,\bar z + w\,\bar w)^{g-2}\,\bar w^2 \tag{1.2}$$

$$\sum_{k=1}^{N+1} \frac{\partial f_k}{\partial z}\frac{\partial f_k}{\partial \bar z} = \frac{1}{2}(z\bar z + w\bar w)^{g-1} + \frac{C}{4}(z\bar z + w\bar w)^{g-2} z\bar z \tag{1.3}$$

$$\sum_{k=1}^{N+1} \frac{\partial f_k}{\partial w}\frac{\partial f_k}{\partial \bar w} = \frac{1}{2}(z\bar z + w\bar w)^{g-1} + \frac{C}{4}(z\bar z + w\bar w)^{g-2} w\bar w \tag{1.4}$$

$$\sum_{k=1}^{N+1} \frac{\partial f_k}{\partial z}\frac{\partial f_k}{\partial w} = \frac{C}{4}(z\bar z + w\bar w)^{g-2}\,\bar z\,\bar w \tag{1.5}$$

$$\sum_{k=1}^{N+1} \frac{\partial f_k}{\partial \bar{z}} \frac{\partial f_k}{\partial w} = \frac{C}{4}(z\bar{z} + w\bar{w})^{g-2} z\bar{w} \tag{1.6}$$

Each of these equations is a polynomial equation which we can solve by equating coefficients. We thus obtain a set of quadratic equations in the coefficients of the polynomials.

Following [To2] we now explain the various rigidity properties of a minimal isometric immersion.

Definition 1.7. *A minimal isometric immersion $f : M^n \hookrightarrow S^m$ is said to be* **rigid** *if whenever $f' : M^n \hookrightarrow S^m$ is another minimal isometric immersion then $f' = A \circ f$ for some $A \in O(m+1)$.*

Remark. Note that immersions which differ by a precomposition with an isometry of the domain may lead to non-congruent immersions. For $M = S^n(1)$, the above notion of rigidity is consistent with the definition of the moduli space B_g by DoCarmo and Wallach. In this case, rigidity of f amongst degree g minimal isometric immersions implies that the codimension m- stratum within B_g collapses to a point. Here the codimension m- stratum is the set of congruence classes of degree g minimal isometric immersions from $S^n(1)$ to $S^m(r_g)$.

There is also a weaker notion of rigidity.

Definition 1.8. *An isometric immersion $f : M^n \hookrightarrow S^m$ is said to be* **linearly rigid** *if whenever there exists $A \in M(m+1, \mathbb{R})$ such that*

(1) *Image($A \circ \hat{f}$) $\subset S^m$ and*
(2) *$A \circ \hat{f} : M^n \hookrightarrow \mathbb{R}^{m+1}$ is an isometric immersion,*

then $A \in O(m+1)$. Here $\hat{f}$ denotes the corresponding map into $\mathbb{R}^{m+1}$.

Note that if f is a *minimal* isometric immersion, then rigidity of f implies linear rigidity. Also by Takahashi's theorem, (2) implies (1) for minimal isometric immersions. Hence to show that the fixed immersion $\tilde{F}$ is linearly rigid, we show that (2) of Definition 1.8 is satisfied.

We will now discuss a third notion of rigidity, namely infinitesimal rigidity. Let $f : M^n \hookrightarrow S^m$ be a minimal isometric immersion. Then one can show [To2, III.1.19] that if $t \to f_t$, $t \in \mathbb{R}$ is a variation of f through minimal isometric immersions then the vector field $v = \frac{\partial f_t}{\partial t}|_{t=0}$ is a Jacobi field along f. In fact [To2] shows that this is true for general harmonic maps $f : M^n \hookrightarrow S^m$. A natural example of such a variation is given by the action of the isometry group $SO(m+1)$, i.e. $f_t = \phi_t \circ f$ where $(\phi_t)_{t\in\mathbb{R}} \subset SO(m+1)$ is a one-parameter subgroup. When derived from such an "orthogonal" variation, the Jacobi field v along f has two properties. First, v is *projectable*, i.e. for x, $y \in M$, $f(x) = f(y)$ implies $v_x = v_y$. This follows from the fact that

