

A classification of S^3 -bundles over S^4

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Abstract

We classify the total spaces of bundles over the four sphere with fiber a three sphere up to orientation preserving and reversing homotopy equivalence, homeomorphism and diffeomorphism. These total spaces have been of interest to both topologists and geometers. It has recently been shown by Grove and Ziller [GZ] that each of these total spaces admits metrics with nonnegative sectional curvature.

1 Introduction

For almost fifty years, 3-sphere bundles over the 4-sphere have been of interest to both topologists and geometers. In 1956, Milnor [M] proved that all S^3 -bundles over S^4 with Euler class $e = \pm 1$ are homeomorphic to S^7 . He also showed that some of these bundles are not diffeomorphic to S^7 and thereby exhibited the first examples of exotic spheres. Shortly thereafter, in 1962, Eells and Kuiper [EK] classified all such bundles with Euler class $e = \pm 1$ up to diffeomorphism and showed that 15 of the 27 seven dimensional exotic spheres can be described as S^3 -bundles over S^4 .

In 1974 Gromoll and Meyer [GM] constructed a metric with non-negative sectional curvature on one of these sphere bundles, exhibiting the first example of an exotic sphere with non-negative curvature. Very recently Grove and Ziller [GZ] showed that the total space of every S^3 -bundle over S^4 admits a metric with non-negative sectional curvature. Motivated by these examples they asked for a classification of these manifolds up to homotopy equivalence, homeomorphism and diffeomorphism (see problem 5.2 in [GZ]). Although partial classifications in various categories appear in [JW2], [S], [T] and [Wi], the complete classification has not been carried out. It is the purpose of this article to do this.

We consider fiber bundles over the 4-sphere S^4 with total space M , fiber the 3-sphere S^3 and structure group $SO(4)$. Equivalence classes of such bundles are in one-to-one correspondence with $\pi_3(SO(4)) \cong \mathbf{Z} \oplus \mathbf{Z}$. Following James and Whitehead [JW2], we choose generators ρ and σ of $\pi_3(SO(4))$ such that

$$\rho(u) v = u v u^{-1}, \quad \sigma(u) v = u v ;$$

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where u and v denote the quaternions with norm 1, i.e. we have identified S^3 with the unit quaternions. With this choice of generators a pair of integers (m, n) gives the element $m\rho + n\sigma \in \pi_3(SO(4))$ and thus determines both a vector bundle $\xi_{m,n}$ and the corresponding sphere bundle $\pi_{m,n} : M_{m,n} := S(\xi_{m,n}) \rightarrow S^4$.

Remark 1.1. *The definition we use is different from the one given by Milnor in [M]. He uses two integers (k, l) corresponding to a different choice of generators of $\pi_3(SO(4)) \cong \mathbf{Z} \oplus \mathbf{Z}$. The two pairs (m, n) and (k, l) are related by $k + l = n$, $l = -m$.*

Note that a change of orientation in the fiber leads to $M_{m,n} \cong M_{m+n,-n}$ whereas a change of orientation in the base leads to $M_{m,n} \cong M_{-m,-n}$. Hence we now always assume that $n \geq 0$. We can also exclude the case of $n = 0$ as $M_{m,0}$ is diffeomorphic or homeomorphic to $M_{m',0}$ only when $m' = \pm m$. This follows from the topological invariance of the rational Pontrjagin classes and the fact that $M_{m,n}$ is diffeomorphic to $M_{-m-n,n}$ for any m and n . James and Whitehead [JW1] proved that $M_{m,0}$ is homotopic to $M_{m',0}$ if and only if $m' \equiv \pm m \pmod{12}$. Lastly, each $M_{m,0}$ has an orientation reversing self-diffeomorphism. Hence we now always assume that $n > 0$.

The first level of classification of the manifolds $M_{m,n}$ is up to homotopy equivalence. In a remarkable paper [JW2] James and Whitehead succeeded in classifying the pairs of manifolds $(M_{m,n}, S^3)$ up to homotopy equivalence except for the case $n = 2$. A few years later Sasao [S] undertook the homotopy classification of the total spaces $M_{m,n}$. Let $M(Z_n, 3)$ denote the Moore space formed by attaching a 4-disc to a 3-sphere with a degree n map of the 3-sphere. Each $M_{m,n}$ has the homotopy type of $M(Z_n, 3)$ with a 7-cell attached. Sasao computed both $\pi_6(M(Z_n, 3))$ and the action of $[M(\mathbf{Z}_n, 3), M(\mathbf{Z}_n, 3)]$ on $\pi_6(M(\mathbf{Z}_n, 3))$. In principle, this should allow for a complete homotopy classification though Sasao did not do the computation. We deduce the homotopy classification as a consequence of the homeomorphism classification and thus avoid the homotopy theory completely.

Theorem 1.1. *Let $n, n' > 0$.*

1. *The manifolds $M_{m',n'}$ and $M_{m,n}$ are orientation preserving homotopy equivalent if and only if $n = n'$ and $m' \equiv \alpha m \pmod{(n, 12)}$ where $\alpha^2 \equiv 1 \pmod{(n, 12)}$.*
2. *Orientation reversing homotopy equivalences between any $M_{m',n}$ and $M_{m,n}$ can only exist when $n = 2^\epsilon p_1^{i_1} \dots p_k^{i_k}$, p_i prime, $p_i \equiv 1 \pmod{4}$ and $\epsilon = 0, 1$. Furthermore if n is of this form with $\epsilon = 0$, then the single oriented homotopy type admits an orientation reversing self homotopy equivalence; if $\epsilon = 1$, $M_{m',n}$ is orientation reversing homotopy equivalent to $M_{m,n}$ if and only if $m' + m \not\equiv 0 \pmod{2}$.*

Here $(n, 12)$ denotes the greatest common divisor of n and 12. Notice that since $H^4(M_{m,n}; \mathbf{Z}) \cong \mathbf{Z}_n$ and we assumed n to be nonnegative, necessarily $n = n'$ if $M_{m,n}$ and $M_{m',n'}$ are homotopy equivalent.

Remark 1.2. *Comparing Theorem 1.1 with [JW2] (for $n > 2$, $(M_{m,n}, S^3)$ is homotopy equivalent to $(M_{m',n}, S^3)$ if and only if $m' \equiv \pm m \pmod{(n, 12)}$) one concludes that the*

homotopy classification of the manifold pairs $(M_{m,n}, S^3)$ differs from the homotopy classification of the total spaces $M_{m,n}$ if and only if n is divisible by 12. In this case the congruence $\alpha^2 \equiv 1 \pmod{(n, 12)}$ has two nontrivial solutions, namely $\alpha \equiv \pm 5 \pmod{12}$.

The next level, the homeomorphism classification, was first studied by Tamura [T], who constructed explicit homeomorphisms between $M_{m,n}$ and $M_{m',n}$ if $m \equiv \pm m' \pmod{n}$. These homeomorphisms were constructed with the aid of specific foliations of the total spaces. However, as handlebody theory developed, general techniques for classifying highly connected manifolds became available and Wall ([Wa1]) was able to classify $(s-1)$ connected $(2s+1)$ manifolds except when $s = 3, 7$. Wilkens [Wi] then extended the techniques of Wall to complete the cases $s = 3, 7$ except for an occasional \mathbf{Z}_2 -ambiguity. We complete Wilkens' classification in the case of S^3 -bundles over S^4 using two topological invariants. The first invariant was used by Wilkens and is the characteristic class of Spin manifolds, $\frac{p_1}{2}$, which is defined as the generator of $H^4(Bspin; \mathbf{Z})$ that has the same sign as the first Pontrjagin class p_1 . We stress that the division by two occurs universally in $H^4(Bspin; \mathbf{Z})$ so that $\frac{p_1}{2}(M_{m,n})$ should not be thought of as $p_1(M_{m,n})/2$ which does not make sense in the presence of 2-torsion. Pulling back a preferred generator of $H^4(S^4; \mathbf{Z})$ to $H^4(M_{m,n}; \mathbf{Z})$ yields a canonical identification of $H^4(M_{m,n}; \mathbf{Z})$ with \mathbf{Z}_n and hence we identify $\frac{p_1}{2}(M_{m,n})$ with an element of \mathbf{Z}_n . The second invariant is the topological Eells-Kuiper invariant $s_1(M_{m,m}) := 28\mu(M_{m,m}) \in \mathbf{Q}/\mathbf{Z}$ which was used in [KS] where it is shown (Theorem 2.5) to be an invariant of topological spin manifolds. We now recall the definition of the μ invariant from [EK].

