

ON INHOMOGENEOUS SPHERICAL SPACE FORMS

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ABSTRACT.

Although much is known about minimal isometric immersions into spheres of *homogeneous* spherical space forms, up to very recently, [E], there were no results in the literature about such immersions in the dominant case of inhomogeneous space forms. For a large class of these, the pq-space forms, we extend the results of [E] to a necessary and sufficient condition for the existence of such an immersion of a given degree. This condition depends only upon the degree and the fundamental group of the space form and is given in terms of an explicitly computable function. We are thus able to construct the first known minimal isometric immersion of an inhomogeneous lens space into a sphere. Moreover, we give a global lower bound for the degree of a minimal isometric immersion of a pq-space form into a sphere. In doing so, we discover several additional conditions for the existence of a such an immersion relating the degree of the immersion and the order of the fundamental group.

1. INTRODUCTION

Many authors ([C],[DW2],[L],[T],[To]) have studied minimal isometric immersions of spherical space forms into spheres. The first fundamental result was a theorem by T. Takahashi [T] in 1966 which establishes a necessary and sufficient condition for the *minimality* of an isometric immersion of a compact Riemannian manifold into a sphere: all components of the immersion f are eigenfunctions of the Laplace operator on M with respect to the same eigenvalue.

Therefore the main idea in constructing minimal isometric immersions of a manifold M into a sphere is to find eigenvalues of the Laplacian on M of sufficiently high multiplicity in order to provide the coordinate functions of the immersions. In the case of M being a sphere the eigenfunctions of the Laplacian are simply harmonic homogeneous polynomials of a fixed degree.

Following this result, one main interest was to analyze possible images under such minimal isometric immersions. Several authors ([DW1],[DW2],[L],[T],[To]) have studied the moduli space of all such immersions, see the introduction of [E] for a detailed description of this development. However, the question remained of just which spherical space forms *do*

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admit minimal isometric immersions into spheres. For the case of the *homogeneous* spherical space forms D. DeTurck and W. Ziller [DZ] showed that in fact all such space forms occur as images of minimal isometric immersions. However, the classification shows that very few space forms are actually homogeneous, see [W, Theorem 2.7.1] for a description. The dominant case of *inhomogeneous* space forms is discussed for the first time in [E]. We were able to give a necessary condition for the existence of minimal isometric immersions of a large class of inhomogeneous space forms into spheres.

The task of this work is to extend these results by proving two conjectures made in [E]. There we introduced a new technique whose various implications will be used in what follows. We will give a necessary *and* sufficient condition for the existence of a minimal isometric immersion of the same large class of inhomogeneous spherical space forms into spheres. Thus we confirm Conjecture 2 of [E]. It turns out that this condition is explicitly computable. We also give a global lower bound on the degree of such an immersion, hence confirming Conjecture 1 of [E].

In [E] the following notation was introduced:

Notation. Denote by $L(p, q)$ the three dimensional lens space generated by the standard action of \mathbb{Z}_p on S^3 . Such a lens space is inhomogeneous if and only if $q \not\equiv \pm 1 \pmod{p}$. We call an inhomogeneous space form a *pq-space form* if it can be covered by some inhomogeneous lens space $L(p, q)$. In [E] we show that virtually all inhomogeneous spherical space forms are in fact pq-space forms ([E, Theorem 1]).

We also introduced $\mathfrak{U}(p, q)$, a function of two natural numbers p and q with image in the natural numbers depending on the image $L(p, q)$ of the immersion which is defined as follows:

Definition 1.1. Fix an inhomogeneous lens space $L(p, q)$ and a degree g . As explained in [E], associate with the isometry condition of a minimal immersion of $L(p, q)$ into a sphere an inhomogeneous linear system $Ax = b$ where $A \in M(6g - 4 \times n, \mathbb{R})$ and $b \in M(6g - 4 \times 1, \mathbb{R})$, $b \neq 0$. The number of equations derived by equating coefficients in the isometry polynomial equations is $6g - 4$ and n is the number of polynomials invariant under the appropriate group action. Define $m(g, p, q)$ to be the number of non-zero rows after row reducing A and $n(g, p, q) := n$. Let

$$\mathfrak{U}(p, q) := \inf_{g > 0} \{g : m(g, p, q) - n(g, p, q) \leq 0\}$$

Using these definitions, the main theorem of this previous paper can be formulated as:

Theorem [E]. If $g < \mathfrak{U}(p, q)$, then no three dimensional pq-space form admits a minimal isometric immersion of degree g into a sphere.

In this work we are able to extend the above result to a necessary and sufficient condition:

Theorem 1. There exists a minimal isometric immersion of degree g of a three dimensional inhomogeneous lens space $L(p, q)$ into a sphere if and only if $g \geq \mathfrak{U}(p, q)$ and the linear system associated to the isometry condition admits a nonnegative solution.

In order to generalize this theorem to the class of pq-space forms we need the following definition:

Definition 1.2. *Let G be a subgroup of $SO(4)$. A solution of the linear system associated to the isometry condition is called a G - solution if it contains the trivial solution for all variables corresponding to polynomials not invariant under the action of G .*

Corollary. *There exists a minimal isometric immersion of degree g of a three dimensional pq-space form, S^3/G , into a sphere if and only if $g \geq \mathfrak{U}(p, q)$ and the linear system associated to the isometry condition of the lens space $L(p, q)$ admits a nonnegative G - solution.*

Proof. By definition, a pq-space form is covered by an inhomogeneous lens space $L(p, q)$. Using a nonnegative G - solution of the linear system associated to $L(p, q)$ we first construct a minimal isometric immersion of $L(p, q)$ into some sphere as in Theorem 1. The additional requirement of being a G - solution implies that the immersion factors through S^3/G . \square

Remark. We were able to compute the condition of Theorem 1 in the case of $p = 8$ and $q = 3$. We obtained that $\mathfrak{U}(8, 3) = 20$, hence the degree g of the immersion must be greater or equal than 20. However, using a procedure of the Simplex method, we found the first nonnegative solution of the linear system for degree $g = 32$. Using the proof of Theorem 1, we can explicitly describe a minimal isometric immersion of $L(8, 3)$ into S^{190} of degree 32. To the author's knowledge, this is the first known example of a minimal isometric immersion of an inhomogeneous space form into a sphere.

We also obtain a global bound for the existence of such immersions:

Theorem 2.

- (1) *If p is odd, then for all degrees g less than 28, no pq-space form admits a minimal isometric immersion of degree g into a sphere.*
- (2) *If p is even, then for all degrees g less than 20, no pq-space form admits a minimal isometric immersion of degree g into a sphere.*

The proof of Theorem 2 heavily relies on the following theorem of a more general nature. In this theorem we describe several necessary conditions for the existence of a minimal isometric immersion of $L(p, q)$ into a sphere which portray an intricate interplay between the degree of the immersion and the order of the fundamental group of the lens space $L(p, q)$.