$$v = \frac{\partial f_t}{\partial t}|_{t=0} = \dot{\phi}_t \circ f = X \circ f$$

where $X \in SO(m+1)$ is the Killing vector field on S^m induced by $(\phi_t)_{t\in\mathbb{R}} \subset SO(m+1)$. Second, the generalized divergence

$$\operatorname{div}_f v = \operatorname{trace}(< f_*, \nabla v >) \in C^\infty(M)$$

must vanish. Here $<,>$ denotes the inner product on M and ∇ the associated connection. To show this, one uses the property that for a Killing vector field X the derivation $A_X = L_X - \nabla_X$ is skew-symmetric with respect to the metric $<,>$.

Going back to the general case of $f : M^n \hookrightarrow S^m$ being a minimal isometric immersion we define $K(f)$ to be the vector space of all divergence free Jacobi fields along f, i.e.

Definition 1.9.

(1) *Let $K(f)$ be the set $\{v \,|\, v$ is a Jacobi field along f and $div_f v = 0\}$.*
(2) *Denote by $PK(f) \subset K(f)$ the linear subspace of projectable elements of $K(f)$.*

As our manifolds are compact, the vector space $K(f)$ will be finite dimensional. From the above argument in the case of the variation being given by actions of the isometry group $SO(m+1)$, we obtain $SO(m+1) \circ f \subseteq PK(f)$.

Definition 1.10. *A harmonic map $f : M^n \hookrightarrow S^m$ is called* **infinitesimally rigid** *if $SO(m+1) \circ f = PK(f)$. If f is not infinitesimally rigid it is called* **infinitesimally flexible**.

2. Proof of Theorem 1

Let $F : S^3(1) \hookrightarrow S^6(\frac{1}{4})$ be an arbitrary minimal isometric immersion of degree 6. Recall that there are $(g+1)^2$ harmonic polynomials of degree g. Thus each component F^k can be written as $F^k = \sum_{i=1}^{49} a_{k,i}\, \mathfrak{p}_i$ with $k = 1, \cdots, 7$ where the $\mathfrak{p}_i$ form a basis of the eigenspace. In order to study the infinitesimal behavior of $\tilde{F}$ we parametrize F and assign

$$\begin{aligned} F_t^k &\longleftrightarrow a_{k,i}(t) \\ \frac{\partial F_t^k}{\partial t} &\longleftrightarrow \frac{\partial a_{k,i}(t)}{\partial t}, \end{aligned}$$

where F_0 is the solution we already know, i.e. $F_0 = \tilde{F}$.

Substituting F_t into the partial differential equations for the isometry condition (1.1) to (1.6) we obtain quadratic equations in the coefficients $a_{k,i}(t)$. Taking the partial derivative of this system with respect to t at $t = 0$ then leads to a linear system of equations. Our first goal is to compute the dimension of this linear system.

Definition 2.1. *Denote by L the solution space of the linear system obtained by taking the derivative with respect to t at $t = 0$ of the system of quadratic equations in $a_{k,i}(t)$ associated to the parametrized isometry condition* (1.1) *to* (1.6).

Proposition 2.2. *The vector space L is 22-dimensional.*

Proof. As in [E1] and [E2] we pick a basis for the vector space of harmonic homogeneous polynomials of degree 6 in two complex variables. With this basis we can describe the components of the function F as:

$$\begin{aligned}
F_t^k = \; & a_{k,1}(t)\,\bar w^6 + a_{k,2}(t)\,\bar z\,\bar w^5 + a_{k,3}(t)\,\bar z^2\,\bar w^4 + a_{k,4}(t)\,\bar z^3\,\bar w^3 + a_{k,5}(t)\,\bar z^4\,\bar w^2 + \\
& a_{k,6}(t)\,\bar z^5\,\bar w + a_{k,7}(t)\,\bar z^6 + a_{k,8}(t)\,(w\,\bar w^5 - 5\,z\,\bar z\,\bar w^4) + \\
& a_{k,9}(t)\,(w\,\bar z\,\bar w^4 - 2\,z\,\bar z^2\,\bar w^3) + a_{k,10}(t)\,(w\,\bar z^2\,\bar w^3 - z\,\bar z^3\,\bar w^2) + \\
& a_{k,11}(t)\,(w\,\bar z^3\,\bar w^2 - \frac{1}{2}\,z\,\bar z^4\,\bar w) + a_{k,12}(t)\,(w\,\bar z^4\,\bar w - \frac{1}{5}\,z\,\bar z^5) + \\
& a_{k,13}(t)\,w\,\bar z^5 + a_{k,14}(t)\,(w^2\,\bar w^4 - 8\,z\,w\,\bar z\,\bar w^3 + 6\,z^2\,\bar z^2\,\bar w^2) + \\
& a_{k,15}(t)\,(w^2\,\bar z\,\bar w^3 - 3\,z\,w\,\bar z^2\,\bar w^2 + z^2\,\bar z^3\,\bar w) + \\
& a_{k,16}(t)\,(w^2\,\bar z^2\,\bar w^2 - \frac{4}{3}\,z\,w\,\bar z^3\,\bar w + \frac{1}{6}\,z^2\,\bar z^4) + a_{k,17}(t)\,(w^2\,\bar z^3\,\bar w - \frac{1}{2}\,z\,w\,\bar z^4) + \\
& a_{k,18}(t)\,w^2\,\bar z^4 + a_{k,19}(t)\,(w^3\,\bar w^3 - 9\,z\,w^2\,\bar z\,\bar w^2 + 9\,z^2\,w\,\bar z^2\,\bar w - z^3\,\bar z^3) + \\
& a_{k,20}(t)\,(w^3\,\bar z\,\bar w^2 - 3\,z\,w^2\,\bar z^2\,\bar w + z^2\,w\,\bar z^3) + a_{k,21}(t)\,(w^3\,\bar z^2\,\bar w - z\,w^2\,\bar z^3) + \\
& a_{k,22}(t)\,w^3\,\bar z^3 + a_{k,23}(t)\,(w^4\,\bar z\,\bar w - 2\,z\,w^3\bar z^2) + a_{k,24}(t)\,w^4\bar z^2 + a_{k,25}(t)\,w^5\,\bar z + \\
& \bar a_{k,1}(t)\,w^6 + \bar a_{k,2}(t)\,z\,w^5 + \bar a_{k,3}(t)\,z^2\,w^4 + \bar a_{k,4}(t)\,z^3\,w^3 + \bar a_{k,5}(t)\,z^4\,w^2 + \\
& \bar a_{k,6}(t)\,z^5\,w + \bar a_{k,7}(t)\,z^6 + \bar a_{k,8}(t)\,(w^5\,\bar w - 5\,z\,w^4\,\bar z) + \\
& \bar a_{k,9}(t)\,(z\,w^4\,\bar w - 2\,z^2\,w^3\,\bar z) + \bar a_{k,10}(t)\,(z^2\,w^3\,\bar w - z^3\,w^2\,\bar z) + \\
& \bar a_{k,11}(t)\,(z^3\,w^2\,\bar w - \frac{1}{2}\,z^4\,w\,\bar z) + \bar a_{k,12}(t)\,(z^4\,w\,\bar w - \frac{1}{5}\,z^5\,\bar z) + \\
& \bar a_{k,13}(t)\,z^5\,\bar w + \bar a_{k,14}(t)\,(w^4\,\bar w^2 - 8\,z\,w^3\,\bar z\,\bar w + 6\,z^2\,w^2\,\bar z^2) + \\
& \bar a_{k,15}(t)\,(z\,w^3\,\bar w^2 - 3\,z^2\,w^2\,\bar z\,\bar w + z^3\,w\,\bar z^2) + \\
& \bar a_{k,16}(t)\,(z^2\,w^2\,\bar w^2 - \frac{4}{3}\,z^3\,w\,\bar z\,\bar w + \frac{1}{6}\,z^4\,\bar z^2) + \bar a_{k,17}(t)\,(z^3\,w\,\bar w^2 - \frac{1}{2}\,z^4\,\bar z\,\bar w) + \\
& \bar a_{k,18}(t)\,z^4\,\bar w^2 + \bar a_{k,20}(t)\,(z\,w^2\,\bar w^3 - 3\,z^2\,w\,\bar z\,\bar w^2 + z^3\,\bar z^2\,\bar w) + \\
& \bar a_{k,21}(t)\,(z^2\,w\,\bar w^3 - z^3\,\bar z\,\bar w^2) + \bar a_{k,22}(t)\,z^3\,\bar w^3 + \\
& \bar a_{k,23}(t)\,(z\,w\,\bar w^4 - 2\,z^2\,\bar z\,\bar w^3) + \bar a_{k,24}(t)\,z^2\,\bar w^4 + \bar a_{k,25}(t)\,z\,\bar w^5
\end{aligned} \tag{2.3}$$