Assume that $M = M^7$ is a closed, smooth spin manifold such that M is a spin boundary, i.e. $M = \partial W^8$, where W^8 is a closed, smooth spin manifold. In addition, we require (W, M) to satisfy the μ -condition which is automatic when M is a rational homology sphere, see [EK]. The μ -condition allows the pullback of the Pontrjagin classes of W to $H^*(W, M)$. Then the μ -invariant is defined as

$$\mu(M^7) \equiv \frac{1}{2^7 \cdot 7} \{p_1^2(W) - 4 \sigma[W]\} \pmod{1} \quad (1)$$

Here $\sigma[W]$ stands for the signature of W . The most important feature of the μ -invariant is that it is an invariant of the diffeomorphism type of M whereas $s_1 := 28\mu$ is an invariant of the homeomorphism type of M , see [KS]. Calculating these invariants for the total spaces $M_{m,n}$ we obtain

$$\frac{p_1}{2}(M_{m,n}) \equiv 2m \pmod{n}; \quad s_1(M_{m,n}) \equiv \frac{1}{8n}(4m(m+n) + n(n-1)) \pmod{1}.$$

Theorem 1.2. *The following are equivalent.*

1. $M_{m',n}$ is orientation preserving [reversing] PL-homeomorphic to $M_{m,n}$.
2. $M_{m',n}$ is orientation preserving [reversing] homeomorphic to $M_{m,n}$.
3. $\frac{p_1}{2}(M_{m',n}) = \alpha \frac{p_1}{2}(M_{m,n})$ and $s_1(M_{m',n}) = s_1(M_{m,n})$ where $\alpha^2 \equiv 1 \pmod{n}$.
 $[\frac{p_1}{2}(M_{m',n}) = \alpha \frac{p_1}{2}(M_{m,n})$ and $s_1(M_{m',n}) = -s_1(M_{m,n})$ where $\alpha^2 \equiv -1 \pmod{n}.$

Combining the congruences we obtain the following version of Theorem 1.2.

Corollary 1.3. *Let $n = 2^a q$, q odd.*

1. $M_{m',n}$ is orientation preserving homeomorphic to $M_{m,n}$ if and only if the following condition holds.
 - $a = 0$: $m' \equiv \alpha m \pmod n$ where $\alpha^2 \equiv 1 \pmod n$.
 - $a = 2$ and m odd, or $a > 2$ and m even: $m' \equiv \alpha m \pmod{\frac{n}{2}}$ where $\alpha^2 \equiv 1 \pmod n$.
 - $a = 1$, or $a = 2$ and m even, or $a > 2$ and m odd: $m' \equiv \alpha m \pmod n$ where $\alpha \equiv \pm 1 \pmod{2^a}$ and $\alpha^2 \equiv 1 \pmod n$.
2. $M_{m',n}$ is orientation reversing homeomorphic to $M_{m,n}$ if and only if the following condition holds.
 - $n = p_1^{i_1} \dots p_k^{i_k}$, $p_i \equiv 1 \pmod 4$ and $m' \equiv \alpha m \pmod n$ where $\alpha^2 \equiv -1 \pmod n$ or
 - $n = 2p_1^{i_1} \dots p_k^{i_k}$, $p_i \equiv 1 \pmod 4$ and $m' \equiv \alpha(m + \frac{n}{2}) \pmod n$ where $\alpha^2 \equiv -1 \pmod n$.

The following corollary of the proof of Theorem 1.2 generalizes a result of Kitchloo and Shankar [KiSh1]. Upon revision we note that Kitchloo and Shankar later included this generalization in their published article [KiSh2] with an alternate proof provided by M. Kreck.

Corollary 1.4. *Any 2-connected 7-manifold M with $H^4(M; \mathbf{Z}) \cong \mathbf{Z}_n$ and linking form isomorphic to $l : \mathbf{Z}_n \times \mathbf{Z}_n \rightarrow \mathbf{Q}/\mathbf{Z}$, $(r, s) \mapsto \frac{rs}{n}$ is homeomorphic to the total space of a 3-sphere bundle over the 4-sphere.*

The diffeomorphism classification of the manifolds $M_{m,n}$ follows the pattern that Eells and Kuiper discovered in the case $n = 1$ for Milnor's exotic spheres. That is, $M_{m',n}$ and $M_{m,n}$ are diffeomorphic if and only if they are homeomorphic and their μ invariants coincide.

Theorem 1.5. *The total space $M_{m',n}$ is orientation preserving [reversing] diffeomorphic to $M_{m,n}$ if and only if the following conditions hold.*

$$\begin{aligned} \frac{p_1}{2}(M_{m',n}) &= \alpha \frac{p_1}{2}(M_{m,n}) \text{ and } \mu(M_{m',n}) = \mu(M_{m,n}) \text{ where } \alpha^2 \equiv 1 \pmod n. \\ \left[\frac{p_1}{2}(M_{m',n}) = \alpha \frac{p_1}{2}(M_{m,n}) \text{ and } \mu(M_{m',n}) = -\mu(M_{m,n}) \text{ where } \alpha^2 \equiv -1 \pmod n.\right] \end{aligned}$$

Using the definition of $\frac{p_1}{2}$ and μ we describe Theorem 1.5 in terms of congruences as follows.

Corollary 1.6. *The total space $M_{m',n}$ is orientation preserving [reversing] diffeomorphic to $M_{m,n}$ if and only if the following conditions hold.*

$$\begin{aligned} m'(n + m') &\equiv m(n + m) \pmod{56n} \text{ and } 2m' \equiv 2\alpha m \pmod n \text{ where } \alpha^2 \equiv 1 \pmod n. \\ \left[m'(n + m') + n(n - 1) &\equiv -m(n + m) - n(n - 1) \pmod{56n} \text{ and } 2m' \equiv 2\alpha m \pmod n \right. \\ &\left. \text{where } \alpha^2 \equiv -1 \pmod n.\right] \end{aligned}$$

Smooth surgery theory (see [MM]) implies that there are exactly 28 different smooth manifolds homeomorphic to $M_{m,n}$. A natural question to ask is which manifolds homeomorphic to $M_{m,n}$ may be represented by manifolds $M_{m',n}$. In section 4 we answer this question for special cases of n . We see that in some cases all 28 differentiable structures are realized by S^3 -bundles over S^4 . We also discuss the case $n = 10$ which includes the seven dimensional Berger manifold as shown in [KiSh2].

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2 Outline of proof

2.1 A note on orientation

Whereas classifications arising from the use of surgery theory are naturally given up to orientation preserving maps, geometers often require classifications irrespective of orientation. For our sphere bundles there are diffeomorphisms that reverse the orientation: of the fiber; of the base; or of both the fiber and the base. These diffeomorphisms give rise to orientation reversing diffeomorphisms from $M_{m,n}$ to $M_{m+n,-n}$ and from $M_{m,n}$ to $M_{m,-n}$ respectively, as well as orientation preserving diffeomorphisms between $M_{m,n}$ and $M_{-m-n,n}$. Hence it is sufficient to work with $n \geq 0$; this is the convention established by James and Whitehead, [JW2]. We now fix a generator ι_4 of $H^4(S^4; \mathbf{Z})$ and use the same notation for the images of ι_4 under the isomorphisms $H^4(S^4; \mathbf{Z}) \cong H_4(S^4; \mathbf{Z}) \cong \pi_4(S^4)$. Using this generator of $\pi_4(S^4)$, we orient the fiber S^3 with $\iota_3 \in \pi_3(S^3)$ such that $\partial(\iota_3) = n \iota_4$ and $n > 0$. With this convention $M_{m,n}$ is an oriented manifold. From now on all manifolds shall be smooth, compact and oriented and maps shall be orientation preserving unless explicitly noted. Thus “ M' and M are homeomorphic” shall usually mean that M' and M are orientation preserving homeomorphic and orientation reversing maps will be considered explicitly as maps from M' to $-M$.

2.2 Structure of proof

We wish to emphasize that our classification does not first solve the problem of homotopy equivalence and then proceed to the finer relations of homeomorphism and diffeomorphism. Instead, we start by completing the PL -homeomorphism classification of Wilkens. We are then able to descend to give a homotopy classification by explicitly exhibiting the entire normal invariant set of any $M_{m,n}$ as fiber homotopy equivalences.

Proposition 2.1. *For all $j \in \mathbf{Z}$ there exist fiber homotopy equivalences $f_j : M_{m+12j,n} \rightarrow M_{m,n}$ with normal invariant $j \in \mathbf{Z}_n$.*

It follows from a simple application of surgery theory that a given seven dimensional manifold M' is homotopy equivalent to $M_{m,n}$ if and only if M is homeomorphic to

$M_{m+12j,n}$ for some j . Given the homeomorphism classification of the bundles this is enough to yield the homotopy classification.