Theorem 3. *If $L(p, q)$ is minimally isometrically immersed into a sphere by a map of degree g , then the following hold:*

- (1) $g \geq p$ for any q ;
- (2) *If p is even, then g must be even;*
- (3) *If g is even, then p must be even or $g \geq 2p$;*

- (4) If g is odd, then p must be odd and $g \geq 3p$ unless $p^2 = \frac{g(g+2)}{3}$;
- (5) If p is odd, q is even and g is even, then $g > 3p$ unless $g^2 + 2g - 2 - 6p^2 + 2q^2 = 0$;
- (6) If p is odd, q is odd and g is even, then $g \geq 4p$ unless $g^2 + 2g - 2 - 12p^2 + 2q^2 = 0$.

Remarks.

- (1) Theorems 1 and 2 were conjectured in [E].
- (2) In the course of proving Theorem 2, we will establish several properties of the function \mathfrak{U} which may be of independent interest (Propositions 4.1.11, 4.1.12). In particular we study how the function $\mathfrak{U}(p, q)$ relates to the degree g of the immersion.
- (3) The proof of Theorem 3 in section 4 consists of proving Propositions 4.1.1, 4.1.2 and 4.1.5.

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2. PROOF OF THEOREM 1

2.1 PRELIMINARIES

In order to *minimally* isometrically immerse any space form into a sphere we need the components of the immersion to consist of harmonic homogeneous polynomials of a given degree in four real variables or alternatively in two complex variables [T]. Following the arguments of section 4.1 in [E], we obtain that the isometry condition translates into the following system of partial differential equations :

$$(2.1.1) \quad \sum_{k=1}^{N+1} \left(\frac{\partial f_k}{\partial z} \right)^2 = \frac{C}{4} (z\bar{z} + w\bar{w})^{g-2} \bar{z}^2$$

$$(2.1.2) \quad \sum_{k=1}^{N+1} \left(\frac{\partial f_k}{\partial w} \right)^2 = \frac{C}{4} (z\bar{z} + w\bar{w})^{g-2} \bar{w}^2$$

$$(2.1.3) \quad \sum_{k=1}^{N+1} \frac{\partial f_k}{\partial z} \frac{\partial f_k}{\partial \bar{z}} = \frac{1}{2} (z\bar{z} + w\bar{w})^{g-1} + \frac{C}{4} (z\bar{z} + w\bar{w})^{g-2} z\bar{z}$$

$$(2.1.4) \quad \sum_{k=1}^{N+1} \frac{\partial f_k}{\partial w} \frac{\partial f_k}{\partial \bar{w}} = \frac{1}{2} (z\bar{z} + w\bar{w})^{g-1} + \frac{C}{4} (z\bar{z} + w\bar{w})^{g-2} w\bar{w}$$

$$(2.1.5) \quad \sum_{k=1}^{N+1} \frac{\partial f_k}{\partial z} \frac{\partial f_k}{\partial w} = \frac{C}{4} (z\bar{z} + w\bar{w})^{g-2} \bar{z} \bar{w}$$

$$(2.1.6) \quad \sum_{k=1}^{N+1} \frac{\partial f_k}{\partial \bar{z}} \frac{\partial f_k}{\partial w} = \frac{C}{4} (z\bar{z} + w\bar{w})^{g-2} z \bar{w}$$

Here the f_k denote the components of $f : S^3 \hookrightarrow S^N$ as a function of two complex variables, g stands for the degree of the polynomials used and C is the constant $C = \frac{3g}{g+2} - 1$.

In our case we want to construct a minimal isometric immersion of an inhomogeneous lens space into a sphere. Therefore we use the above mentioned construction but additionally require the harmonic polynomials to be invariant under the corresponding group action.

Consider the following action of $G \cong \mathbb{Z}_p$ on $S^3 \subset \mathbb{C}^2$. Let $\omega = e^{\frac{2\pi}{p}i}$ be a generator of \mathbb{Z}_p . Then the group action is given by $\omega \cdot (z, w) = (e^{\frac{2\pi}{p}i} z, e^{\frac{2\pi q}{p}i} w)$ and the resulting quotient space S^3/G is the lens space $L(p, q)$. In the above complex notation the action of \mathbb{Z}_p on $\mathfrak{p} \in \mathbb{C}[z, w]$ is given by

$$\begin{aligned} \omega \cdot \mathfrak{p}(z, w, \bar{z}, \bar{w}) &= \mathfrak{p}(\omega z, \omega^q w, \bar{\omega} \bar{z}, \bar{\omega}^q \bar{w}) \\ &= \mathfrak{p}(e^{\frac{2\pi}{p}i} z, e^{\frac{2\pi q}{p}i} w, e^{-\frac{2\pi}{p}i} \bar{z}, e^{-\frac{2\pi q}{p}i} \bar{w}). \end{aligned}$$

Hence in order for a monomial $z^a w^b \bar{z}^c \bar{w}^d$ to be invariant under this action, we need that p divides $a + qb - c - qd$. Moreover, a set of polynomials is invariant under this particular action if and only if the corresponding set of generating monomials is invariant under the action.

Conclusion. *A set of harmonic homogeneous polynomials is invariant under the above described action if and only if all generating monomials $z^a w^b \bar{z}^c \bar{w}^d$ satisfy the condition :*

$$a + qb - c - qd \equiv 0 \pmod{p}.$$

Notation. Whenever we speak of the ‘‘action of \mathbb{Z}_p generating $L(p, q)$ ’’ we mean the fixed action of \mathbb{Z}_p described above which induces the inhomogeneous lens space $L(p, q)$ as image of the immersion.

As in [E], we use a fixed basis of the $(g+1)^2$ -dimensional vector space of all harmonic homogeneous polynomials of degree g . In this basis we can express the components of f in the following manner:

$$f_k = \sum_{i=1}^{n_g} a_{ki} \mathfrak{p}_i + \bar{a}_{ki} \bar{\mathfrak{p}}_i, \quad \text{with } \mathfrak{p}_i \in P^g \quad \text{and} \quad n_g = \begin{cases} \frac{(g+1)^2}{2} & \text{if } g \text{ odd} \\ \frac{(g+1)^2+1}{2} & \text{if } g \text{ even} \end{cases}$$

Remark. To simplify the notation we are using the above more general expression for the components f_k for both cases of g even and g odd. In the case of g even, one of the polynomials \mathfrak{p}_i has the property that $\mathfrak{p}_i = \bar{\mathfrak{p}}_i$. In that case the real coefficient a_{ki} is used only once without its complex conjugate. For more details, see section 4 in [E].