As a next step we compute the partial derivatives $\dfrac{\partial F_t^k}{\partial z}$, $\dfrac{\partial F_t^k}{\partial w}$, $\dfrac{\partial F_t^k}{\partial \bar z}$, and $\dfrac{\partial F_t^k}{\partial \bar w}$ and substitute those into the isometry partial differential equations (1.1) to (1.6). In our case the range of F will be $S^6(\frac{1}{4}) \subset \mathbb{R}^7$, hence we assume that $N+1 = 7$. We obtain a system of polynomial equations in degree 6 which we solve by coefficient comparison. As in [E1] and [E2] we obtain a system of quadratic equations in the coefficients $a_{k,i}(t)$. This system consists of linear combinations of terms of the form $\sum_{k=1}^{7} a_{k,i}(t)\, a_{k,j}(t)$, see [E1] for more details. In fact, in [E1] we used the corresponding *linear* system in the variables $\sum_{k=1}^{7} a_{k,i}\, a_{k,j}$. As this linear system has real coefficients yet the variables $a_{k,i}(t)$ are complex numbers we obtain the following condition.

Condition 1. *Let $I = \{1, \cdots, 25\}$. Then the following hold:*

$$\sum_{k=1}^{7} a_{k,i}(t)\, a_{k,j}(t) = \sum_{k=1}^{7} \overline{a_{k,i}(t)}\ \overline{a_{k,j}(t)} \text{ for all } i, j \in I\,,$$
$$\sum_{k=1}^{7} a_{k,i}(t)\, \overline{a_{k,j}(t)} = \sum_{k=1}^{7} \overline{a_{k,i}(t)}\, a_{k,j}(t) \text{ for all } i, j \in I\,.$$

As the next step we differentiate the system of quadratic equations with respect to t at $t = 0$. We obtain a linear system in the $a'_{k,i}(0)$, as $\sum_{k=1}^{7} a_{k,i}\, a_{k,j}$ transforms to

$$\sum_{k=1}^{7} a'_{k,i}(0)\, a_{k,j}(0) + \sum_{k=1}^{7} a_{k,i}(0)\, a'_{k,j}(0)$$

and we substitute the values for $a_{k,i}(0)$ as the coefficients of the fixed immersion $\tilde{F}$. Using the basis as described above, the coefficients of $\tilde{F}$ are

$$a_{k,j}(0) = 0 \text{ for all } j \notin \{2, 6, 8, 12, 17, 21, 23\}\,;$$

$$a_{k,2}(0) = \begin{cases} \frac{\sqrt{6}}{8}\,, & \text{for } k = 1\,; \\ -\frac{\sqrt{6}}{8}\,\imath\,, & \text{for } k = 2\,; \\ 0\,, & \text{for } k \neq 1, 2\,; \end{cases} \qquad \text{and} \qquad a_{k,6}(0) = -a_{k,2}(0)\,.$$

$$a_{k,8}(0) = \begin{cases} -\frac{1}{8}\,, & \text{for } k = 3\,; \\ \frac{1}{8}\,\imath\,, & \text{for } k = 4\,; \\ 0\,, & \text{for } k \neq 3, 4\,; \end{cases} \qquad \text{and} \qquad a_{k,12}(0) = -5\, a_{k,8}(0)\,.$$

$$a_{k,17}(0) = \begin{cases} -\frac{\sqrt{10}}{4}\,, & \text{for } k = 5\,; \\ \frac{\sqrt{10}}{4}\,\imath\,, & \text{for } k = 6\,; \\ 0\,, & \text{for } k \neq 5, 6\,; \end{cases} \qquad \text{and} \qquad a_{k,23}(0) = \overline{a_{k,17}(0)}\,.$$

$$a_{k,21}(0) = \begin{cases} \frac{\sqrt{15}}{4}\,\imath\,, & \text{for } k = 7\,; \\ 0\,, & \text{for } k \neq 7\,. \end{cases}$$