3 Homeomorphism classification

The elementary algebraic topology of our manifolds is determined by the Euler number of the bundle $e(\xi_{m,n}) = n$. From now on we use $H^i(M)$ to denote the i -th cohomology group of M with integer coefficients.

$$\begin{aligned} H^0(M_{m,n}) &\cong H^7(M_{m,n}) \cong \mathbf{Z} \\ H^4(M_{m,n}) &\cong \mathbf{Z}_n \\ H^i(M_{m,n}) &\cong 0 \text{ for all } i \neq 0, 4, 7. \end{aligned}$$

One sees from the Serre spectral sequence of the fibration $S^3 \hookrightarrow M_{m,n} \xrightarrow{\pi} S^4$ that $\pi^* : H^4(S^4) \rightarrow H^4(M_{m,n})$ is a surjection. Let $\kappa_4 := \pi^*(\iota_4)$ and use κ_4 to identify $H^4(M_{m,n})$ with \mathbf{Z}_n . The bundle structure ensures that the linking form of $M_{m,n}$, $lk(M_{m,n})$, is isomorphic to the standard form, l , for all $m \in \mathbf{Z}$. Thus,

$$\begin{aligned} lk(M_{m,n}) : H^4(M_{m,n}) \cong \mathbf{Z}_n \times H^4(M_{m,n}) \cong \mathbf{Z}_n &\longrightarrow \mathbf{Q}/\mathbf{Z} \\ (r, s) &\longmapsto l(r, s) = \frac{rs}{n}. \end{aligned}$$

To see this, note that the linking form $lk(M_{m,n})$ is induced from the intersection form of any coboundary W . For sphere bundles we may choose $W = W_{m,n}$ to be the associated disk bundle, $D^4 \hookrightarrow W_{m,n} \xrightarrow{\pi_W} S^4$, $\partial W_{m,n} = M_{m,n}$. The long exact sequence of the pair $(W_{m,n}, M_{m,n})$ yields

$$0 \rightarrow H^4(W_{m,n}, M_{m,n}) \xrightarrow{j^*} H^4(W_{m,n}) \xrightarrow{i^*} H^4(M_{m,n}) \rightarrow 0.$$

Let $x = \pi_W^* \iota_4 \in H^4(W_{m,n}) \cong \mathbf{Z}$ and let y be a generator of $H^4(W_{m,n}, M_{m,n}) \cong \mathbf{Z}$ such that $j^*(y) = nx$. Since $\kappa_4 = i^*(x)$, the self linking number of the class κ_4 is defined as follows,

$$lk(\kappa_4, \kappa_4) = \frac{1}{n} \langle y \cup x, [W_{m,n}, M_{m,n}] \rangle = \frac{1}{n}.$$

Bilinearity is now enough to ensure that $lk(M_{m,n}) = l$. Note that the linking form $lk(M_{m,n})$ may be computed in a variety of ways. See for example [KiSh2] for a different method.

We now describe the invariants required for our classification in some detail.

3.1 The invariants μ and s_1

The invariant $\mu(M_{m,n}) \equiv \mu(W, M_{m,n}) \pmod{1}$ is computed for any spin coboundary W which satisfies the μ -condition. We choose $W = W_{m,n}$ as above and note that the pair

$(W_{m,n}, M_{m,n})$ satisfies the μ -condition. This coboundary was already used in [M] and [T] and we use it to obtain for the μ -invariant of the total spaces $M_{m,n}$

$$\mu(M_{m,n}) \equiv \frac{1}{2^7 \cdot 7} \{p_1^2(W_{m,n}) - 4 \sigma[W_{m,n}]\} \pmod{1}, \quad (2)$$

where $\sigma[W_{m,n}]$ is the signature of the quadratic form given by

$$\begin{array}{ccc} H^4(W_{m,n}, \partial W_{m,n}) & \longrightarrow & \mathbf{Q} \\ v & \longrightarrow & \langle v \cup v, [W_{m,n}, \partial W_{m,n}] \rangle. \end{array}$$

and $(p_1(W_{m,n}))^2$ is the characteristic number

$$\langle (j^*)^{-1} p_1(W_{m,n}) \cup p_1(W_{m,n}), [W_{m,n}, \partial W_{m,n}] \rangle$$

where $j^* : H^4(W_{m,n}, \partial W_{m,n}; \mathbf{Q}) \longrightarrow H^4(W_{m,n}; \mathbf{Q})$. As $n > 0$ we obtain for the signature $\sigma(W_{m,n}) = 1$. Furthermore, we chose y such that $j^*(y) = nx$. Hence $(j^*)^{-1}(x) = \frac{1}{n}y$. Finally, it is well known (see [M]) that $p_1(W_{m,n}) = 2(n+2m)x$ and thus

$$\begin{aligned} & \langle (j^*)^{-1} p_1(W_{m,n}) \cup p_1(W_{m,n}), [W_{m,n}, \partial W_{m,n}] \rangle \\ &= \langle \frac{1}{n} 2(n+2m)y \cup 2(n+2m)x, [W_{m,n}, \partial W_{m,n}] \rangle \\ &= \frac{4(n+2m)^2}{n}. \end{aligned} \quad (3)$$

Hence

$$\mu(M_{m,n}) \equiv \frac{1}{2^5 \cdot 7 \cdot n} \{(n+2m)^2 - n\} \pmod{1}$$

and therefore

$$\begin{aligned} s_1(M_{m,n}) = 28\mu(M_{m,n}) & \equiv \frac{1}{8 \cdot n} \{(n+2m)^2 - n\} \pmod{1} \\ & \equiv \frac{m^2}{2n} + \frac{m}{2} + \frac{n-1}{8} \pmod{1}. \end{aligned} \quad (4)$$

As noted above, it is proven in [KS] that s_1 is an invariant of topological spin manifolds.

We would like to point out that there are mistakes in the literature in calculations of the μ -invariant for the manifolds $M_{m,n}$. In particular, an incorrect formula for the μ -invariant [D, p. 67] is applied to prove [D, Theorem 2.1]. In fact, the statement [D, Theorem 2.1] is true under use of the true μ -invariant as W. Ziller pointed out to us. However, Ziller tells us that the proof of [Ma, Theorem A] does not work with the true μ -invariant. We do not know if [Ma, Theorem A] is true.

3.2 The invariant $\frac{p_1}{2}$

Since the total spaces $M_{m,n}$ are 2-connected they all have unique spin structures and we work with spin characteristic classes for simplicity. The Hopf bundle $\pi_{0,1} : \xi_{0,1} \longrightarrow S^4$ defines a generator of $\pi_4(BSpin)$ but $p_1(\xi_{0,1}) = \pm 2 \in H^4(S^4)$. It follows that half the first Pontrjagin class, $\frac{p_1}{2}$, is a generator of $H^4(BSpin)$ and thus is always integral. Moreover,

$\frac{p_1}{2}$ is a topological invariant since the canonical map $H^4(BSpin) \rightarrow H^4(BTop)$ is an injection. We refer the reader to [KS, Lemma 6.5] for a discussion of this map. In [M] Milnor showed how to compute p_1 for the manifolds $M_{m,n}$. One first observes that the stable tangent bundle of a sphere bundle is a pullback of bundles over the base space. As we shall need the corresponding fact for the stable normal bundle we give both results now. Let τ_B denote the stable tangent bundle of the base space B and let ν_B denote the corresponding stable normal bundle.

Fact 3.1. *Let $S(\psi)$ be the sphere bundle associated to a vector bundle ψ over a manifold B and let $-\psi$ denote the stable inverse of ψ . Then*

$$\nu_{S(\psi)} = \pi_\psi^*(\nu_B \oplus -\psi) \text{ and } \tau_{S(\psi)} = \pi_\psi^*(\tau_B \oplus \psi).$$

Recalling that $p_1(\xi_{m,n}) = 2(n+2m)\iota_4$ we obtain that

$$\begin{aligned} \frac{p_1}{2}(M_{m,n}) &= \frac{p_1}{2}(\tau_{M_{m,n}}) = \frac{p_1}{2}(\pi^*(\tau_{S^4} \oplus \xi_{m,n})) = \pi^*\left(\frac{p_1}{2}(\xi_{m,n})\right) = \pi^*((n+2m)\iota_4) \\ &= 2m\kappa_4 \in \mathbf{Z}_n. \end{aligned} \tag{5}$$