Here we use the same notation as in [E]:

Definition 2.1.7.

$$\begin{aligned}
M^g &:= \{m_i \mid m_i \text{ all monomials in two complex variables of degree } g\} \\
&= \{z^a w^b \bar{z}^c \bar{w}^d \mid a + b + c + d = g\} \\
M_G^g &:= \{m_i \mid m_i \text{ all monomials of degree } g \text{ invariant under action of } G \cong \mathbb{Z}_p \text{ as above}\} \\
&= \{z^a w^b \bar{z}^c \bar{w}^d \mid a + b + c + d = g, a + bq - c - dq \equiv 0 \pmod{p}\} \\
P^g &:= \{\mathfrak{p}_i \mid \mathfrak{p}_i \text{ a harmonic homogeneous polynomials of degree } g, \\
&\quad i = 1, \dots, (g+1)^2\} \\
P_G^g &:= \{\mathfrak{p}_i \mid \mathfrak{p}_i \text{ a harmonic homogeneous polynomials of degree } g \text{ invariant under} \\
&\quad \text{the action of } G \cong \mathbb{Z}_p\} \\
n &:= \dim(P_G^g)
\end{aligned}$$

Note that from here on, we will always use the above mentioned fixed basis for P^g and P_G^g . Each of the original set of equations (2.1.1) to (2.1.6) is a polynomial equation which we can solve by equating coefficients. We thus obtain a set of quadratic equations in the coefficients a_{ki} . Let

$$\begin{aligned}
AA_{ij} &:= \sum_{k=1}^{N+1} a_{ki} a_{kj}; \quad A\bar{A}_{ij} := \sum_{k=1}^{N+1} a_{ki} \bar{a}_{kj} \\
\mathfrak{K} &:= \{AA_{ij}, A\bar{A}_{ij} \mid i \leq j, i = 1, \dots, n_g\}.
\end{aligned}$$

Consider the system of quadratic equations in a_{ki} as a linear system in elements of \mathfrak{K} . This linear system has the property that it splits into an inhomogeneous and a homogeneous system in a natural way:

Lemma 2.1.8 [E]. *The linear system in elements of \mathfrak{K} splits into an inhomogeneous system in the variables $A\bar{A}_{ii}$ and a homogeneous system in $\mathfrak{K} \setminus \{A\bar{A}_{ii} \mid i = 1, \dots, n_g\}$.*

This inhomogeneous linear system is the one used in the definition of the function $\mathfrak{U}(p, q)$.

2.2 PROOF OF THEOREM 1

One direction of the theorem follows from [E]:

\Leftarrow : Assume there exists a minimal isometric immersion of degree g of a three dimensional inhomogeneous lens space $L(p, q)$ into a sphere. The Theorem mentioned in the introduction implies that then $g \geq \mathfrak{U}(p, q)$. But we actually know more about the solutions of the isometry equations: If there exists such an immersion, then the isometry equations and hence the inhomogeneous part of the associated linear system has a solution. But this solution must clearly be nonnegative as the variables $\sum_{k=1}^n a_{ki} \bar{a}_{ki}$ are sums of real squares.

We now prove

\implies : Assume that $g \geq \mathfrak{U}(p, q)$ and that the linear system admits a nonnegative solution. Using this solution of the linear system, we construct a minimal isometric immersion: We can discard those variables $A\bar{A}_{ii}$ with $A\bar{A}_{ii} = 0$, as $A\bar{A}_{ii} = 0$ implies that $a_{kj} = 0$ for all k and hence the corresponding polynomial is not to be used in the construction of the immersion f . Recall that n is the number of polynomials invariant under the group action generating the inhomogeneous lens space $L(p, q)$. Denote by n_+ the number of strictly positive variables and let

$$\{A\bar{A}_{ii} = c_i \quad \text{for positive real numbers } c_i \in \mathbb{R}, i = 1, \dots, n_+\}$$

be a positive solution to the linear system. As stated in Lemma 2.1.8 the linear system splits into an inhomogeneous linear system in the variables $A\bar{A}_{ii}$ and a homogeneous system in the remaining variables. To construct the immersion f , we pick the trivial solution of the homogeneous system. Then let

$$\begin{cases} a_{l,j} &= \sqrt{c_j} \\ a_{l+1,j} &= \iota\sqrt{c_j} \\ a_{l,k} &= 0 \quad \text{for all } k \neq l, l+1, \end{cases}$$

where $j = 1, \dots, n_+$ and $l = 1, 3, \dots, 2n_+ - 1$ and $\iota = \sqrt{-1} \in \mathbb{C}$. The immersion f constructed in this way has the property that $A\bar{A}_{ii} = c_i$ for all i and both $AA_{ij} = 0$ for all i, j and $A\bar{A}_{ij} = 0$ for all $i \neq j$. Hence f is a solution to the isometry equations and by construction its components consist of eigenfunctions of the Laplacian invariant under the appropriate group action. It follows that f is a minimal isometric immersion $f : L(p, q) \hookrightarrow S^{2n}$.

This completes the proof of Theorem 1. \square

3. APPLICATION OF THEOREM 1

Proposition 3.1. *There exists a minimal isometric immersion of a three-dimensional inhomogeneous spherical space form into a sphere.*

Proof. We prove this Proposition by directly and explicitly constructing such an immersion. In doing so, we use the necessary and sufficient condition of Theorem 1. Note that by Theorem 1 and the proof of Theorem 1, we know that if we are able to find a nonnegative solution to the linear system, we can explicitly construct a minimal isometric immersion. In the case of the inhomogeneous lens space $L(8, 3)$, we computed $\mathfrak{U}(8, 3)$ to be 20 and found a nonnegative solution of the linear system in degree 32. This solution is quite large and is available at <http://www.orst.edu/~escher/Research/Program>. For this solution, $n = 95$, hence we obtain a minimal isometric immersion of $L(8, 3)$ into S^{190} .

Remark. Calculations show that there are no nonnegative solutions in smaller degrees for this particular example, hence $g = 32$ is the smallest possible degree for a minimal isometric immersion of $L(8, 3)$ into a sphere. Note, however, that the codimension of the sphere (190) might not be optimized by this particular immersion.

4. PROOF OF THEOREMS 2 AND 3

In this section, we provide the proofs of Theorem 2 and Theorem 3. The proof of Theorem 3 is given by Propositions 4.1.1, 4.1.2 and 4.1.5.

4.1 GROUP ACTIONS AND ISOMETRIES

To prove Theorem 2, we need several propositions concerning the group action generating the inhomogeneous lens spaces and the system of partial differential equations describing the isometry condition. These propositions portray an interesting interplay between the order of the group and the degree of the immersion.