Before we solve this linear system using a mathematical software package (Maple) we observe the following relation between taking derivatives and complex conjugation. Let $a_{k,i}(t) = Re(a_{k,i}(t)) + \imath\, Im(a_{k,i}(t)) = a^R_{k,i}(t) + \imath\, a^I_{k,i}(t)$. Then

$$\begin{aligned} \frac{\partial a_{k,i}(t)}{\partial t} &= \frac{\partial a^R_{k,i}(t)}{\partial t} + \imath\, \frac{\partial a^I_{k,i}(t)}{\partial t} \text{ and} \\ \overline{\frac{\partial a_{k,i}(t)}{\partial t}} &= \frac{\partial a^R_{k,i}(t)}{\partial t} - \imath\, \frac{\partial a^I_{k,i}(t)}{\partial t} \\ &= \frac{\partial (a^R_{k,i}(t) - \imath\, a^I_{k,i}(t))}{\partial t} \\ &= \frac{\partial (\overline{a}_{k,i}(t))}{\partial t} \end{aligned}$$

Hence we obtain a second condition which shows that the operations "complex conjugation" and "$\frac{\partial}{\partial t}|_{t=0}$" are commutative.

Condition 2. $\dfrac{\partial(\overline{a}_{k,i}(t))}{\partial t}|_{t=0} = \overline{\dfrac{\partial a_{k,i}(t)}{\partial t}|_{t=0}}$.

Using these two conditions in solving the linear system, one obtains that the dimension of the solution space is 22. □

Remarks.

(1) Using the same conditions and successive elimination of variables to reduce the system, one can also solve this system by hand. As the system of linear equations is quite large (175 complex variables) this is a lengthy calculation. The answer is, of course, the same.
(2) Even using mathematical software packages such as Mathematica or Maple causes some difficulties with this system of equations as both of these packages cannot handle symbolic complex conjugation. In order to avoid this problem we identified $\mathbb{C}$ with $\mathbb{R}^2$ and obtained a system in the real and imaginary parts of the original variables. The program and an explicit basis for the solution space is available at http://www.orst.edu/ escherc.
(3) Note that $dim(L) = 22 > dim(SO(7)) = 21$.

For the remainder of the proof of the theorem we use the following proposition by Toth [To2].

Proposition [To2, Ch. 5, 2.3]**.** *For any harmonic map* $f : M \hookrightarrow S^n$, $dim(PK(f)) \geq (n - \frac{r}{2})\,(r+1)$, *where* $r+1$ *is the dimension of the linear subspace* $span_{\mathbb{R}}(Im(f)) \subset \mathbb{R}^{n+1}$. *Furthermore equality holds if and only if* f *is infinitesimally rigid.*

Recall from the previous section (Definition 1.7) that $PK(f)$ is the vector space of projectable, divergence free Jacobi fields along f. Applying the proposition in our case $(n = 6\,,\, r = 6)$ we obtain $\dim(PK(f)) \geq 21$.

Proposition 2.3. *The solution space* L *is a subspace of the vector space* $PK(\tilde{F})$ *.*

Note that this concludes the proof of the theorem: as both L and $PK(\tilde{F})$ are finite dimensional vector spaces, Proposition 2.3 implies $\dim(PK(\tilde{F})) \geq \dim(L) = 22$ where the last equality follows from Proposition 2.2. But then the above proposition [To2, Ch. 5, 2.3] implies that $\tilde{F}$ cannot be infinitesimally rigid.

Proof of Proposition 2.3.