The characteristic class $\frac{p_1}{2}$ is identical to the invariant $\hat{\beta}$ defined in [Wi] as the obstruction to the tangent bundle being trivial over the 4-skeleton. Wilkens' full invariant for a 2-connected 7-manifold with torsion fourth cohomology group is the triple $(H^4(M), lk(M), \frac{p_1}{2}(M))$ where $lk(M)$ denotes the linking form of M on $H^4(M)$. He denoted such a triple abstractly as (G, l, β) and defined two triples (G, l, β) and (G', l', β') to be the same invariant if there is an isomorphism $h: G \rightarrow G'$ preserving the linking form and sending β to β' . For $M = M_{m,n}$ we have identified $H^4(M_{m,n})$ with \mathbf{Z}_n by taking $\pi_{m,n}^*\iota_4 = \kappa_4$ as a generator of $H^4(M_{m,n})$. With this identification the linking form becomes $lk(M_{m,n}) = l$ where $l(r, s) = \frac{rs}{n}$. Hence when we speak of the invariant $\frac{p_1}{2}$ we interpret it as part of the Wilkens triple $(\mathbf{Z}_n, l, \frac{p_1}{2}(M_{m,n}) = 2m)$. Note that the automorphisms of \mathbf{Z}_n which preserve the linking form l can be described as elements of the set $A^+(n) := \{\alpha \in \mathbf{Z}_n : \alpha^2 = 1\}$. Therefore we say that $M_{m',n}$ and $M_{m,n}$ have the same $\frac{p_1}{2}$ when there is an $\alpha \in A^+(n)$ such that $\frac{p_1}{2}(M_{m',n}) = \alpha \frac{p_1}{2}(M_{m,n}) \in \mathbf{Z}_n$. In the orientation reversing case the invariant $\frac{p_1}{2}$ is unchanged but for the linking form we obtain $lk(-M_{m,n}) = -lk(M_{m,n}) = -l$. We must therefore consider the automorphisms of \mathbf{Z}_n which send l to $-l$. These form the set $A^-(n) := \{\alpha \in \mathbf{Z}_n : \alpha^2 = -1\}$. Thus we say that two manifolds $M_{m',n}$ and $-M_{m,n}$ have the same $\frac{p_1}{2}$ when there exists an $\alpha \in A^-(n)$ such that $\frac{p_1}{2}(M_{m',n}) = \alpha \frac{p_1}{2}(M_{m,n}) \in \mathbf{Z}_n$. Elementary considerations in number theory lead to the following lemma describing the automorphisms α , see for example [L, Theorem 5.2].

Lemma 3.2. (a) *The congruence $\alpha^2 \equiv 1 \pmod n$ has 2^{r+u} solutions, where r is the number of distinct odd prime divisors of n and u is 0, 1 or 2 according to 4 does not divide n , 2^2 exactly divides n or 8 divides n .*

(b) *The congruence $\alpha^2 \equiv -1 \pmod n$ is solvable if and only if $n = p_1^{i_1} \dots p_k^{i_k}$, $p_i \equiv 1 \pmod 4$, p_i prime or $n = 2p_1^{i_1} \dots p_k^{i_k}$, $p_i \equiv 1 \pmod 4$, p_i prime. In this case one has 2^k solutions.*

3.3 Orientation Preserving Homeomorphism

Let G^* be the torsion subgroup of G . Then Wilkens called a symmetric bilinear form $b : G^* \times G^* \rightarrow \mathbf{Q}/\mathbf{Z}$ *irreducible* if it cannot be written as the proper sum of two bilinear forms and he defined M to be *indecomposable* if $H^4(M)$ is finite and $lk(M)$ is irreducible or if $H^4(M) \cong \mathbf{Z}$. Wall ([Wa2]) had already proven that finite irreducible forms only exist for $G \cong \mathbf{Z}_{p^a}$ or $\mathbf{Z}_{2^a} \oplus \mathbf{Z}_{2^a}$ for p prime. Modulo a \mathbf{Z}_2 ambiguity in some cases Wilkens showed that the triple $(H^4(M), lk(M), \frac{p_1}{2}(M))$ classifies indecomposable 2-connected 7-manifolds up to oriented diffeomorphism and connected sum with an exotic 7-sphere. In the case of $M = M_{m,n}$ we obtain $H^4(M_{m,n}) \cong \mathbf{Z}_n \cong \bigoplus_{i=1}^n \mathbf{Z}_{q_i^{e_i}}$ where $n = \prod_{i=1}^k q_i^{e_i}$ is the prime decomposition of n . Then any automorphism of $H^4(M_{m,n})$ must preserve the decomposition into irreducible forms, there being no nontrivial homomorphisms between cyclic groups of coprime orders. Hence any homeomorphism $h : M_{m',n} \rightarrow M_{m,n}$ must preserve the decompositions of $M_{m',n}$ and $M_{m,n}$ into indecomposable manifolds. It follows that the triple $(\mathbf{Z}_n, l, 2m)$ classifies the total spaces $M_{m,n}$ up to an occasional \mathbf{Z}_2 ambiguity.

A homeomorphism of manifolds $f : M' \rightarrow M$ is called an almost diffeomorphism if there is an exotic sphere Σ such that $f : M' \rightarrow M \sharp \Sigma$ is a diffeomorphism. As PL/O is 6-connected, general smoothing theory implies that the manifolds $M_{m,n}$ are almost-diffeomorphic if and only if they are PL -homeomorphic [MM]. Below we show that Wilkens' ambiguous cases have distinct values for s_1 . It follows that manifolds $M_{m',n}$ and $M_{m,n}$ are PL -homeomorphic if and only if they have the same invariants $\frac{p_1}{2}$ and s_1 . But these are topological invariants and so $M_{m',n}$ and $M_{m,n}$ are homeomorphic if and only if they are PL -homeomorphic.

Remark 3.1. *When n is even, $H^4(M_{m,n}; \mathbf{Z}_2) \cong \mathbf{Z}_2$ and thus Kirby-Siebenmann theory guarantees that there is a pair of non-concordant PL -structures on $M_{m,n}$. The above argument shows that these PL structures are PL -homeomorphic.*

Let $n = 2^a q$ with q odd. It follows directly from Wilkens that $\frac{p_1}{2}$ classifies the manifolds $M_{m,n}$ in the following circumstances: n odd or $n = 4q$ and m odd or $n = 2^a q$, $a > 2$ and m even. However, there remained a \mathbf{Z}_2 ambiguity in the cases: $n = 2q$ with any m , $n = 4q$ with m even and $a > 2$ with m odd.

We now complete the orientation preserving homeomorphism classification by showing that the invariant s_1 distinguishes manifolds $M_{m,n}$ with the same $\frac{p_1}{2}$ invariant in the remaining cases.

Let $m' = m - \frac{n}{2}$. Then $\frac{p_1}{2}(M_{m',n}) = \frac{p_1}{2}(M_{m,n})$ and

$$\begin{aligned} s_1(M_{m,n}) - s_1(M_{m',n}) &\equiv \frac{m-m'}{2} + \frac{m^2-m'^2}{2n} \pmod{1} \\ &\equiv \frac{m}{2} + \frac{n}{8} \pmod{1} \\ &\equiv \frac{m}{2} + \frac{2^a q}{8} \pmod{1} \not\equiv 0 \pmod{1} \end{aligned} \tag{6}$$

for the cases of $a = 1$ or $a = 2$ and m even or $a > 2$ and m odd. We have thus proven Theorem 1.2 in the orientation preserving case.

Now, Wilkens showed that every $\frac{p_1}{2}(M)$ of a 2-connected 7-manifold must be divisible by two and since $\frac{p_1}{2}(M_{m,n}) = 2m$, S^3 -bundles over S^4 realize every Wilkens triple $(H^4(M), lk(M), \frac{p_1}{2}(M))$ with linking form $lk(M) = l, l(r, s) = \frac{rs}{n}$. Moreover, we have just seen that each of Wilkens' ambiguous cases is realized by such a bundle. It follows then from Wilkens' classification that any 2-connected 7-manifold with linking form l is almost diffeomorphic, and hence homeomorphic, to an S^3 -bundle over S^4 . This is precisely the statement of Corollary 1.4.