Proposition 4.1.1. *If $p > g$, then there does not exist a degree g minimal isometric immersion of $L(p, q)$ into a sphere for any q .*

Proof. Assume $p > g$. We know by [T] that every minimal isometric immersion of $L(p, q)$ into a sphere must have the property that all of its components consist of harmonic homogeneous polynomials invariant under the corresponding group action. We show that a map whose components consist of such polynomials can not satisfy the isometry equations (2.1.1) to (2.1.6) under the hypothesis.

Case 1 g odd.

As in [E] we use the representation of polynomials as pairs $(a - c, b - d) = (l, m)$: We chose a basis for the vector space of all harmonic homogeneous polynomials in two complex variables such that $a - c$ and $b - d$ is constant for all monomials $z^a w^b \bar{z}^c \bar{w}^d$ in each basis polynomial. As well, $l + m$ is odd if and only if g is odd and $l + m$ is even if and only if g is even. Moreover $l + m \leq g$. Also note that the polynomial $(-l, -m)$ corresponds to the complex conjugate of the polynomial (l, m) .

We use the first of the isometry equations (2.1.1) and study the coefficients of $z^{g-2} \bar{z}^g$. This monomial has non-zero coefficients on the right hand side of the equation. We show that the corresponding coefficients on the left hand side vanish under the assumption that $p > g$. Therefore there is no solution to (2.1.1) and hence none to the isometry equations.

On the left hand side, all monomials $z^a w^b \bar{z}^c \bar{w}^d$ of the immersion which lead to $z^{g-2} \bar{z}^g$ in (2.1.1) have the property that $b - d = 0$:

Let $f_k = \sum_{i=1}^{n_g} a_{ki} \mathfrak{p}_i$. Then $\frac{\partial f_k}{\partial z} = \sum_{i=1}^{n_g} a_{ki} \frac{\partial \mathfrak{p}_i}{\partial z}$. Using $\mathfrak{p}_i = \sum_{s=1}^{t_g} \lambda_{is} m_s$, $(\frac{\partial f_k}{\partial z})^2$ consists of linear combinations of monomials $\frac{\partial m_r}{\partial z} \cdot \frac{\partial m_s}{\partial z}$. Hence all monomials on the left hand side of (2.1.1) are of the form $\frac{\partial m_r}{\partial z} \cdot \frac{\partial m_s}{\partial z}$ for some r and s . Let $m_r = z^{a_r} w^{b_r} \bar{z}^{c_r} \bar{w}^{d_r}$ and $m_s = z^{a_s} w^{b_s} \bar{z}^{c_s} \bar{w}^{d_s}$. As we are looking for monomials m_r and m_s such that

$$\frac{\partial m_r}{\partial z} \cdot \frac{\partial m_s}{\partial z} = z^{g-2} \bar{z}^g,$$

we need that

$$\begin{cases} a_r + a_s - 2 = g - 2 \\ b_r + b_s = 0 \\ c_r + c_s = g \\ d_r + d_s = 0. \end{cases}$$

All of the above variables are nonnegative integers, as they are powers of monomials. Hence we obtain $b_r = b_s = d_r = d_s = 0$ and therefore $b_r - d_r = 0$ and $b_s - d_s = 0$ for all monomials m_r and m_s with the property that

$$\frac{\partial m_r}{\partial z} \cdot \frac{\partial m_s}{\partial z} = z^{g-2} \bar{z}^g.$$

But this means that the only polynomials appearing on the left hand side are of the form $(l, m) = (\pm 1, 0), (\pm 3, 0), \dots, (\pm g, 0)$ as g was assumed to be odd. Now, recall from section 2.1 that if a polynomial (l, m) is invariant under the action of \mathbb{Z}_p to generate the lens space $L(p, q)$, then l and m satisfy $l + qm \equiv 0 \pmod{p}$. In our case $l = 0$ and we obtain $m \equiv 0 \pmod{p}$ with $m = \pm 1, \pm 3, \dots, \pm g$. However, we assumed that $p > g$ which implies that none of these polynomials are invariant under the group action. Therefore none of the invariant polynomials appear on the left hand side of (2.1.1) corresponding to $z^{g-2} \bar{z}^g$ in the coefficient comparison. Hence equation (2.1.1) and, therefore, the whole system is insoluble for the case of the lens spaces $L(p, q)$.

Case 2 g even

Repeat the same argument. As g is even, the only polynomials which consist of monomials m_r and m_s with

$$\frac{\partial m_r}{\partial z} \cdot \frac{\partial m_s}{\partial z} = z^{g-2} \bar{z}^g$$

are of the form $(l, m) = (0, 0), (\pm 2, 0), \dots, (\pm g, 0)$. As $p > g$ only the real polynomial $(0, 0)$ is invariant under the group action: $m \equiv 0 \pmod{p}$ with $m = 0, \pm 2, \dots, \pm g$ and $g < p$. Using our fixed basis this real polynomial can be described as

$$\mathfrak{r} = d_{k1} \left(w^{\frac{g}{2}} \bar{w}^{\frac{g}{2}} - \frac{g^2}{4} z w^{\frac{g}{2}-1} \bar{z} \bar{w}^{\frac{g}{2}-1} + \dots - \frac{g^2}{4} z^{\frac{g}{2}-1} w \bar{z}^{\frac{g}{2}-1} \bar{w} + z^{\frac{g}{2}} \bar{z}^{\frac{g}{2}} \right)$$

Proceeding with the corresponding coefficient comparison, we obtain the equation $DD_{11} = \frac{C}{g^2}$ where $C = \frac{3g}{g+2} - 1$. But if we repeat the same argument for the coefficients of $z^{g-1} \bar{z}^{g-1}$ in equation (2.1.3) we again conclude that only the real polynomial is invariant under the group action and we obtain $DD_{11} = \frac{C}{g^2} + \frac{2}{g^2}$. But this is impossible and hence there is also no solution to the isometry equation in the case of g even. \square

Proposition 4.1.2. *Fix an inhomogeneous lens space $L(p, q)$.*

- (1) *If p is even and g is the degree of a minimal isometric immersion of $L(p, q)$ into some sphere, then g is even.*
- (2) *If the degree g of such an immersion is even, then either p is even or $g \geq 2p$.*
- (3) *If g is odd, then p is odd by (1) and $g \geq 3p$ unless $p^2 = \frac{g(g+2)}{3}$.*

Proof. (1) Let $z^a w^b \bar{z}^c \bar{w}^d \in M_G^g$. As this monomial is invariant under the group action generating the lens space $L(p, q)$, we obtain

$$a + qb - c - qd \equiv 0 \pmod{p}.$$

As this monomial is of degree g , we conclude

$$(*) \quad g + b(q - 1) - 2c - d(q + 1) \equiv 0 \pmod{p}.$$

But if p is even, then q must be odd as p and q are relatively prime and hence both $q + 1$ and $q - 1$ are even numbers. But then $(*)$ implies that g must be even.