Recall the definition of the harmonic variation F_t of the fixed minimal isometric immersion $\tilde{F}$.

$$F_t \;=\; (\sum_{j=1}^{49} a_{1,j}(t)\,p_j, \cdots, \sum_{j=1}^{49} a_{7,j}(t)\,p_j)\,.$$

In the process of computing the dimension of L we formed the vector field $v = \dfrac{\partial F_t}{\partial t}|_{t=0}$. That v is a Jacobi field along $\tilde{F}$ follows directly from [To2, Proposition III.1.19]. There Toth proves that for a general harmonic map $f : M \to N$ between Riemannian manifolds M and N, the vector field $v = \dfrac{\partial f_t}{\partial t}|_{t=0}$ formed by a variation of f through harmonic maps is a Jacobi field along f. In our case we additionally require the maps F_t to be isometries. Hence we can describe the solution space L in the following way.

$$\begin{aligned} L &= \{v = \frac{\partial F_t}{\partial t}|_{t=0} \ : F_t \text{ as above and } F_t \text{ an isometry}\} \\ &= \{v = \frac{\partial F_t}{\partial t}|_{t=0} \ : F_t \text{ as above and } F_t \text{ is a solution to the partial differential} \\ &\qquad\qquad \text{equations [1.1] to [1.6] associated to the isometry condition.}\} \end{aligned}$$

Now let $v \in L$. Our goal is to show that $v \in PK(\tilde{F})$. We use the following necessary and sufficient condition for a vector field to belong to $K(f)$.

Proposition [V.2.5, To2]**.** *Let v be a vector field along $f : M \to S^n$, a harmonic map. Then*

$$v \in K(f) \Longleftrightarrow \triangle \hat{v} = 2\, e(f)\, \hat{v}\,,$$

where $e(f)$ is the energy density of f and $\hat{v}$ is the induced vector field $\hat{v} : M \to \mathbb{R}^{n+1}$ via the canonical identification $\hat{}: T(\mathbb{R}^{n+1}) \to \mathbb{R}^{n+1}$.

Note that in our case f has constant energy density, in particular $e(f) = \frac{\lambda}{2} \in \mathbb{R}$. Hence in this case the necessary and sufficient condition becomes

$$v \in K(f) \Longleftrightarrow \triangle \hat{v} = \lambda\, \hat{v}\,.$$

Therefore each component of $\hat{v}$ belongs to the eigenspace V_λ corresponding to the eigenvalue λ if and only if $v \in K(f)$. But the components of $\hat{v}$ are all eigenfunctions of the Laplacian on S^3 with respect to the same eigenvalue λ. Hence $v \in K(\tilde{F})$. The definition of $PK(f)$ directly implies that if f is an embedding, then $PK(f) = K(f)$. Since $\tilde{F}$ is an embedding [DZ], we are done.

This concludes the proof of Proposition 2.3 and hence the proof of the theorem. $\square$

3. Proof of Theorem 2

We need to show that for an arbitrary $A \in M(7,\mathbb{R})$ such that $A \circ \tilde{F}$ is an isometric immersion, we have $A \cdot A^T = \mathrm{Id}$. Using $S^3 \subset \mathbb{C}^2$ we obtain as coordinates of $A \circ \tilde{F}$: $A \circ \tilde{F}\,(z, w, \bar{z}, \bar{w}) = (a_{11}\,\tilde{F}_1 + \cdots a_{17}\,\tilde{F}_7, \cdots, a_{71}\,\tilde{F}_1 + \cdots a_{77}\,\tilde{F}_7)$ where $A = (a_{ij})_{i,j=1}^7$ and the $\tilde{F}_k$ denote the components of $\tilde{F}$. This is exactly how we constructed a general minimal isometric immersion in [E1,E2] except that we used complex coefficients. To translate to real coefficients, let $\{p_2, p_6, p_8, p_{12}, p_{17}, p_{21}, p_{23}\}$ be the basis polynomials as described in (2.3). Then the fixed immersion can be written as