We now proceed to demonstrate the algebraic formulation of Theorem 1.2 given in Corollary 1.3. The first two cases of the orientation preserving homeomorphism condition of Corollary 1.3 immediately follow from the fact that in those cases the invariant $\frac{p_1}{2}$ is enough to classify the manifolds. For the last case we have to prove that the conditions $\frac{p_1}{2}(M_{m',n}) = \alpha \frac{p_1}{2}(M_{m,n})$ and $s_1(M_{m',n}) = s_1(M_{m,n})$ where $\alpha^2 \equiv 1 \pmod{n}$ are equivalent to the conditions $m' \equiv \alpha m \pmod{n}$ and $\alpha \equiv \pm 1 \pmod{2^a}$ where $\alpha^2 \equiv 1 \pmod{n}$. Let $\alpha^2 = kn + 1$ for some $k \in \mathbf{Z}$ and let $m' \equiv \alpha m + \epsilon \frac{n}{2} \pmod{n}$ where $\epsilon = 0, 1$. Then we calculate the difference of the s_1 invariants:

$$\begin{aligned} s_1(M_{m,n}) - s_1(M_{m',n}) &\equiv \frac{\alpha m - m}{2} + \frac{\epsilon n}{4} + \frac{(\alpha m + \frac{\epsilon n}{2})^2}{2n} - \frac{m^2}{2n} \pmod{1} \\ &\equiv \frac{km^2}{2} + \frac{3\epsilon n}{8} + \frac{\alpha m \epsilon}{2} \pmod{1}. \end{aligned}$$

Now if $m' \equiv \alpha m \pmod{n}$ and $\alpha \equiv \pm 1 \pmod{2^a}$ where $\alpha^2 \equiv 1 \pmod{n}$, then the $\frac{p_1}{2}$ invariants must be the same. But the s_1 invariants also must coincide as $\alpha \equiv \pm 1 \pmod{2^a}$ implies that k is even and the $\frac{p_1}{2}$ condition implies that $\epsilon = 0$. For the converse we argue case by case. In the first case of $a = 1$ we need to show that $m' \equiv \alpha m \pmod{2q}$ where $\alpha \equiv 1 \pmod{2}$. But here $0 \equiv s_1(M_{m,n}) - s_1(M_{m',n}) \equiv \frac{km^2}{2} + \frac{3\epsilon q}{4} + \frac{\alpha m \epsilon}{2} \pmod{1}$ which implies that $\epsilon = 0$ and hence $m' \equiv \alpha m \pmod{2q}$. Also $\alpha^2 \equiv 1 \pmod{2q}$ reduces modulo 2 to $\alpha^2 \equiv 1 \pmod{2}$ and hence $\alpha \equiv 1 \pmod{2}$. In the second case of $a = 2$, m even we need to show that $m' \equiv \alpha m \pmod{4q}$ and $\alpha \equiv \pm 1 \pmod{4}$. Again $0 \equiv \frac{km^2}{2} + \frac{3\epsilon q}{2} + \frac{\alpha m \epsilon}{2} \equiv \frac{3\epsilon q}{2} \pmod{1}$ implies that $\epsilon = 0$ and hence $m' \equiv \alpha m \pmod{4q}$. Also $\alpha^2 \equiv 1 \pmod{4q}$ reduces modulo 4 to $\alpha^2 \equiv 1 \pmod{4}$ and hence $\alpha \equiv \pm 1 \pmod{4}$. For the last case of $a > 2$, m odd we need to show that $m' \equiv \pm \alpha m \pmod{2^a q}$ and $\alpha \equiv 1 \pmod{2^a}$. Again $0 \equiv \frac{km^2}{2} + \frac{3\epsilon 2^a q}{8} + \frac{\alpha m \epsilon}{2} \equiv \frac{km^2}{2} + \frac{\alpha m \epsilon}{2} \equiv \frac{k+\epsilon}{2} \pmod{1}$ as m is odd. By Lemma 3.2 we know that there are 4 possible cases for α after reduction modulo 2^a , namely $\alpha \equiv 1, 2^{a-1} - 1, 2^{a-1} + 1, 2^a - 1 \pmod{2^a}$, $a > 2$. For $\alpha \equiv 2^{a-1} - 1, 2^{a-1} + 1 \pmod{2^a}$ we obtain that k must be odd as $\alpha^2 \equiv 1 \pmod{n}$. Hence ϵ must be odd as well by the s_1 condition. But then we obtain that $m' \equiv \pm m \pmod{2^a}$ and hence $m' \equiv \bar{\alpha} m \pmod{n}$ with $\bar{\alpha} \equiv \pm 1 \pmod{2^a}$. In the case $\alpha \equiv 1, 2^a - 1 \pmod{2^a}$ we obtain that k must be even and hence $\epsilon = 0$ by the s_1 condition. Again we conclude that $m' \equiv \alpha m \pmod{n}$ and $\alpha^2 \equiv \pm 1 \pmod{2^a}$.

3.4 Orientation Reversing Homeomorphism

There is an orientation reversing homeomorphism between manifolds M and M' exactly when there is an orientation preserving homeomorphism between M and $-M'$. Since the linking form of $-M_{m,n}$ is $-l$ we now seek automorphisms of \mathbf{Z}_n which send l to $-l$. Such automorphisms α are precisely the elements of $A^-(n) \subset \mathbf{Z}_n$. Now $\frac{p_1}{2}$ is unchanged

under changes of orientation whereas $s_1(-M) = -s_1(M)$. We conclude that $M_{m',n}$ is orientation reversing homeomorphic to $M_{m,n}$ if and only if $s_1(M_{m',n}) = -s_1(M_{m,n})$ and $\frac{p_1}{2}(M_{m',n}) = \alpha \frac{p_1}{2}(M_{m,n})$ for some $\alpha \in A^-(n)$. Applying Lemma 3.2 shows that $A^-(n)$ is nonempty only if $n = 2^a q$ where $a = 0$ or 1 and q is a product of powers of primes which are congruent to $1 \pmod{4}$. When $a = 0$, $\frac{p_1}{2}$ alone classifies the manifolds $M_{m,n}$ and Theorem 1.2 follows in the orientation reversing case. When $a = 1$, $\frac{p_1}{2}$ is ambiguous and we resort to s_1 to settle the issue. Assume that $\frac{p_1}{2}(M_{m',n}) = \alpha \frac{p_1}{2}(M_{m,n})$ so that $m' \equiv \alpha m + \epsilon q \pmod{n}$ where $\epsilon = 0$ or 1 . One calculates

$$s_1(M_{m',n}) + s_1(M_{m,n}) = \frac{\epsilon}{2} + \frac{\epsilon^2}{4} + \frac{1}{4} \pmod{1}.$$

Thus $s_1(M_{m',n}) \equiv -s_1(M_{m,n}) \pmod{1}$ if and only if $\epsilon = 1$ and the orientation reversing case of Theorem 1.2 follows.

4 Diffeomorphism classification

From section 3.3 we know that the manifolds $M_{m,n}$ and $M_{m',n}$ are PL -homeomorphic if and only if they are almost diffeomorphic. In [EK] it is shown that the invariant μ is additive with respect to connected sums and that it distinguishes all exotic 7-spheres. Combining these facts with the PL -homeomorphism classification we obtain that the total spaces $M_{m,n}$ and $M_{m',n}$ are diffeomorphic if and only if they are PL -homeomorphic and their μ -invariants coincide. Hence they are diffeomorphic if and only if they have the same invariants $\frac{p_1}{2}$ and μ .

Smooth surgery theory (see [MM]) implies that there are exactly 28 different smooth manifolds homeomorphic to $M_{m,n}$. A natural question to ask is which manifolds homeomorphic to $M_{m,n}$ may be represented by total spaces of S^3 -bundles over S^4 , i.e. by manifolds $M_{m',n}$. In order to answer this question for some examples of small n we introduce the following integer valued functions.

- Definition 4.1.**
1. Let $\text{Hom}^+(n)$ [$\text{Hom}(n)$] be the number of orientation preserving [orientation preserving and reversing] homeomorphism types of the total space of an S^3 -bundle over S^4 with Euler class n .
 2. Let $\text{Diff}^+(m, n)$ [$\text{Diff}(m, n)$] be the number of distinct orientation preserving [orientation preserving and reversing] classes of smooth manifolds represented by S^3 -bundles over S^4 in each homeomorphism class of $M_{m,n}$.

Using the congruences of Theorems 1.1 and 1.2 we obtain the following results for a selection of values of n with $n \leq 16$. Note that in this range orientation reversing homeomorphism and diffeomorphisms only exist for $n = 1, 2, 5, 10, 13$.

- $n = 1$
 $\text{Hom}^+(1) = 1$ and $\text{Diff}^+(m, 1) = 16$;
 $\text{Hom}(1) = 1$ and $\text{Diff}(m, 1) = 11$;