(2) Let g be even. We need to show that $g \geq 2p$ in the case of p odd. Use the representation of all polynomials of degree g in terms of ordered pairs (l, m) as before. Then, as we saw earlier, $|l + m| \leq g$ and $l + m$ must be even as g was assumed to be even. Now consider polynomials of the form $(l, 0)$. They are invariant under the action of \mathbb{Z}_p generating the inhomogeneous lens space $L(p, q)$ if and only if $l \equiv 0 \pmod{p}$. Hence the polynomials invariant under the group action are $(l, m) = (0, 0), (\pm p, 0), (\pm 2p, 0), \dots$. But, p is odd, hence only the following polynomials are invariant: $(l, m) = (0, 0), (\pm 2p, 0), (\pm 4p, 0), \dots$. Hence for degrees $g < 2p$ the only polynomial of the form $(l, 0)$ invariant under the group action is the real polynomial $(0, 0)$. But, as we saw in (Case 2) of Proposition 4.1.1, the isometry equations are insoluble in this case.

(3) Assume both g and p are odd. As in (2), consider polynomials of the form $(l, 0)$. As g is odd, l must also be odd. Again such polynomials are invariant if and only if $l \equiv 0 \pmod{p}$. Hence the polynomials of the form $(l, 0)$ invariant under the group action are $(\pm p, 0), (\pm 3p, 0), (\pm 5p, 0), \dots$. Hence for degrees $g < 3p$ the only polynomial of the form $(l, 0)$ invariant under the group action is $(\pm p, 0)$. We show that there can be no solution to the isometry equations in this case. As in (Case 1) of Proposition 4.1.1, we use equations (2.1.1) and look for monomials m_r and m_s such that

$$\frac{\partial m_r}{\partial z} \cdot \frac{\partial m_s}{\partial z} = z^{g-2} \bar{z}^g.$$

Following the same argument, we obtain that the only polynomials appearing on the left hand side are of the form $(\pm 1, 0), (\pm 3, 0), \dots, (\pm g, 0)$ and as $g < 3p$, $(\pm p, 0)$ is the only invariant polynomial on the left hand side. Analyzing the coefficient comparison using (Case 1) of Proposition 4.1.1, we find that

$$\begin{cases} m_r = z^{a_r} \bar{z}^{c_r} & \text{with } a_r + c_r = g \quad \text{and} \\ m_s = z^{a_s} \bar{z}^{c_s} & \text{with } a_s + c_s = g. \end{cases}$$

As we are using only the inhomogeneous part of the linear system, we conclude that $m_r = \bar{m}_s = z^{c_s} \bar{z}^{a_s}$. Using that $a_s - c_s = p$ by the invariance condition, we obtain

$$\begin{cases} a_s = \frac{g+p}{2}, & a_r = \frac{g-p}{2}, \\ c_s = \frac{g-p}{2}, & c_r = \frac{g+p}{2}. \end{cases}$$

With λ denoting the coefficient of m_s in the polynomial $(p, 0)$, we obtain on the left hand side:

$$\begin{aligned} \lambda \frac{\partial m_r}{\partial z} \cdot \lambda \frac{\partial m_s}{\partial z} + \lambda \frac{\partial m_s}{\partial z} \cdot \lambda \frac{\partial m_r}{\partial z} &= 2\lambda^2 \frac{\partial m_r}{\partial z} \cdot \frac{\partial m_s}{\partial z} = 2\lambda^2 a_r a_s z^{a_r+a_s-2} \bar{z}^{c_r+c_s} \\ &= 2\lambda^2 \frac{(g^2 - p^2)}{4} z^{g-2} \bar{z}^g. \end{aligned}$$

Equating with the corresponding coefficients on the right hand side of (2.1.1) we obtain

$$2\lambda^2 \frac{(g^2 - p^2)}{4} \cdot A\bar{A}_{pp} = \frac{C}{4}$$

where $A\bar{A}_{pp} = \sum_{k=1}^{N+1} a_{kp} \bar{a}_{kp}$ and a_{kp} is the variable coefficient of the polynomial $(p, 0)$. Hence

$$(4.1.3) \quad A\bar{A}_{pp} = \frac{C}{g^2 - p^2} \cdot \frac{1}{2\lambda^2}.$$

Repeating the same argument with equations (2.1.3) we obtain

$$\begin{aligned} \lambda \frac{\partial m_r}{\partial z} \cdot \lambda \frac{\partial m_s}{\partial \bar{z}} + \lambda \frac{\partial m_s}{\partial z} \cdot \lambda \frac{\partial m_r}{\partial \bar{z}} &= \lambda^2 (a_r c_s + a_s c_r) z^{a_r+a_s-1} \bar{z}^{c_r+c_s-1} \\ &= \lambda^2 \left(\frac{(g+p)^2}{4} + \frac{(g-p)^2}{4} \right) z^{g-1} \bar{z}^{g-1} \\ &= \lambda^2 \frac{(g^2 + p^2)}{2} z^{g-1} \bar{z}^{g-1}. \end{aligned}$$

Again coefficient comparison yields

$$(4.1.4) \quad A\bar{A}_{pp} = \frac{C}{2\lambda^2(g^2 + p^2)} + \frac{1}{\lambda^2(g^2 + p^2)}.$$

Comparing (4.1.3) and (4.1.4) we obtain

$$\begin{aligned} \frac{C}{g^2 - p^2} \cdot \frac{1}{2\lambda^2} &= \frac{C}{2\lambda^2(g^2 + p^2)} + \frac{1}{\lambda^2(g^2 + p^2)} \\ \iff p^2 &= \frac{g(g+2)}{3} \quad \text{with} \quad C = \frac{3g}{g+2} - 1. \end{aligned}$$

This completes the proof of Proposition 4.1.2. \square

Remark. The condition $p^2 = \frac{g(g+2)}{3}$ only rarely holds. The smallest degree g satisfying the condition is $g = 25$ in which case $p = 15$. We will deal with this case separately in the proof of Theorem 2. The next largest degree is $g = 361$ with $p = 209$.