$$
\begin{aligned}
\tilde{F}\,:\, S^3(1) &\hookrightarrow S^6(\frac{1}{4})\\
(z,w,\bar{z},\bar{w}) &\mapsto [\frac{\sqrt{6}}{8}\,(p_2+\bar{p}_2-p_6-\bar{p}_6),\frac{\sqrt{6}}{8\imath}\,(p_2-\bar{p}_2-p_6+\bar{p}_6),\\
&\quad -\frac{1}{8}\,(p_8+\bar{p}_8)+\frac{5}{8}(p_{12}+\bar{p}_{12}),-\frac{1}{8\imath}\,(p_8-\bar{p}_8)+\frac{5}{8\imath}(p_{12}-\bar{p}_{12}),\\
&\quad -\frac{\sqrt{10}}{4}\,(p_{17}+\bar{p}_{17})-\frac{\sqrt{10}}{8}(\bar{p}_{23}+p_{23}),\\
&\quad -\frac{\sqrt{10}}{4\imath}\,(p_{17}-\bar{p}_{17})-\frac{\sqrt{10}}{8\imath}(\bar{p}_{23}-p_{23}),-\frac{\sqrt{15}}{4}\,\imath\,(p_{21}-\bar{p}_{21})\,]\,.
\end{aligned}
$$

Composing with an arbitrary matrix

$$
\begin{pmatrix}
a_{11} & a_{12} & \dots & a_{17}\\
a_{21} & a_{22} & \dots & a_{27}\\
\vdots & \vdots & \ddots & \vdots\\
a_{71} & a_{72} & \dots & a_{77}
\end{pmatrix}
$$

with real entries we obtain

$$
\begin{aligned}
\tilde{F}\,:\, S^3(1) &\hookrightarrow S^6(\frac{1}{4})\\
(z,w,\bar{z},\bar{w}) &\mapsto [\frac{\sqrt{6}}{8}\,(a_{11}-\imath\,a_{12})\,p_2+\frac{\sqrt{6}}{8}\,(a_{11}+\imath\,a_{12})\,\bar{p}_2+\frac{\sqrt{6}}{8}\,(-a_{11}+\imath\,a_{12})\,p_6\\
&\quad +\frac{\sqrt{6}}{8}\,(-a_{11}-\imath\,a_{12})\,\bar{p}_6-\frac{1}{8}\,(a_{13}-\imath\,a_{14})\,p_8-\frac{1}{8}\,(a_{13}+\imath\,a_{14})\,\bar{p}_8\\
&\quad +\frac{5}{8}\,(a_{13}-\imath\,a_{14})\,p_{12}+\frac{5}{8}\,(a_{13}+\imath\,a_{14})\,\bar{p}_{12}-\frac{\sqrt{10}}{4}\,(a_{15}-\imath\,a_{16})\,p_{17}\\
&\quad -\frac{\sqrt{10}}{4}\,(a_{15}+\imath\,a_{16})\,\bar{p}_{17}-\frac{\sqrt{10}}{8}\,(a_{15}+\imath\,a_{16})\,p_{23}\\
&\quad -\frac{\sqrt{10}}{8}\,(a_{15}-\imath\,a_{16})\,\bar{p}_{23}-\frac{\sqrt{15}}{4}\,\imath\,a_{17}\,p_{21}+\frac{\sqrt{15}}{4}\,\imath\,a_{17}\bar{p}_{21}\,,\;\cdots,\\
&\quad \frac{\sqrt{6}}{8}\,(a_{71}-\imath\,a_{72})\,p_2+\frac{\sqrt{6}}{8}\,(a_{71}+\imath\,a_{72})\,\bar{p}_2+\frac{\sqrt{6}}{8}\,(-a_{71}+\imath\,a_{72})\,p_6\\
&\quad +\frac{\sqrt{6}}{8}\,(-a_{71}-\imath\,a_{72})\,\bar{p}_6-\frac{1}{8}\,(a_{73}-\imath\,a_{74})\,p_8-\frac{1}{8}\,(a_{73}+\imath\,a_{74})\,\bar{p}_8\\
&\quad +\frac{5}{8}\,(a_{73}-\imath\,a_{74})\,p_{12}+\frac{5}{8}\,(a_{73}+\imath\,a_{74})\,\bar{p}_{12}-\frac{\sqrt{10}}{4}\,(a_{75}-\imath\,a_{76})\,p_{17}\\
&\quad -\frac{\sqrt{10}}{4}\,(a_{75}+\imath\,a_{76})\,\bar{p}_{17}-\frac{\sqrt{10}}{8}\,(a_{75}+\imath\,a_{76})\,p_{23}\\
&\quad -\frac{\sqrt{10}}{8}\,(a_{75}-\imath\,a_{76})\,\bar{p}_{23}-\frac{\sqrt{15}}{4}\,\imath\,a_{77}\,p_{21}+\frac{\sqrt{15}}{4}\,\imath\,a_{77}\bar{p}_{21}]\,.
\end{aligned}
$$