- $n = 2$
 $\text{Hom}^+(2) = 2$, namely $m \equiv 0, 1 \pmod{2}$;
 $\text{Diff}^+(m, 2) = 8$ in each of the two homeomorphism classes;
 $\text{Hom}(2) = 1$, namely $m \equiv 0$ or $1 \pmod{2}$;
 $\text{Diff}(m, 2) = 13$;
- $n = 5$
 $\text{Hom}^+(5) = 3$, namely $m \equiv 0, \pm 1, \pm 2 \pmod{5}$;
 $\text{Diff}^+(m, 5) = 16$ in each homeomorphism class;
 $\text{Hom}(5) = 2$, namely $m \equiv 0 \pmod{5}$ and
 $m \equiv \pm 1$ or $\pm 2 \pmod{5}$;
 $\text{Diff}(m, 5) = 12$ for $m \equiv 0 \pmod{5}$;
 $\text{Diff}(m, 5) = 24$ for $m \not\equiv 0 \pmod{5}$;
- $n = 10$
 $\text{Hom}^+(10) = 6$, namely $m \equiv 0, \pm 1, \pm 2, \pm 3, \pm 4, 5 \pmod{10}$;
 $\text{Diff}^+(m, 10) = 8$ in each homeomorphism class;
 $\text{Hom}(10) = 3$, namely $m \equiv 0$ or $5 \pmod{10}$ and
 $m \equiv \pm 1$ or $\pm 2 \pmod{10}$ and
 $m \equiv \pm 3$ or $\pm 4 \pmod{10}$;
 $\text{Diff}(m, 10) = 14$ in each homeomorphism class;
- $n = 7$
 $\text{Hom}^+(7) = 4$, namely $m \equiv 0, \pm 1, \pm 2, \pm 3 \pmod{7}$;
 $\text{Diff}^+(m, 7) = 4$ for $m \equiv 0 \pmod{7}$;
 $\text{Diff}^+(m, 7) = 28$ for $m \not\equiv 0 \pmod{7}$;
- $n = 14$
 $\text{Hom}^+(14) = 8$, namely $m \equiv 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, 7 \pmod{14}$;
 $\text{Diff}^+(m, 14) = 2$ for $m \equiv 0 \pmod{14}$ and
for $m \equiv 7 \pmod{14}$;
 $\text{Diff}^+(m, 14) = 14$ for all others;
- $n = 4$
 $\text{Hom}^+(4) = 3$, namely $m \equiv 0, \pm 1, 2 \pmod{4}$;
 $\text{Diff}^+(m, 4) = 4$ for $m \equiv 0 \pmod{4}$;
 $\text{Diff}^+(m, 4) = 8$ for $m \equiv 2 \pmod{4}$;
 $\text{Diff}^+(m, 4) = 16$ for $m \equiv \pm 1 \pmod{4}$;

- $n = 8$

$$\text{Hom}^+(8) = 4, \text{ namely } \begin{array}{l} m \equiv 0 \text{ or } 4 \pmod{8} \text{ and} \\ m \equiv \pm 1, \pm 2, \pm 3 \pmod{8} \end{array}$$

$$\text{Diff}^+(m, 8) = 8 \quad \begin{array}{l} \text{for } m \equiv 0 \text{ or } 4 \pmod{8} \text{ and} \\ \text{for } m \equiv \pm 2 \pmod{8}; \end{array}$$

$$\text{Diff}^+(m, 8) = 16 \quad \text{for } m \equiv \pm 1, \pm 3 \pmod{8};$$

- $n = 12$

$$\text{Hom}^+(12) = 6, \text{ namely } \begin{array}{l} m \equiv \pm 1 \text{ or } \pm 5 \pmod{12} \text{ and} \\ m \equiv 0, \pm 2, \pm 3, \pm 4, 6 \pmod{12} \end{array}$$

$$\text{Diff}^+(m, 12) = 4 \quad \text{for } m \equiv 0, \pm 4 \pmod{12};$$

$$\text{Diff}^+(m, 12) = 8 \quad \text{for } m \equiv \pm 2, 6 \pmod{12};$$

$$\text{Diff}^+(m, 12) = 16 \quad \begin{array}{l} \text{for } m \equiv \pm 1 \text{ or } \pm 5 \pmod{12} \text{ and} \\ \text{for } m \equiv \pm 3 \pmod{12}; \end{array}$$

- $n = 16$

$$\text{Hom}^+(16) = 7, \text{ namely } \begin{array}{l} m \equiv 0 \text{ or } 8 \pmod{16} \text{ and} \\ m \equiv \pm 2 \text{ or } \pm 6 \pmod{16} \text{ and} \\ m \equiv \pm 1, \pm 3, \pm 4, \pm 5, \pm 7 \pmod{16} \end{array}$$

$$\text{Diff}^+(m, 16) = 4 \quad \text{for } m \equiv \pm 4 \pmod{16};$$

$$\text{Diff}^+(m, 16) = 8 \quad \text{for } m \equiv 0 \text{ or } 8 \pmod{16};$$

$$\text{Diff}^+(m, 16) = 16 \quad \begin{array}{l} \text{for } m \equiv \pm 2 \text{ or } \pm 6 \pmod{16} \text{ and} \\ \text{for } m \equiv \pm 1, \pm 3, \pm 5, \pm 7 \pmod{16}. \end{array}$$

For completeness we include the specific μ -values for the case $n = 10$ which includes the seven dimensional Berger manifolds. We list the μ -values modulo $224n = 2240$ in the orientation preserving and reversing case.

- $m \equiv 0 \text{ or } 5 \pmod{10}$:

$$\mu \in \{90, 250, 390, 410, 890, 950, -10, -230, -330, -570, -650, -710, -870, -1030\}$$

- $m \equiv \pm 1 \text{ or } \pm 2 \pmod{10}$:

$$\mu \in \{26, 54, 134, 186, 454, 666, 694, 774, 986, 1014, -506, -906, -934, -1094\}$$

- $m \equiv \pm 3 \text{ or } \pm 4 \pmod{10}$:

$$\mu \in \{6, 246, 326, 474, 566, 634, 886, 1114, -314, -394, -486, -646, -806, -954\}$$

5 Classification up to homotopy equivalence

We now describe the simply connected surgery exact sequence in order to deduce the homotopy classification of the total spaces from their homeomorphism classification. Recall the following definitions of the simply connected, oriented structure set and normal

invariant set. We refer the reader to [MM] Chapter 2 and [Wa3] Chapter 10 for further details. Let Cat stand for either the smooth (O), piecewise linear (PL), or topological (Top) category and let M be an oriented compact Cat -manifold with boundary ∂M (which may be empty). The Cat -structure set of M , $\mathcal{S}^{\text{Cat}}(M)$, consists of orientation preserving equivalence classes of homotopy equivalences of pairs $f : (L, \partial L) \rightarrow (M, \partial M)$ from a Cat -manifold L relative its boundary to $(M, \partial M)$. Two homotopy equivalences (L_1, f_1) and (L_2, f_2) are equivalent in $\mathcal{S}^{\text{Cat}}(M)$ if there exists a Cat h-cobordism W with $\partial W = L_1 \cup L_2 \cup N$, where N is an h-cobordism from ∂L_1 to ∂L_2 , as well as a homotopy equivalence $F : (W, \partial W) \rightarrow (M \times I, \partial(M \times I))$ such that $F|_{L_i} = f_i$ for $i = 1, 2$. Equivalently, one may ask for a Cat -isomorphism $g : L_1 \rightarrow L_2$ such that f_1 is homotopy equivalent to $f_2 \circ g$. The set of Cat -structures of M maps to the set of degree one normal maps to M modulo normal cobordisms, $\mathcal{N}^{\text{Cat}}(M)$. A degree one normal map to $(M, \partial M)$ is a bundle map (f, \hat{f})

$$\begin{array}{ccc} \nu_L & \xrightarrow{\hat{f}} & \nu \\ \downarrow & & \downarrow \\ (L, \partial L) & \xrightarrow{f} & (M, \partial M) \end{array}$$

from the normal bundle ν_L of a Cat -manifold L to a Cat -bundle over M such that the induced map on homology carries the fundamental class of L to that of M . The equivalence relation between degree one normal maps is precisely cobordism with boundary over the bundle $\nu \rightarrow M$. Specifically, two degree one normal maps

$$\begin{array}{ccc} \nu_{L_1} & \xrightarrow{\hat{f}_1} & \nu_1 \\ \downarrow & & \downarrow \\ (L_1, \partial L_1) & \xrightarrow{f_1} & (M, \partial M) \end{array} \quad \text{and} \quad \begin{array}{ccc} \nu_{L_2} & \xrightarrow{\hat{f}_2} & \nu_2 \\ \downarrow & & \downarrow \\ (L_2, \partial L_2) & \xrightarrow{f_2} & (M, \partial M) \end{array}$$

are normally cobordant if there exists a compact manifold W and a degree one normal map (F, \hat{F})

$$\begin{array}{ccc} \nu_W & \xrightarrow{\hat{F}} & \nu \times I \\ \downarrow & & \downarrow \\ W & \xrightarrow{F} & M \times I \end{array}$$

with the following two properties:

(i) $\partial W = L_1 \cup L_2 \cup N$ where N is a compact manifold with $\partial N = (\partial L_1 \cup \partial L_2) \cap N = \partial L_1 \cup \partial L_2$

(ii) $(F|_{L_1}, \hat{F}|_{\nu_{L_1}}) = (f_1, \hat{f}_1)$ and $(F|_{L_2}, \hat{F}|_{\nu_{L_2}}) = (f_2, b \circ \hat{f}_2)$ where $b : \nu_2 \rightarrow \nu_1$ is a bundle isomorphism.