Proposition 4.1.5. *Fix an inhomogeneous lens space $L(p, q)$. Let*

$$\begin{aligned} \mathfrak{E}_1(p, q, g) &:= g^2 + 2g - 2 - 6p^2 + 2q^2 \\ \mathfrak{E}_2(p, q, g) &:= g^2 + 2g - 2 - 12p^2 + 2q^2. \end{aligned}$$

- (1) *If p is odd, q is even and g is the even degree of a minimal isometric immersion of $L(p, q)$ into some sphere, then $g > 3p$ unless $\mathfrak{E}_1(p, q, g) = 0$.*
- (2) *If p is odd, q is odd and g is even, then $g \geq 4p$ unless $\mathfrak{E}_2(p, q, g) = 0$.*

Proof. Using the same methods as before we use five equations of the coefficient comparison in (2.1.1)-(2.1.6). We show that these five equations cannot be solved for $g < 3p$ in the case of (1) unless $\mathfrak{E}_1(p, q, g)$ vanishes and for $g < 4p$ in the case of (2) unless $\mathfrak{E}_2(p, q, g)$ vanishes.

Assume that p is odd, q and the degree g are even and that $g < 3p$. As in (2) of Proposition 4.1.2 we obtain that the only polynomials of the form $(l, 0)$ and of degree g , invariant under the group action, consist of $(0, 0), (\pm 2p, 0)$. Let d_{k1} be the coefficient of the real polynomial $(0, 0)$ and a_{k1} the one of $(2p, 0)$. Then \bar{a}_{k1} , the complex conjugate of a_{k1} is the coefficient of $(-2p, 0)$. Using our fixed basis the corresponding polynomials are:

$$\begin{aligned}\mathfrak{p}_1 &= a_{k1} (z^{\frac{g}{2}-p} \bar{z}^{\frac{g}{2}+p} - (\frac{g^2}{4} - p^2) z^{\frac{g}{2}-p-1} w \bar{z}^{\frac{g}{2}+p-1} \bar{w} + \dots) \\ \bar{\mathfrak{p}}_1 &= \bar{a}_{k1} (z^{\frac{g}{2}+p} \bar{z}^{\frac{g}{2}-p} - (\frac{g^2}{4} - p^2) z^{\frac{g}{2}+p-1} w \bar{z}^{\frac{g}{2}-p-1} \bar{w} + \dots) \\ \mathfrak{r} &= d_{k1} (w^{\frac{g}{2}} \bar{w}^{\frac{g}{2}} - \frac{g^2}{4} z w^{\frac{g}{2}-1} \bar{z} \bar{w}^{\frac{g}{2}-1} + \dots - \frac{g^2}{4} z^{\frac{g}{2}-1} w \bar{z}^{\frac{g}{2}-1} \bar{w} + z^{\frac{g}{2}} \bar{z}^{\frac{g}{2}})\end{aligned}$$

Comparing coefficients in equations (2.1.1) and (2.1.3) and using the monomial $z^{g-2} \bar{z}^g$ as comparison monomial, we know that all monomials which lead to $z^{g-2} \bar{z}^g$ on the left hand side of (2.1.1) and (2.1.3) have to be of the form $(l, 0)$, see Proposition 4.1.2. Hence only the monomials of $\mathfrak{p}_1, \bar{\mathfrak{p}}_1$ and \mathfrak{r} occur and we obtain the following two equations:

$$(4.1.6) \quad \frac{g^2}{4} DD_{11} + \frac{1}{2} (g^2 - 4p^2) A\bar{A}_{11} = \frac{C}{4}$$

$$(4.1.7) \quad \frac{g^2}{4} DD_{11} + \frac{1}{2} (g^2 + 4p^2) A\bar{A}_{11} = \frac{C}{4} + \frac{1}{2}$$

Now consider polynomials of the type $(l, \pm 1)$. They are invariant under the action of \mathbb{Z}_p if and only if $l \pm q \equiv 0 \pmod{p}$. Hence $l = \mp q + p, \mp q + 2p, \mp q + 3p, \dots$. As we assumed q to be even, p odd and g to be even and since $l + m$ is even, we conclude that l must be odd. Hence we are left with $l = \mp q + p, \mp q + 3p, \dots$. But as $g < 3p$, $l = \mp q + p$ is the only remaining case. In our fixed basis the pairs $(-q \pm p, 1), (q \pm p, -1)$ correspond to the following two polynomials and their conjugates:

$$\mathfrak{p}_2 = a_{k2} (z^{\frac{g+p+q-1}{2}} \bar{z}^{\frac{g-p-q-1}{2}} \bar{w} + \dots)$$

$$\bar{\mathfrak{p}}_2 = \bar{a}_{k2} (z^{\frac{g-p-q-1}{2}} w \bar{z}^{\frac{g+p+q-1}{2}} + \dots)$$

$$\mathfrak{p}_3 = a_{k3} (z^{\frac{g+p-q-1}{2}} w \bar{z}^{\frac{g-p+q-1}{2}} + \dots)$$

$$\bar{\mathfrak{p}}_3 = \bar{a}_{k3} (z^{\frac{g-p+q-1}{2}} \bar{z}^{\frac{g+p-q-1}{2}} \bar{w} + \dots)$$

Now we use the monomial $z^{g-3} w \bar{z}^{g-3} \bar{w}$ in the coefficient comparison (2.1.1) to obtain:

$$(4.1.8) \quad \frac{(g+p-q-1)(g-p+q-1)}{2} A\bar{A}_{33} + \frac{(g+p+q-1)(g-p-q-1)}{2} A\bar{A}_{22} - \frac{1}{4} (g^2 - 4p^2) (g^2 - 2g - 4p^2) A\bar{A}_{11} - \frac{g^3}{8} (g-2) DD_{11} = \frac{C}{4} (g-2)$$

Repeating the same calculation using the monomial $z^{g-1} \bar{z}^{g-1}$ in the coefficient comparison of (2.1.4) yields

$$(4.1.9) \quad A\bar{A}_{22} + A\bar{A}_{33} = \frac{1}{2}$$

The fifth and last equation comes from using the same calculation again for the coefficient comparison of (2.1.5) with the monomial $z^{g-2} \bar{z}^{g-1} \bar{w}$.

$$(4.1.10) \quad \frac{g-p+q-1}{2} A\bar{A}_{33} - \frac{g^3}{8} DD_{11} + \frac{g+p+q-1}{2} A\bar{A}_{22} - \frac{1}{4} (g^2 - 4p^2) g A\bar{A}_{11} = \frac{C}{4}$$

Thus we have constructed five linear equations in the four unknowns $A\bar{A}_{11}, A\bar{A}_{22}, A\bar{A}_{33}, DD_{11}$. One easily checks that there is a solution for this system if and only if the polynomial $\mathfrak{E}_1(p, q, g)$ vanishes.