Hence the complex coefficients translate to real coefficients in the following way.

$$
\begin{aligned}
u_{k2} &= \frac{\sqrt{6}}{8}\,(a_{k1} - \imath\, a_{k2})\,;\; u_{k6} = \frac{\sqrt{6}}{8}\,(-a_{k1} + \imath\, a_{k2})\,;\\
u_{k8} &= -\frac{1}{8}\,(a_{k3} - \imath\, a_{k4})\,;\; u_{k12} = \frac{5}{8}\,(a_{k3} - \imath\, a_{k4})\,;\\
u_{k17} &= -\frac{\sqrt{10}}{4}\,(a_{k5} - \imath\, a_{k6})\,;\; u_{k23} = -\frac{\sqrt{10}}{8}\,(a_{k5} + \imath\, a_{k6})\,;\\
u_{k21} &= -\frac{\sqrt{15}}{4}\,\imath\, a_{k7}\,;
\end{aligned}
\tag{3.1}
$$

Using our algorithm for existence, see [E1,E2], for a general immersion with components $F_k = \sum_{i\in I} u_{ki}\, p_i$ where $I = \{2, 6, 8, 12, 17, 21, 23\}$ we obtain

$$
\begin{aligned}
&\sum_{k=1}^{7} u_{ki}\, u_{kj} = \sum_{k=1}^{7} \bar{u}_{ki}\, \bar{u}_{kj} = 0 \;\text{ for all }\; (i,j) \neq (17,23), (21,21)\,;\\
&\sum_{k=1}^{7} u_{ki}\, \bar{u}_{kj} = \sum_{k=1}^{7} \bar{u}_{ki}\, u_{kj} = 0 \;\text{ for all }\; (i,j) \neq (2,6), (8,12) \;\text{ and }\; i \neq j\,;\\
&\sum_{k=1}^{7} u_{k2}\, \bar{u}_{k2} = \sum_{k=1}^{7} u_{k6}\, \bar{u}_{k6} = \frac{3}{16}\,;\; \sum_{k=1}^{7} u_{k8}\, \bar{u}_{k8} = \frac{1}{25}\; \sum_{k=1}^{7} u_{k12}\, \bar{u}_{k12} = \frac{1}{32}\,;\\
&\sum_{k=1}^{7} u_{k17}\, \bar{u}_{k17} = 4 \sum_{k=1}^{7} u_{k23}\, \bar{u}_{k23} = \frac{5}{4}\,;\; \sum_{k=1}^{7} u_{k21}\, \bar{u}_{k21} = \frac{15}{16}\\
&\sum_{k=1}^{7} u_{k2}\, \bar{u}_{k6} = \frac{1}{5} \sum_{k=1}^{7} {u_{k21}}^2\,;\; \sum_{k=1}^{7} u_{k8}\, \bar{u}_{k12} = \frac{1}{6} \sum_{k=1}^{7} {u_{k21}}^2\,;\\
&\sum_{k=1}^{7} u_{k17}\, u_{k23} = -\frac{2}{3} \sum_{k=1}^{7} {u_{k21}}^2\,.
\end{aligned}
\tag{3.2}
$$

Translating equations (3.2) into real equations using relations (3.1) yields

$$
\begin{aligned}
&\sum_{k=1}^{7} {a_{ki}}^2 = 1 \;\text{ for all }\; i = 1\,, \cdots, 7\,;\\
&\sum_{k=1}^{7} a_{ki}\, a_{kj} = 0 \;\text{ for all }\; i \neq j\,.
\end{aligned}
$$

But this implies that $A^T \cdot A = \mathrm{Id}$, hence $A \in O(7)$.

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