In order to use the central results of simply connected surgery theory we first define the following abelian groups (where e stands for the trivial group).

$$L_k(e) = \left\{ \begin{array}{ll} \mathbf{Z}, & \text{if } k = 4j; \\ \mathbf{Z}_2, & \text{if } k = 4j + 2; \\ 0, & \text{otherwise.} \end{array} \right\}, \quad L_k(e \rightarrow e) = 0 \text{ for all } k.$$

The following two facts are well-known, see [Wa3] or [MM].

Fact 5.1. (1) Let G/Cat be the fiber of the map $B\text{Cat} \rightarrow BG$. Then $\mathcal{N}^{\text{Cat}}(M) \cong [M, G/\text{Cat}]$.

(2) For $k > 5$ there is a pair of linked exact sequences of sets

$$\begin{array}{ccccccc} L_{k+1}(e \rightarrow e) & \longrightarrow & \mathcal{S}^{\text{Cat}}(M^k) & \xrightarrow{\eta} & \mathcal{N}^{\text{Cat}}(M^k) & \longrightarrow & L_k(e \rightarrow e) \\ \downarrow & & i^* \downarrow & & i^* \downarrow & & \downarrow \\ L_k(e) & \longrightarrow & \mathcal{S}^{\text{Cat}}((\partial M)^{k-1}) & \xrightarrow{\eta} & \mathcal{N}^{\text{Cat}}((\partial M)^{k-1}) & \longrightarrow & L_{k-1}(e) \end{array}$$

We now apply surgery theory in the PL category when the manifold pair is the disc-bundle, sphere-bundle pair of a vector bundle over S^4 , $(M, \partial M) = (W_{m,n}, M_{m,n})$. It is well known that η is injective for simply connected PL -surgery and in this case $k = 8$, $L_7(e) = 0$ and η is also surjective. Thus the PL -structure set and PL -normal invariant set coincide for both $W_{m,n}$ and $M_{m,n}$. Moreover, both $[W_{m,n}, G/PL]$ and $[M_{m,n}, G/PL]$ can be identified, via the primary obstruction to null homotopy, with $H^4(W_{m,n}; \pi_4(G/PL)) \cong H^4(W)$ and $H^4(M_{m,n}; \pi_4(G/PL)) \cong H^4(M)$ respectively. We therefore obtain the following commutative diagram.

$$\begin{array}{ccccccc} \mathcal{S}^{PL}(W_{m,n}) & \xrightarrow{\eta} & \mathcal{N}^{PL}(W_{m,n}) & \cong & H^4(W_{m,n}) & \cong & \mathbf{Z} \\ i^* \downarrow & & i^* \downarrow & & i^* \downarrow & & \downarrow \\ \mathcal{S}^{PL}(M_{m,n}) & \xrightarrow{\eta} & \mathcal{N}^{PL}(M_{m,n}) & \cong & H^4(M_{m,n}) & \cong & \mathbf{Z}_n \end{array}$$

(7)

We emphasize that the first two vertical maps i^* are given by restricting a PL -structure or PL -normal invariant to the PL -structure or PL -normal invariant on the boundary.

A class of homotopy equivalences between the total spaces $M_{m,n}$ is provided by the set of fiber homotopy equivalences of the associated spherical fibrations. We shall demonstrate that all PL -structures in $M_{m,n}$ have representatives which are fiber homotopy equivalences. We let $SG(4)$ denote the topological monoid of orientation preserving self homotopy equivalences of S^3 which acts as the structure “group” for spherical fibrations with fiber S^3 . There is a natural inclusion $i_4 : SO(4) \hookrightarrow SG(4)$ obtained by restricting each element of $SO(4)$ to $S^3 \hookrightarrow \mathbf{R}^4$. Moreover, from Theorem 6.2 and Corollary 7.4 of [DL] we deduce that if ξ and η are elements of $\pi_3(SO(4))$ then the corresponding sphere bundles, $S(\xi)$ and $S(\eta)$, are fiber homotopy equivalent if and only if $i_{4*}(\xi) = i_{4*}(\eta) \in \pi_3(SG(4))$.

Lemma 5.2. *There exist fiber homotopy equivalences $f_j : M_{m+12j,n} \rightarrow M_{m,n}$ for all $j \in \mathbf{Z}$.*

Proof. Recall that $M_{m,n} = S(\xi_{m,n})$ where we use the same notation as before for group elements $\xi_{m,n} \in \pi_3(SO(4))$. The proof of this lemma follows immediately from the

preceding remarks and the following fact. There is an isomorphism $\pi_3(SG(4)) \cong \mathbf{Z}_{12} \oplus \mathbf{Z}$ such that

$$i_{4*}(\xi_{m,n}) = (m \bmod 12, n)$$

We shall give a proof of this fact since we have not found a reference to it in the literature.

Let $SF(3)$ denote the subspace of $SG(4)$ consisting of orientation preserving homotopy equivalences of S^3 which fix a point p . By the usual adjoint correspondence there is an isomorphism $A_{3,3} : \pi_3(SF(3)) \cong \pi_6(S^3)$. We relate the homotopy groups of $SF(3)$ to $SG(4)$ by noting that $SF(3)$ is the fiber of the fibration

$$\begin{array}{ccc} ev : SG(4) & \rightarrow & S^3 \\ f & \mapsto & f(p). \end{array}$$

Moreover, $ev|_{SO(4)} : SO(4) \rightarrow S^3$ is the usual fibration with fiber $SO(3) \subset SF(3)$. The fibration $ev|_{SO(4)}$ has a section $s : S^3 \rightarrow SO(4)$ given by $s(x) = L_x$ where $L_x(y) = x \cdot y$ where we regard $x, y \in S^3$ as unit quaternions and \cdot denotes quaternionic multiplication. Applying the long exact homotopy sequence to this pair of fibrations at π_3 we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} \oplus \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \pi_3(SO(3)) & \xrightarrow{i_*} & \pi_3(SO(4)) & \xrightarrow{ev|_{SO(4)*}} & \pi_3(S^3) & \longrightarrow & 0 \\ & & \downarrow i_{3*} & & \downarrow i_{4*} & & \downarrow Id & & \\ 0 & \longrightarrow & \pi_3(SF_3) & \longrightarrow & \pi_3(SG(4)) & \xrightarrow{ev_*} & \pi_3(S^3) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathbf{Z}_{12} & \longrightarrow & \mathbf{Z}_{12} \oplus \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \end{array}$$

To conclude the proof we observe the following. Firstly, if ι_3 and ρ denote suitable generators of $\pi_3(S^3)$ and $\pi_3(SO(3))$ respectively, then for integers n and m ,

$$s_*(n\iota_3) = \xi_{0,n} \quad \text{and} \quad i_*(m\rho) = \xi_{m,0}.$$

Secondly, if $i_3 : SO(3) \hookrightarrow SF(3)$ denotes the inclusion of one fiber into the other, then the homomorphism $A_{3,3} \circ i_{3*} : \pi_3(SO(3)) \rightarrow \pi_6(S^3)$ is the usual J -homomorphism (see [Wh] 502-4). But the J -homomorphism $J_{3,3} : \pi_3(SO(3)) \rightarrow \pi_6(S^3)$ is known to be the surjection $\mathbf{Z} \rightarrow \mathbf{Z}_{12}$ ([JW1]). \square

Lemma 5.3. *The fiber homotopy equivalences $f_j : M_{m+12j,n} \rightarrow M_{m,n}$ have normal invariant $\eta(f_j) = j \in \mathcal{N}^{PL}(M_{m,n}) \cong \mathbf{Z}_n$.*

Proof. For any map of sphere bundles $f : S(\xi) \rightarrow S(\zeta)$ we may define the cone on f to be the map of disc bundles $F : D(\xi) \rightarrow D(\zeta)$ which for $0 \leq s \leq 1$ and $s \cdot v \in D(\xi)$ takes the value $F(s \cdot v) = s \cdot f(v)$. One can then check that the pair (F_j, f_j) defines a fiber homotopy equivalence of pairs, $(F_j, f_j) : (W_{m+12j,n}, M_{m+12j,n}) \rightarrow (W_{m,n}, M_{m,n})$. Now recall that the vertical maps i^* in the surgery exact sequences (7) are given by restricting

a homotopy equivalence of pairs to the boundary component. Thus $\eta(f_j) = i^*\eta(F_j)$ and it suffices to prove that $\eta(F_j) \in \mathcal{N}^{PL}(W_{m,n})$ takes on the value $j \in \mathbf{Z}$.