Now assume that p and q are odd, the degree g is even and $g < 4p$. As the construction of the first two equations, (4.1.6) and (4.1.7), was independent of q they remain the same. For the third equation, (4.1.8) we again consider polynomials of the form $(l, \pm 1)$ and we obtain $l = \mp q + p, \mp q + 2p, \mp q + 3p, \dots$. Now we assumed p and q to be odd and g to be even. Again using that l must be odd we are left with $l = \mp q + 2p, \mp q + 4p, \dots$. But as $g < 4p$ this reduces to $l = \mp q + 2p$ and we obtain the following two polynomials and their conjugates:

$$\begin{aligned} \mathfrak{p}_2 &= a_{k2} \left(z^{\frac{g+2p+q-1}{2}} \bar{z}^{\frac{g-2p-q-1}{2}} \bar{w} + \dots \right) \\ \bar{\mathfrak{p}}_2 &= \bar{a}_{k2} \left(z^{\frac{g-2p-q-1}{2}} w \bar{z}^{\frac{g+2p+q-1}{2}} + \dots \right) \\ \mathfrak{p}_3 &= a_{k3} \left(z^{\frac{g+2p-q-1}{2}} \bar{z}^{\frac{g-2p+q-1}{2}} \bar{w} + \dots \right) \\ \bar{\mathfrak{p}}_3 &= \bar{a}_{k3} \left(z^{\frac{g-2p+q-1}{2}} w \bar{z}^{\frac{g+2p-q-1}{2}} + \dots \right) \end{aligned}$$

Using these polynomials we follow the same arguments as in the case of (1) and obtain

$$\begin{aligned} & \frac{(g+2p-q-1)(g-2p+q-1)}{2} A\bar{A}_{33} + \frac{(g+2p+q-1)(g-2p-q-1)}{2} A\bar{A}_{22} - \\ & \frac{1}{4} (g^2 - 4p^2) (g^2 - 2g - 4p^2) A\bar{A}_{11} - \frac{g^3}{8} (g-2) DD_{11} = \frac{C}{4} (g-2) \\ & A\bar{A}_{22} + A\bar{A}_{33} = \frac{1}{2} \\ & \frac{g-2p+q-1}{2} A\bar{A}_{33} - \frac{g^3}{8} DD_{11} + \frac{g+2p+q-1}{2} A\bar{A}_{22} - \frac{1}{4} (g^2 - 4p^2) g A\bar{A}_{11} = \frac{C}{4} \end{aligned}$$

Again one easily checks that there is a solution for this system if and only if the polynomial $\mathfrak{E}_2(p, q, g)$ vanishes. \square

Proposition 4.1.11. *If $\alpha > 1$ and $\tilde{q} \equiv q \pmod{p}$ then $\mathfrak{U}(\alpha p, \tilde{q}) \geq \mathfrak{U}(p, q)$.*

Proof. First consider the case of $\tilde{q} = q$. We use that $\mathbb{Z}_p \subset \mathbb{Z}_{\alpha p}$. If a monomial $m = z^a w^b \bar{z}^c \bar{w}^d$ is invariant under $\mathbb{Z}_{\alpha p}$ to generate $L(\alpha p, q)$, then $a + b q - c - d q \equiv 0 \pmod{\alpha p}$. But this implies that $a + b q - c - d q \equiv 0 \pmod{p}$ and m is also invariant under \mathbb{Z}_p to generate $L(p, q)$. Notice that this step remains the same if we replace q by $\tilde{q} \equiv q \pmod{p}$. Recall that by definition $n(g, p, q)$ is the number of polynomials of degree g invariant under the group action. As the invariance of monomials implies the invariance of the corresponding polynomials, we obtain $n(g, \alpha p, q) < n(g, p, q)$. In the case of $\tilde{q} \equiv q \pmod{p}$, we obtain that $a + b \tilde{q} - c - d \tilde{q} \equiv 0 \pmod{\alpha p}$ induces $a + b \tilde{q} - c - d \tilde{q} \equiv a + b q - c - d q \equiv 0 \pmod{p}$. Hence $n(g, \alpha p, \tilde{q}) < n(g, p, q)$ holds as well.

Now assume we computed $\mathfrak{U}(\alpha p, q)$ for $L(\alpha p, q)$. For all $g \geq \mathfrak{U}(\alpha p, q)$ the corresponding linear system has a non-trivial solution. But we can also derive the same linear system by first computing the row reduced linear system for $L(p, q)$ and then setting the corresponding columns zero. These columns correspond to the coefficients of those polynomials which are invariant under the action of \mathbb{Z}_p but not under $\mathbb{Z}_{\alpha p}$. As the original system had the property that $m(g, \alpha p, q) < n(g, \alpha p, q)$, setting those columns zero will not change the number of non-zero rows after a row reduction of the coefficient matrix and hence $m(g, \alpha p, q) = m(g, p, q)$. Again recall that $m(g, p, q)$ was the number of non-zero rows after row reduction of the coefficient matrix. We conclude $m(g, \alpha p, q) - n(g, \alpha p, q) \geq m(g, p, q) - n(g, p, q)$ and therefore by taking the infimum over all possible degrees g : $\mathfrak{U}(\alpha p, q) \geq \mathfrak{U}(p, q)$. For the case of $\tilde{q} \equiv q \pmod{p}$, observe that the equality for the function m also holds, i.e. $m(g, \alpha p, \tilde{q}) = m(g, p, q)$. Combining this with the corresponding inequality for the function n yields $\mathfrak{U}(\alpha p, \tilde{q}) \geq \mathfrak{U}(p, q)$ for $\alpha > 1$ where $\tilde{q} \equiv q \pmod{p}$. \square

Proposition 4.1.12. *If $L(p, q')$ is homeomorphic to $L(p, q)$, then $\mathfrak{U}(p, q) = \mathfrak{U}(p, q')$.*

Proof. Let $L(p, q')$ be homeomorphic to $L(p, q)$. It is well known that then $q \cdot q' \equiv \pm 1 \pmod{p}$. Recall the invariance condition for a monomial $z^a w^b \bar{z}^c \bar{w}^d$ corresponding to the lens space $L(p, q)$:

$$(*) \quad a - c + q(b - d) \equiv 0 \pmod{p} \iff l + qm \equiv 0 \pmod{p} \quad \text{with} \quad l = a - c, \quad m = b - d$$

Again use the representation of harmonic polynomials in terms of pairs (l, m) as above. Rewriting (*) yields

$$q^{-1}l + m \equiv 0 \pmod{p} \quad \text{or} \quad -q^{-1}l - m \equiv 0 \pmod{p}$$

As $q' \equiv \pm q^{-1} \pmod{p}$, we obtain

$$q'l + m \equiv 0 \pmod{p} \quad \text{or} \quad q'l - m \equiv 0 \pmod{p}.$$

This implies that a polynomial (l, m) is invariant under \mathbb{Z}_p to generate $L(p, q)$ if and only if (m, l) (or $(-m, l)$) is invariant under \mathbb{Z}_p to generate $L(p, q')$ where $q \cdot q' \equiv 1 \pmod{p}$ (or $q \cdot q' \equiv -1 \pmod{p}$). This implies that $n(g, p, q) = n(g, p, q')$.