To compute the normal invariant of F_j we first define $i_{m,n}$ to be the inclusion of the zero section S^4 into $W_{m,n}$. Then $i_{m,n}$ is a homotopy equivalence onto its image. Now let $j : G/PL \rightarrow BPL$ denote the standard inclusion of the fiber of the canonical map $BPL \rightarrow BG$. We obtain the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{N}^{PL}(W_{m,n}) & \cong & [W_{m,n}, G/PL] & \xrightarrow{j_*} & [W_{m,n}, BPL] \\ \downarrow & & i_{m,n}^* \downarrow & & i_{m,n}^* \downarrow \\ \mathcal{N}^{PL}(S^4) & \cong & [S^4, G/PL] & \xrightarrow{j_*} & [S^4, BPL] \end{array}$$

Each vertical arrow induces a bijection since $i_{m,n}$ is a homotopy equivalence. The bottom j_* is the map $j_* : \pi_4(G/PL) \rightarrow \pi_4(BPL)$ which is well known to be multiplication by 24 between two infinite cyclic groups. For any compact space X we may regard $[X, BPL]$ as formal differences of stable PL -bundles over X . Hence for $X = W_{m,n}$ we obtain

$$j_*(\eta(F_j)) = \nu(W_{m,n}) - F_j^{-1*}(\nu(W_{m+12j,n})).$$

Now recall Fact 3.1 from Section 3 which has a counterpart for disc-bundles implying that $\nu(W_{m,n}) = \pi_{m,n}^*(\nu_{S^4} \oplus -\xi_{m,n}) = \pi_{m,n}^*(-\xi_{m,n})$. Since F_j commutes with $\pi_{m,n}$ and $\pi_{m+12j,n}$, we may choose F_j^{-1} to similarly commute up to homotopy. Thus

$$\begin{aligned} i_{m,n}^*(j_*(\eta(F_j))) &= i_{m,n}^*(\nu(W_{m,n}) - F_j^{-1*}(\nu(W_{m+12j,n}))) \\ &= i_{m,n}^*(\pi_{m,n}^*(-\xi_{m,n}) - F_j^{-1*}(\pi_{m+12j,n}^*(-\xi_{m+12j,n}))) \\ &= i_{m,n}^*(\pi_{m,n}^*(-\xi_{m,n} + \xi_{m+12j,n})) \\ &= \xi_{m+12j,n} - \xi_{m,n} \\ &= 2(m + 12j) + n - (2m + n) && \in \mathbf{Z} \\ &= 24j && \in \mathbf{Z} \end{aligned}$$

Thus $\eta(F_j) = 24j/24 = j \in \mathcal{N}^{PL}(W_{m,n})$ and so $\eta(f_j) = j \in \mathcal{N}^{PL}(M_{m,n})$. \square

Proof of Theorem 1.1 The description of the PL -structure set in 5.3 implies that a given seven dimensional PL -manifold M is homotopy equivalent to $M_{m,n}$ if and only if M is PL -homeomorphic to $M_{m+12j,n}$ for some j . By Theorem 1.2 M is PL -homeomorphic to $M_{m+12j,n}$ for some j if and only if M is homeomorphic to $M_{m+12j,n}$ for some j . In Corollary 1.3 we gave necessary and sufficient conditions on m and m' for $M_{m,n}$ and $M_{m',n}$ to be homeomorphic. It remains to check that applying Corollary 1.3 yields the relation given in Theorem 1.1 and we do this now.

Case $n = 2^0 q$, q odd: $\frac{q-1}{2}$ classifies.

$$\begin{aligned} M_{m',n} \simeq M_{m,n} &\Leftrightarrow M_{m',n} \cong M_{m+12j,n} \text{ for some } j. \\ &\Leftrightarrow \alpha m' \equiv m + 12j \pmod{n} \text{ for some } j \text{ and some } \alpha \in A^+(n). \\ &\Leftrightarrow \alpha m' \equiv m \pmod{(n, 12)} \text{ for some } \alpha \in A^+(n). \\ &\Leftrightarrow \alpha m' \equiv m \pmod{(n, 12)} \text{ for some } \alpha \in A^+((n, 12)). \end{aligned}$$

Case $n = 2^a q$, $a = 2$, m odd or $a > 2$, m even: $\frac{p_1}{2}$ classifies and $M_{m+\epsilon \frac{n}{2}, n} \cong M_{m,n}$ for $\epsilon = 0, 1$.

$$\begin{aligned}
M_{m',n} \simeq M_{m,n} &\Leftrightarrow M_{m',n} \cong M_{m+12j+\epsilon \frac{n}{2}, n} \text{ for some } j \text{ and for } \epsilon = 0, 1. \\
&\Leftrightarrow \alpha m' \equiv m + 12j + \epsilon \frac{n}{2} \pmod{\frac{n}{2}} \text{ for some } j, \epsilon = 0, 1, \alpha \in A^+(n). \\
&\Leftrightarrow \alpha m' \equiv m + 12j \pmod{n} \text{ for some } \alpha \in A^+(n). \\
&\Leftrightarrow \alpha m' \equiv m \pmod{(n, 12)} \text{ for some } \alpha \in A^+((n, 12)).
\end{aligned}$$

Case $n = 2^a q$, $a = 1$ or $a = 2$ and m even or $a > 2$ and m odd: $\frac{p_1}{2}$ does not classify.

$$\begin{aligned}
M_{m',n} \simeq M_{m,n} &\Leftrightarrow M_{m',n} \cong M_{m+12j,n} \text{ for some } j. \\
&\Leftrightarrow \alpha m' \equiv m + 12j \pmod{n} \text{ for some } j, \alpha \in A^+(n), \alpha \equiv \pm 1 \pmod{2^a}. \\
&\Rightarrow \alpha m' \equiv m \pmod{(n, 12)} \text{ for some } \alpha \in A^+(n). \\
&\Leftrightarrow \alpha m' \equiv m \pmod{(n, 12)} \text{ for some } \alpha \in A^+((n, 12)).
\end{aligned}$$

To reverse the third implication we must show that if $\alpha m' \equiv m + 12j \pmod{n}$ for some $\alpha \in A^+(n)$ then $\bar{\alpha}$ and \bar{j} can be chosen so that $\bar{\alpha} m' \equiv m + 12\bar{j} \pmod{n}$ and $\bar{\alpha} \equiv \pm 1 \pmod{2^a}$. This is evident for $a = 1, 2$ so we assume that $a > 2$ and $\alpha \not\equiv \pm 1 \pmod{n}$. But now $\frac{n}{2} \equiv 12k \pmod{n}$ for some k and, since m is odd, $(\alpha + \frac{n}{2})m' \equiv m + 12(j+k) \pmod{n}$. Moreover $(\alpha + \frac{n}{2})^2 \equiv 1 \pmod{n}$, hence $\bar{\alpha} = \alpha + \frac{n}{2}$ and $\bar{j} = j + k$ suffice.

In the orientation reversing case, $M_{m',n} \simeq -M_{m,n}$ if and only if $M_{m',n} \cong -M_{m+12j,n}$ for some j . But we know from Theorem 1.2 that this only happens when $n = 2^\epsilon p_1^{i_1} \dots p_k^{i_k}$, $p_i \equiv 1 \pmod{4}$ and $\epsilon = 0, 1$. When $\epsilon = 0$ there is only one oriented homotopy type and it admits an orientation reversing homotopy equivalence by Theorem 1.2. When $\epsilon = 1$ there are two oriented homotopy types and these are orientation reversing homotopy equivalent again by Theorem 1.2. This concludes the proof of Theorem 1.1.

6 Remarks on the classification

6.1 The integer 4 is not favorable

The significance of orientation in the classification is strikingly illustrated by the pair of manifolds $M_{-1,2}$ and $M_{0,2}$. The first is the total space of the unit tangent bundle of S^4 and the second is the total space of “twice” the Hopf fibration. It follows from Theorem 1.3 that these bundles are orientation reversing diffeomorphic, a fact well known to geometers, see for example [Ri, p.192, example g], but by Theorem 1.1 they are not orientation preserving homotopy equivalent. Notice this situation implies that the oriented homotopy types represented by $M_{-1,2}$ and $M_{0,2}$ do not admit orientation reversing homotopy equivalences. If they did then they would be orientation preserving homotopy equivalent. James and Whitehead ([JW2, p.151]) called an integer k favorable when the existence of an orientation reversing homotopy equivalence between $(k-1)$ -sphere bundles over S^k with Euler number 2 implies the existence of an orientation preserving homotopy equivalence between the bundles. We have thus shown that 4 is not favorable.

6.2 The total spaces $M_{m,n}$ versus the pairs $(M_{m,n}, S^3)$

We noted above that the homotopy classification of the total spaces and the homotopy classification of the pairs differ if and only if $(n, 12) = 12$. For example, $M_{1,12}$ and $M_{5,12}$ are homeomorphic but the pairs $(M_{1,12}, S^3)$ and $(M_{5,12}, S^3)$ are not homotopy equivalent. Algebraically, the homeomorphism takes a fiber to five times a fiber and thus pays no respect to the bundle structures of domain and range. In [JW0, Theorem 3.7] James and Whitehead give connectivity conditions under which maps of bundle-like spaces are homotopic to maps of bundle fiber pairs. This example shows that their results are sharp.

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