In the case of $q \cdot q' \equiv 1 \pmod{p}$ consider the isometry $T(z, w) = (w, z)$ of S^3 . Applying T to a monomial we obtain $T(z^a w^b \bar{z}^c \bar{w}^d) = z^b w^a \bar{z}^d \bar{w}^c$. But we saw in [E] that replacing f by $T(f)$ yields exactly the same linear system, hence the dimension of the solution set of the inhomogeneous part of the linear system L_f is the same as that of $L_{T(f)}$. Hence $m(g, p, q) = m(g, p, q')$ for $q \cdot q' \equiv 1 \pmod{p}$. For the case of $q \cdot q' \equiv -1 \pmod{p}$ repeat the argument with the isometry $S(z, w) = (w, \bar{z})$ of S^3 .

Taking the infimum over all degrees g we conclude that $\mathfrak{U}(p, q) = \mathfrak{U}(p, q')$ for $q \cdot q' \equiv \pm 1 \pmod{p}$. \square

Remark. In fact, we used a slightly different isometry of S^3 in [E]. However all arguments can be repeated for this particular T (or S).

Heuristic remark. It seems difficult to prove more generally that the function \mathfrak{U} is increasing with p . One can easily show that $n(g, p_1, q) < n(g, p_2, q)$ for $p_1 > p_2$, but it is not clear why $m(g, p_1, q) > m(g, p_2, q)$ should be true. In all examples to date, our computations indicate that $m(g, p, q) = 3g - 1$, independent of p and q . Note that $3g - 1$ is the number of independent equations of the general system without any invariance under a group action.

4.2 PROOF OF THEOREM 2

Our goal is to show that $\mathfrak{U}(p, q) \geq 28$ in the case of p odd and $\mathfrak{U}(p, q) \geq 20$ in the case of p even. Using Proposition 4.1.1 we can reduce the number of cases to $8 \leq p \leq 18$ in the case of p even and to $5 \leq p \leq 27$ in the case of p odd.

First consider the case of p being even. Recall that we already computed $\mathfrak{U}(8, 3) = 20$ in [E]. By Proposition 4.1.12 we know that $\mathfrak{U}(8, 3) = \mathfrak{U}(8, 5)$ and hence $\mathfrak{U}(8, q) = 20$. For the case of $\mathfrak{U}(10, q)$ we use Proposition 4.1.11: $\mathfrak{U}(10, \tilde{q}) \geq \mathfrak{U}(5, q)$ where $\tilde{q} \equiv q \pmod{5}$. This implies that $\mathfrak{U}(10, 3) \geq \mathfrak{U}(5, 3)$ and $\mathfrak{U}(10, 7) \geq \mathfrak{U}(5, 2)$. As $\mathfrak{U}(5, 3) = \mathfrak{U}(5, 2) = 28$ by Proposition 4.1.12 and [E], we obtain that $\mathfrak{U}(10, q) \geq 28 > 20$.

Using the same technique we conclude that $\mathfrak{U}(14, q) \geq \mathfrak{U}(7, 2)$ as $\mathfrak{U}(7, 2) = \mathfrak{U}(7, 3) = \mathfrak{U}(7, 4) = \mathfrak{U}(7, 5)$ and we will see in the case of p odd that $\mathfrak{U}(7, 2) \geq 28$.

Similarly $\mathfrak{U}(16, q) \geq \mathfrak{U}(8, 3) = 20$ for all q except $q = 7$. Also $\mathfrak{U}(18, q) \geq \mathfrak{U}(9, 2)$ as $\mathfrak{U}(9, 2) = \mathfrak{U}(9, 4) = \mathfrak{U}(9, 5) = \mathfrak{U}(9, 7)$ and again we will see that $\mathfrak{U}(9, 2) \geq 28$.

Combining the above, the following cases remain to be calculated: $\mathfrak{U}(12, 5)$ and $\mathfrak{U}(16, 7)$. Recall that by Proposition 4.1.2 we only need to check even degrees g . Therefore it is sufficient to compute $m(p, q, g) - m(p, q, g)$ for $g = 12, 14, 16, 18$ in the case of $p = 12, q = 5$ and for $g = 16, 18$ in the case of $p = 16, q = 7$. Doing so, one obtains in both cases for the mentioned degrees that $m(p, q, g) - m(p, q, g) > 0$. Hence $\mathfrak{U}(12, 5) \geq 20$ and $\mathfrak{U}(16, 7) \geq 20$ as claimed.

Now consider the case of p being odd. Recall that we already computed $\mathfrak{U}(5, 2) = 28$. Again Proposition 4.1.12 implies that $\mathfrak{U}(5, 2) = \mathfrak{U}(5, 3)$ and hence $\mathfrak{U}(5, q) = 28$. There are two possibilities for the remaining cases:

- (1) g even

By Proposition 4.1.2 we know that $g \geq 2p$. However we utilize Proposition 4.1.5 to eliminate the remaining cases. Using Proposition 4.1.12 we can reduce the calculations to the case of q being odd and are left with computations for $q = 3$ with $p = 7, 9, 11, 13$, $q = 5$ with $p = 11, 13$ and $q = 7$ with $p = 13$. Then Proposition 4.1.5 implies that $g > 4p$ unless $\mathfrak{E}_2(p, q, g) = 0$. One easily verifies that the polynomial \mathfrak{E}_2 never vanishes for the above mentioned cases. Hence for even degrees g and odd order p of the group we always obtain $\mathfrak{U}(p, q) \geq 28$.

(2) g odd

In this case Proposition 4.1.2 implies that $g \geq 3p$ except for $p = 15$ and $g = 25$. Combining Propositions 4.1.11 and 4.1.12 we can reduce this exceptional case to $p = 15$ and $q = 4$. We are still left with a calculation of $m(p, q, g) - m(p, q, g)$ for $g = 21, 23, 25, 27$ in the case of $p = 7, q = 2$ and for $g = 27$ in the case of $p = 9, q = 2$ as well as $g = 25$ in the case of $p = 15, q = 4$. Doing so, one obtains in all cases for the mentioned degrees that $m(p, q, g) - m(p, q, g) > 0$. Hence $\mathfrak{U}(7, 2) \geq 28$, $\mathfrak{U}(9, 2) \geq 28$ and $\mathfrak{U}(15, 4) \geq 28$ as claimed.

Hence we showed that $\mathfrak{U}(p, q) \geq 28$ for all odd p and $\mathfrak{U}(p, q) \geq 20$, for all even p which completes the proof of Theorem 2. \square

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